CAMBRIDGE LECTURE NOTES IN PHYSICS 4

Diagrammatica

The Path to Feynman Diagrams

MARTINUS VELTMAN

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The Path to Feynman Rules

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Introduction

In recent years particle theory has been very successful. The theory agrees with the data wherever it could be tested, and while the theory has its weak spots, this numerical agreement is a solid fact. Physics is a quantitative science, and such agreement defines its validity.

It is a fact that the theory, or rather the successful part, is perturbation theory. Up to this day the methods for dealing with non-perturbative situations are less than perfect. No one, for example, can claim to understand fully the structure of the proton or the pion in terms of quarks. The masses and other properties of these particles have not really been understood in any detail. It must be added that there exists, strictly speaking, no sound starting point for dealing with non-perturbative situations.

Perturbation theory means Feynman diagrams. It appears therefore that anyone working in elementary particle physics, experimentalist or theorist, needs to know about these objects. Here there is a most curious situation: the resulting machinery is far better than the originating theory. There are formalisms that in the end produce the Feynman rules starting from the basic ideas of quantum mechanics. However, these formalisms have flaws and defects, and no derivation exists that can be called satisfactory. The more or less standard formalism, the operator formalism, uses objects that can be proven not to exist. The way that Feynman originally found his diagrams, by using path integrals, can hardly be called satisfactory either: on what argument rests the assumption that a path integral describes nature? What is the physical idea behind that formalism? Path integrals are objects very popular among mathematically oriented theorists, but just try to sell them to an experimentalist. However, to be more positive, given that one believes Feynman diagrams, path integrals may be considered a very valuable tool to understand properties of these diagrams. They are justified by the result, not by their definition. They are mathematical tools.

Well, things are as they are. In this book the object is to derive Feynman rules, but there is no good way to do that. The physicist may take a pragmatic attitude: as long as it works, so what. Indeed, that is a valid attitude. But that is really not enough. Feynman rules have a true physics content, and the physicist must understand that. He/she must know how Lorentz invariance, conservation of probability, renormalizability reflect themselves in the Feynman rules. In other words, even if there is no rigorous foundation for these rules, the physical principles at stake must be understood.

This then is the aim: to make it clear which principles are behind the rules, and to define clearly the calculational details. This requires some kind of derivation. The method used is basically the canonical formalism, but anything that is not strictly necessary has been cut out. No one should have an excuse not understanding this book. Knowing about ordinary non-relativistic quantum mechanics and classical relativity one should be able to understand the reasoning.

This book is somewhat unusual in that I have tried very hard to avoid numbering the equations and the figures. This has forced me to keep all derivations and arguments closed in themselves, and the reader needs not to have his fingers at eleven places to follow an argument.

I am indebted to my friends and colleagues R. Akhoury, F. Erné, P. Federbush, P. Van Nieuwenhuizen and F.J. Yndurain. They have read the manuscript critically and suggested many improvements.

The help of M. Jezabek in unraveling the complications of metric usage is gratefully acknowledged. I have some hope that this matter can now finally be put to rest, by providing a very simple translation dictionary.

Ann Arbor, December 1993

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1
Lorentz and Poincaré Invariance

1.1 Lorentz Invariance

To begin with we will very briefly review some aspects of Einstein's theory of relativity that are of particular importance here.

The theory of relativity states that physical laws are the same in two systems that move with respect to each other with uniform velocity. Furthermore, the speed of light is a constant. These two statements lead to the concept of invariance under Lorentz transformations. We now first investigate Lorentz transformations in some detail. Some of the rather basic mathematics involved is summarized in an appendix.

Lorentz transformations can be understood as rotations in four-dimensional space (three-dimensional space + time). A rotation can be specified by a matrix L with the property that $\widetilde{L}$ (the reflected of L) is its inverse:

$$\widetilde{L} = L^{-1} \qquad \text{or} \qquad L\widetilde{L} = 1 \,.$$

Writing indices explicitly:

$$L_{\mu\nu}\widetilde{L}_{\nu\lambda} = \delta_{\mu\lambda} \quad \text{or} \quad L_{\mu\nu}L_{\lambda\nu} = \delta_{\mu\lambda} \,.$$

We have used here Einstein's summation convention: twice occurring indices (such as ν here) are summed over (in this case $\nu = 1, \ldots, 4$). At this point we must also settle some conventions. We take the fourth dimension as imaginary, $x_4 = ict$. This leads to the fact that the matrix-elements of a Lorentz transformation are imaginary if one (but not both) of the indices is four. With this convention a particle at rest has the four-momentum $(0, 0, 0, iM)$, where c has already been taken to be one.

Let us emphasize that there is no physics in the choice of metric. Some physicists prefer to work with real space/time but define their dot-product with a metric involving minus signs. It is really of no relevance where you hide your minus signs, at most it is a matter of convenience. Which is usually what you are used to. It is a matter though that you can debate hotly at lunch time (real time). See appendix on metric.

Examples of Lorentz transformations are:

- ordinary rotations in three dimensions such as a rotation over an angle ϕ around the third axis;

$$L = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- transformation to a system moving with velocity v along the first axis:

$$L = \begin{bmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{bmatrix}$$

where θ is imaginary and such that $\sin\theta = iv/c\beta$, $\beta = \sqrt{1 - v^2/c^2}$. It follows that $\cos\theta = \sqrt{1 - \sin^2\theta} = 1/\beta$. This transformation is a rotation over an imaginary angle.

In addition to the Lorentz transformations that have determinant 1, such as the ordinary rotations and the velocity transformations there are also transformations with determinant -1. These are the space or time reflections. These are not transformations that you can actually do: nobody has ever managed to reflect himself, transforming himself from, say, a right handed person into a left handed person. In particle physics it has been discovered that the laws of nature are not invariant with respect to these reflections, although large parts of the interactions are. The reflections remain therefore important tools in classifying interactions and establishing selection rules.

A reflection is the combination of any ordinary Lorentz transformation and a space reflection P or time reversal T:

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

There are a number of fundamental differences about rotations over real versus imaginary angles. Rotating over a real angle of 2π gives the identity, because $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$. Thus two successive rotations may lead back to the original (for example a rotation of 30° followed by a rotation of 330°). Since no such thing holds for imaginary angles this is not true for the rotations over imaginary angles. In fact:

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right),$$

and if θ is imaginary, $\theta = i\alpha$ with real α, then

$$\sin\theta = \frac{1}{2i}\left(e^{-\alpha} - e^{\alpha}\right),$$

which grows exponentially for θ going to either $\pm i\infty$. Thus while the domain of the angle ϕ for spatial rotations is finite, $0 \leq \phi < 2\pi$, the domain of θ is infinite. The rotations in three dimensions form a compact group (finite domain of the parameters) while the full set of Lorentz transformations is non-compact. To judge on compact versus non-compact on the basis of the domain of the parameters one must first specify how the parameters are to be chosen; the requirement is that they be chosen such that two successive applications of the same transformation described by the parameters $\alpha_1, \ldots, \alpha_n$ must be given by the transformation described by the parameters $2\alpha_1, \ldots, 2\alpha_n$. For example, two successive rotations over an angle θ equals a rotation over an angle 2θ, and that means that the parameter θ is appropriate for a judgment on compactness. The importance of compactness relates to group theory: compact groups have unitary representations. The Lorentz group is non-compact, and its representations are not necessarily unitary. At this point there is no need to understand this mathematically in any detail.

A general Lorentz transformation can be seen as a combination of a rotation in three dimensional space followed by a transformation to a system moving with some velocity in some direction. A rotation can be specified by three parameters, for example by a vector whose direction is the axis of rotation while its magnitude equals the magnitude of the rotation. Thus the vector $(0, 0, \pi/2)$ specifies a rotation over 90° around the third axis. The "velocity" transformation can be specified by giving the velocity, which is also a three component vector. It would be nice if we could talk in terms of an axis and an imaginary angle also in this case, but that is not important at this moment. The important point is that we observe that a Lorentz transformation is specified by six parameters. Three have a finite, three an infinite domain. The Lorentz-transformations form a six-parameter group.

Since any finite rotation can be seen as an infinite sequence of infinitesimal rotations it is sufficient for most purposes to understand infinitesimal Lorentz transformations. Let us first consider a rotation over an angle ϕ around the third axis. Its form has been given above, and we will denote it by $L(\phi)$. This rotation can be obtained also by applying n times a rotation over an angle ϕ/n :

$$L(\phi) = \left[L\left(\frac{\phi}{n}\right)\right]^n .$$

Let us now consider a rotation over an angle ϕ/n with very large n. We may then expand $\sin(\phi/n)$ and $\cos(\phi/n)$ to get:

$$\begin{aligned} L(\phi/n) &= \begin{bmatrix} 1 & \phi/n & 0 & 0 \\ -\phi/n & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \mathcal{O}\left(\phi^2/n^2\right) \\ &= I + \frac{\phi}{n} L_3 + \mathcal{O}\left(\phi^2/n^2\right) \end{aligned}$$

with

$$L_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and I denoting the unit matrix. In the limit of large n:

$$L(\phi) = \lim_{n\to\infty} \left[L\left(\frac{\phi}{n}\right)\right]^n = e^{\phi L_3} .$$

Exercise 1.1 Read the appendix on matrices or else show that

$$\left[1 + \frac{\alpha}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]^n = e^{\alpha} + \mathcal{O}\left(\frac{1}{n}\right) .$$

We have now written this Lorentz transformation in exponential form. The great advantage is that the parameter ϕ is directly visible, and the property $L(\phi)L(\phi) = L(2\phi)$ is manifest:

$$e^{\phi L_3} e^{\phi L_3} = e^{2\phi L_3}.$$

Similarly other rotations may be treated. A general infinitesimal rotation in three dimensions differs infinitesimally from the unit matrix:

$$R = \begin{bmatrix} 1+g & a & b & 0 \\ d & 1+h & c & 0 \\ e & f & 1+k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $a \dots k$ infinitesimal. For a rotation the equation $R\tilde{R} = 1$ holds, and this leads to the result $h = g = k = 0$, $d = -a$, $e = -b$ and $f = -c$ (ignoring higher order terms in a, b, etc.).

Exercise 1.2 Prove this assertion.

We therefore can write:

$$R = I + cL_1 - bL_2 + aL_3$$

with

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reason for the sign choices above will become clear shortly. Since a finite transformation can be obtained by exponentiation of an infinitesimal one we so find a representation in terms of three parameters for any rotation in three dimensions:

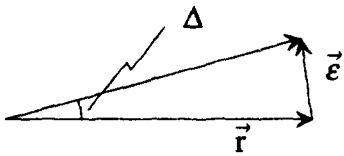
$$R = e^{\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3} = e^{\alpha_i L_i}.$$

The three quantities α_i are precisely equal to the vector used to describe rotations introduced above. Thus the direction of α is the axis of rotation, the magnitude is the magnitude of the rotation in radians. The sense of the rotation is this: if $\vec{\alpha}$ points upwards, along the positive third axis, then a small rotation will turn a vector along the positive first axis into a vector having a small negative second component. From the above this connection is obvious for the special cases of rotations around first, second, or third axis. For the general case this becomes obvious by considering an infinitesimal rotation:

$$\begin{bmatrix} 1 & \alpha_3 & -\alpha_2 & 0 \\ -\alpha_3 & 1 & \alpha_1 & 0 \\ \alpha_2 & -\alpha_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \alpha_i \text{ infinitesimal.}$$

This matrix describes an infinitesimal rotation over an angle $\Delta = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$ around an axis in the direction $\vec{\alpha}$. The choice of signs for the L_i was made such as to obtain this.

Exercise 1.3 Verify the above by showing that a vector in the direction of $\vec{\alpha}$ is invariant, while a vector perpendicular to $\vec{\alpha}$, for example the unit vector $\vec{r}$ with components $\lambda(\alpha_2,-\alpha_1,0,0)$ with $\lambda = 1/\sqrt{(\alpha_1^2 + \alpha_2^2)}$, is changed by an amount corresponding to a rotation over an angle Δ. Thus compute the effect of the infinitesimal rotation on $\vec{r}$, writing the result in the form $\vec{r}+\vec{\epsilon}$. Show that $\vec{\epsilon}$ is orthogonal to $\vec{r}$ and $\vec{\alpha}$, and has the magnitude Δ.



This treatment can be extended trivially to include the "velocity" transformations. A general "velocity" transformation will be of the form:

$$V = e^{\beta_1 M_1 + \beta_2 M_2 + \beta_3 M_3}$$

with imaginary β_1, β_2 and β_3 and real M_1, M_2 and M_3:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Of course we could equally well have used real $\vec{\beta}$ and imaginary

M, but we will do it this way. Writing $\vec{\beta} = i\vec{v}$, a particle of mass m at rest is transformed into a particle moving with momentum $\vec{p} = -m\vec{v}$. Note that the $\vec{v}$ here is the conventional velocity divided by the relativistic factor β.

The general Lorentz transformation is of the form

$$L = e^{\alpha_i L_i + \beta_i M_i}$$

but the interpretation of the α_i and β_i in terms of axis of rotation or velocity is no more as easy as above.

At this point it is necessary to introduce a new and sometimes slightly confusing notation. We write:

$$L = e^{\frac{1}{2}\alpha_{\mu\nu}K_{\mu\nu}}, \quad \mu, \nu = 1, \ldots 4.$$

The matrices K are defined by the prescription that $K_{\mu\nu}$ is a matrix with 1 in row μ, column ν, and -1 in row ν, column μ. Otherwise its elements are zero. Note that $K_{\mu\nu} = -K_{\nu\mu}$. The $\alpha_{\mu\nu}$ are chosen such as to give the correct result. Thus, given that $L_1 = K_{23}$, $L_2 = -K_{13}$ and $L_3 = K_{12}$ the correspondence is:

$$\begin{array}{ll} \alpha_1 \leftrightarrow \alpha_{23} & \beta_1 \leftrightarrow \alpha_{14} \\ \alpha_2 \leftrightarrow \alpha_{31} & \beta_2 \leftrightarrow \alpha_{24} \\ \alpha_3 \leftrightarrow \alpha_{12} & \beta_3 \leftrightarrow \alpha_{34} \end{array}$$

while the remaining α are defined by $\alpha_{\mu\nu} = -\alpha_{\nu\mu}$.

The confusion may arise by not being careful about indices. The K are 4×4 matrices, the α are numbers with $\alpha_{\mu\nu}$ real if $\mu, \nu = 1, 2, 3$, or $\mu = \nu = 4$, and imaginary if μ or $\nu = 4$. To be very explicit, the matrix-element i, j of the matrix K_{13} could be written as

$$(K_{13})_{ij}\,.$$

1.2 Structure of the Lorentz Group

We must now study the structure of the Lorentz group, by which we mean the following. Two successive Lorentz transformations equals another Lorentz transformation, and we must understand this connection in terms of the parameters $\alpha_{\mu\nu}$. Thus, let there be given two Lorentz transformations described by parameters $\alpha_{\mu\nu}$ and $\beta_{\mu\nu}$ respectively. The product of these two is another Lorentz transformation described by parameters $\gamma_{\mu\nu}$ and we would like to