On the Growth of Laplace-Stieltjes Transforms and the Singular Direction of Complex Analysis

Kong Yinying（孔荫莹）& Hong Yong（洪勇）

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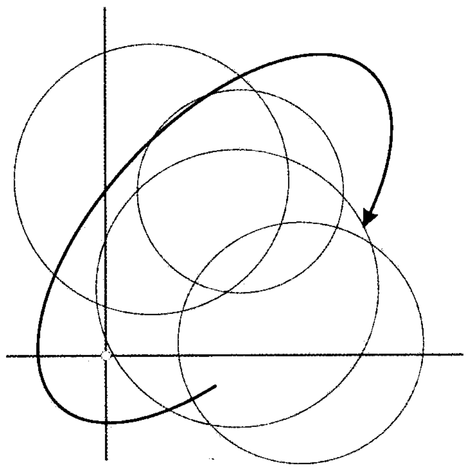

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序

20世纪20年代，芬兰著名数学家R. Nevanlinna创立了亚纯函数Nevanlinna值分布理论，为Dirichlet级数的研究注入了新的活力。30年代，法国著名数学家S. Mandelbrojt与G. Valiron等人首先对Dirichlet级数所定义的整函数的值分布进行研究，开创了Dirichlet级数研究的新方向。50年代初，余家荣先生留法回国，在国内开创了对Dirichlet级数、随机Dirichlet级数定义的解析函数的增长性、值分布及边界性质的研究，引起了国内外数学界的重视。80年代以来，经过以余家荣教授为代表的、以他的众多学生为主的中国数学工作者的努力，在这方面取得了丰富的成果，使我国在该领域的研究得到国际的重视。

Laplace - Stieltjes变换是一类在理论和应用上都十分重要的函数，在某种意义上，Dirichlet级数可以看作是它的一种特例。同时Laplace - Stieltjes变换也是Laplace变换的推广，而后者在通信传播技术等方面有广泛的应用，故其研究有助于应用学科的发展。对于这类函数的分析性质的研究，可追溯到20世纪30年代。1963年余家荣教授首先对Laplace - Stieltjes变换定义的整函数的增长性及值分布方面的研究作了一些奠基性工作。近年来，C. J. K. Batty、高宗升、尚丽娜和作者等人继续对这类函数作了研究，取得了一些很好的结果。

本书以作者的研究工作为基础，前两章主要介绍关于Laplace - Stieltjes变换增长性的研究工作，中间两章讨论了复函数奇异方向的研究工作，都有很重要的意义。最后一章介绍Dirichlet级数最新的研究成果，有很大的研究价值。本书为有志探讨单复变函数值分布理论，尤其是Dirichlet级数的学者提供了一个新的参考和一个新的研究途径。凡具备“复变函数”和“实变函数”等大学本科知识的读者，都可以读懂本书。

孙道椿

2010年3月3日

Preface

The major purpose of this book is to introduce the growth of Laplace-Stieltjes transforms and the singular direction of complex analysis. The book contains five chapters, each chapter is self-contained. Our main contribution here is to provide some new research directions on Dirichlet series, Laplace transforms and the distribution of values of complex analysis for the readers.

Chapter 1 is concerned with the basic fundamentals of Laplace-Stieltjes (L-S) transforms and provides a convenient introduction to such transforms, which is the extension of Dirichlet series. We define the type, the precise order and the Type-function $U(r)$ of L-S transforms convergent in the complex plane and investigate the growth and the distribution of values of such functions by using Valiron-Knopp-Bohr formulas. At last, we study the abscissa of convergence of L-S transforms independent on the sequence $\{\lambda_n\}$, which renew some results of Dirichlet series.

In Chapter 2, the growth and the distribution of value of the L-S transforms convergent in the right half-plane will be studied. By the introduction of the exponential order and the exponential low order, we discuss some problems on the low order of L-S transforms with zero order, establish some relations of the exponential order, the exponential low order and A_n^* and extend the relative results of Dirichlet series. Moreover, some properties of the L-S transforms and its relative transforms are obtained. In addition, we prove the existence of Borel points of such analytic functions with the finite positive order and the infinite order.

In Chapter 3, by Ahlfors's covering surface method, some Type-functions' singular directions of k-quasimeromorphic mappings are studied. The largest type Borel direction(with their multiple values) on the Type-function $U(r)$ is also a Borel direction. In addition, the above results have been extended to the case of the unit disc, the existence theorem of a singular radius, named S-radius (T direction in the case of the complex plane for meromorphic functions) has been established. Furthermore, the existence of filling discs in Borel radius of the quasimeromorphic mapping with finite order in the unit disc is proved, which briefly extend the results of A. Rauch.

In Chapter 4, some properties of meromorphic functions have been generalized to multi-valued algebroidal functions, an extension theorem for the algebroidal function, which is the connection between the Nevanlinna characteristic function and the maximum modulus, is obtained. By which, we show that the order of the entire algebroidal function is equal to that of its derived function. Some singular directions of the algebroidal function, such as T-radius, Borel direction and the filling discs are introduced.

The existence theorems and their relations are obtained. Moreover, some research on the algebroidal function and its derived function in the unit disc and the normality criteria for families have been shown.

In Chapter 5, some interesting relations on the maximum modulus, the maximum term, the rank of maximum term and the coefficients of integral functions defined by Dirichlet series or the random Dirichlet series have been studied. For the random Dirichlet series of infinite order, we replace exceptional values by exceptional small functions, and show that every horizontal line is a strong Borel line almost surely(a.s.) without exceptional small functions. Finally, we study the pits on some entire Dirichlet series and give an estimation of the upper and the lower bounds of the generalized order and the generalized type of a new product function, named Dirichlet-Hadamard product function.

We wish to thank the support of Guangdong University of Business Studies Grant Center(广东商学院学术专著基金) and the Foundation for Distinguished Young Talents in Higher Education of Guangdong, China(广东高校优秀青年创新人才培养计划项目资助, No. LYM08060) and the Specialized Research Fund for the Doctoral Program of Higher Education(高等学校博士学科点专项科研基金, No. 20050574002), which enabled us to do research jointly for past 2 years and complete the writing of the book at GUBS. I would like to acknowledge my great debt to previous works on this subject by Yu Jia-rong, Sun Dao-chun and Gao Zong-sheng. Last but not the least, I give thanks to my supervisor, Sun Dao-chun, for his patience and support throughout the task.

Kong Yin-ying
Guangzhou, China 2009

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Contents

Chapter 1

On the growth of the Laplace-Stieltjes transform convergent in the complex plane

1.1 Entire functions represented by the L-S transform

1.1.1 The introduction of the Laplace-Stieltjes transform

The L-S transform, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. For real-valued functions, it is the Laplace transform of a Stieltjes measure, however it is often defined for functions with values in a Banach space. It is useful in a number of areas of mathematics, including functional analysis, and certain areas of theoretical and applied probability. The Laplace-Stieltjes transform of a real-valued function g is given by a Lebesgue-Stieltjes integral of the form

$$\int e^{-sx} dg(x)$$

for s a complex number. As with the usual Laplace transform, one gets a slightly different transform depending on the domain of integration, and for the integral to be defined, one also needs to require that g be of bounded variation on the region of integration. The most common are:

The bilateral (or two-sided) Laplace-Stieltjes transform is given by

$$\{L^* g\}(s) = \int_{-\infty}^{\infty} e^{-sx} dg(x).$$

The unilateral (one-sided) Laplace-Stieltjes transform is given by

$$\{L^*g\}(s) = \int_{0^-}^{\infty} e^{-sx} dg(x),$$

where the lower limit 0^- means that $\lim_{\varepsilon\to 0^-} \int_{-\varepsilon}^{\infty}$. This is necessary to ensure that the transform captures a possible jump in $g(x)$ at $x = 0$, as is needed to make sense of the Laplace transform of the Dirac delta function. More general transforms can be considered by integrating over a contour in the complex plane.

1.1.2 Valiron-Knopp formula on the abscissas of convergent

Laplace-Stieltjes transform is the extension of Laplace transform, at the same time it is the extension of Dirichlet series. Especially, Laplace transform makes a contribution towards the technique of electric wave transmission, so the research of the L-S transform helps to do some development of applied studies. *Dirichlet series* was introduced by L.Dirichlet in 19th century and it has the following form:

$$f(s) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n s} (s = \sigma + \mathrm{i}t; \sigma, t \in \mathbf{R}),$$

where $\{a_n\} \subset \mathbf{C}, 0 < \lambda_1 < \lambda_2 \cdots < \lambda_n \uparrow +\infty$. The growth and the value distribution of such series were investigated for a long time by Yu [148], Sun [154] and Gao [32]. Recently, Shang Lina and authors continued doing some correlative research on this respect and obtained other interesting results in [67] and [111] . Dirichlet series was regarded as a special example of the L-S transform in [59] and [151] .

Some problems on Borel-line of integral functions defined by Dirichlet series were firstly studied by G.Valiron [23]. His studies based on a summing way of a series from Borel's. Yu [151] studied this problem and related problems largely according to Valiron's way. But in the research of Valiron and Yu's, they need to add a condition to the series

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\lambda_n} < +\infty. \tag{1.0}$$

By applying the methods of Knopp [59], Kojima and Fujiwara, they extended Cauchy-Hadamard formula of power series to that of Dirichlet series. However, when Tanaka [11] studied Borel-line of integral functions defined by general Dirichlet series, he canceled the condition (1.0) and obtained a more universal result.

By moving the condition (1.0) to the L-S transform, Yu first studied Borel-line of integral functions defined by the L-S transform in [151], where the L-S transform was actually a transform very close to Dirichlet series. In 1963, Yu studied a more general Borel-line of integral functions defined by the L-S transform [146] and greatly improved some results of [151]. This is what we will introduce in this section. The methods here are a little different from that of Tanaka. Further, we want to establish the theorem of Liouville type for integral functions defined by the L-S transform.

In this section, we will combine the ideas of Valiron [23] and Knopp [59], and extend Cauchy-Hadamard formula of power series (or Dirichlet series) to the L-S transform. Yu [152] has already published corresponding result of two-fold Dirichlet series and two-fold L-S transforms. The results and proofs in this section are similar to what was published in [152], hence we ignore some details.

Consider a Laplace-Stieltjes transform of the form [19]

$$F(s) = \int_0^{+\infty} \mathrm{e}^{-sx} \mathrm{d}\alpha(x) \qquad (s = \sigma + it), \tag{1.1}$$

where $\alpha(x)$ is a defined real-valued or complex-valued function with $x \geq 0$, and it is of bounded variation on any closed interval $[0, X](0 < X < +\infty)$. In this book, we will make such hypothesis for $\alpha(x)$ at all times. We denote by $V(x)$ the total variation on closed interval $[0, X]$. If $\alpha(x)$ satisfies certain conditions, then (1.1) reduce to Dirichlet series $f(s)$, so it is regarded as a special case of the L-S transform.

We define the *abscissa of convergence* σ_c^F, the *abscissa of uniform convergence* σ_u^F and the *abscissa of absolute convergence* σ_a^F of the L-S transform $F(s)$ respectively as

$\sigma_c^F = \inf\{\sigma_0;\ F(s)$ is convergent when $\sigma > \sigma_0\}$;

$\sigma_u^F = \inf\{\sigma_1;\ F(s)$ is uniformly convergent when $\sigma \geq \sigma_1\}$;

$\sigma_a^F = \inf\{\sigma_2;\ F(s)$ is absolutely convergent when $\sigma \geq \sigma_2\}$.

Next we will study these abscissas and their relations.

Put a sequence $\{\lambda_n\}$:

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow +\infty, \tag{1.2}$$

which satisfies the following conditions:

$$\varlimsup_{n\to\infty} (\lambda_{n+1} - \lambda_n) < +\infty, \qquad \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n} = D < +\infty. \tag{1.3}$$

Set a *comparison series*

$$g(s) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n s}.$$

According to Valiron formula [23], we can easily show that the abscissa of convergence σ_c^g, the abscissa of absolute convergence σ_a^g and the abscissa of uniform convergence σ_u^g of this series satisfy the following relations

$$0 \leq \sigma_c^g \leq \sigma_u^g \leq \sigma_a^g \leq D.$$

By using the comparison series, we can introduce the following theorem.

Theorem 1.1 *Suppose that the L-S transform* (1.1) *satisfies* (1.2) *and* (1.3), *then*

$$\overline{\lim_{n\to\infty}}\frac{\ln A_n}{\lambda_n} \le \sigma_c^F \le \overline{\lim_{n\to\infty}}\frac{\ln A_n}{\lambda_n} + D, \tag{1.4}$$

where $A_n = \sup\limits_{\lambda_n < x \le \lambda_{n+1}} |\alpha(x) - \alpha(\lambda_n)|$.

Proof. We only prove the case when σ_c^F and $\frac{\ln A_n}{\lambda_n} = l$ are finite. The other cases can be easily proved by the similar method.

Firstly, we prove the latter half of formula (1.4). For any given $\varepsilon > 0$, when n is sufficiently large, $A_n < \mathrm{e}^{\lambda_n(l+\varepsilon)}$. Notice (1.3), we can deduce that there exists a constant $K > 0$, such that

$$0 < \lambda_{n+1} - \lambda_n \le K \quad (n = 1, 2, 3, \cdots), \tag{1.5}$$

and we have

$$\int_{\lambda_n}^{\lambda_{n+1}} \mathrm{e}^{-\sigma x}\mathrm{d}\alpha(x) = \mathrm{e}^{-\sigma x}[\alpha(x) - \alpha(\lambda_n)]\,|_{\lambda_n}^{\lambda_{n+1}} + \sigma\int_{\lambda_n}^{\lambda_{n+1}} \mathrm{e}^{-\sigma x}[\alpha(x) - \alpha(\lambda_n)]\mathrm{d}x.$$

Therefore

$$|\int_{\lambda_n}^{\lambda_{n+1}} \mathrm{e}^{-\sigma x}\mathrm{d}\alpha(x)| \le A_n(\mathrm{e}^{-\lambda_{n+1}\sigma} + |\mathrm{e}^{-\lambda_n\sigma} - \mathrm{e}^{-\lambda_{n+1}\sigma}|)$$

$$\le \begin{cases} 2A_n\mathrm{e}^{-\lambda_n\sigma}, & \textit{when} \quad \sigma \ge 0, \\ 2A_n\mathrm{e}^{-\lambda_{n+1}\sigma} \le 2A_n\mathrm{e}^{-(\lambda_n+K)\sigma}, & \textit{when} \quad \sigma < 0. \end{cases}$$

When n is sufficiently large, it follows that

$$|\int_{\lambda_n}^{\lambda_{n+1}} \mathrm{e}^{-\sigma x}\mathrm{d}\alpha(x)| \le K'(\sigma)A_n\mathrm{e}^{-\lambda_n\sigma} < K'(\sigma)A_n\mathrm{e}^{-\lambda_n(\sigma-l-\varepsilon)},$$

where $K'(\sigma)$ is 2 when $\sigma \ge 0$, and $2\mathrm{e}^{-K\sigma}$ when $\sigma < 0$.

Since

$$\int_0^x \mathrm{e}^{-sy}\mathrm{d}\alpha(y) = \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \mathrm{e}^{-sy}\mathrm{d}\alpha(y) + \int_{\lambda_n}^{x} \mathrm{e}^{-sy}\mathrm{d}\alpha(y) \quad (\lambda_n \le x < \lambda_{n+1}),$$

and the series $\sum \mathrm{e}^{-\lambda_n(\sigma-l-\varepsilon)}$ is absolutely convergent when $\sigma - l - \varepsilon > D$. It follows that the transform (1.1) is convergent when $s = \sigma > l + D + \varepsilon$. Since ε is arbitrary, the latter half of formula (1.4) is proved.

Now suppose that the transform (1.1) is convergent when $s = \sigma_0$, here σ_0 is a real number. So there exists a finite constant $M > 0$, such that

$$|\int_0^x \mathrm{e}^{-\sigma_0 y}\mathrm{d}\alpha(y)| \le M$$

holds for arbitrary $x > 0$. Put

$$I_k(x;\sigma_0) = \int_{\lambda_k}^{x} \mathrm{e}^{-\sigma_0 y}\mathrm{d}\alpha(y) \qquad (x > \lambda_k).$$

Then we can see that $|I_k(x;\sigma_0)| \leq 2M \quad (n = 1, 2, 3, \cdots)$. Now suppose that $\lambda_n < x \leq \lambda_{n+1}$, we have

$$\begin{aligned}\int_{\lambda_n}^{x} \mathrm{d}\alpha(y) &= \int_{\lambda_n}^{x} \mathrm{e}^{\sigma_0 y}\mathrm{e}^{-\sigma_0 y}\mathrm{d}\alpha(y) = \int_{\lambda_n}^{x} \mathrm{e}^{\sigma_0 y}\mathrm{d}I_n(y;\sigma_0)\\ &= \mathrm{e}^{\sigma_0 y} I_n(y;\sigma_0)|_{\lambda_n}^{x} - \sigma_0 \int_{\lambda_n}^{x} \mathrm{e}^{\sigma_0 y} I_n(y;\sigma_0)\mathrm{d}y\\ &= \mathrm{e}^{x\sigma_0} I_n(x;\sigma_0) - \sigma_0 \int_{\lambda_n}^{x} \mathrm{e}^{\sigma_0 y} I_n(y;\sigma_0)\mathrm{d}y.\end{aligned}$$

Therefore

$$|\alpha(x) - \alpha(\lambda_n)| \leq 2M(\mathrm{e}^{x\sigma_0} + |\mathrm{e}^{x\sigma_0} - \mathrm{e}^{\lambda_n\sigma_0}|) \leq K_2(\sigma_0)\mathrm{e}^{\lambda_n\sigma_0}.$$

Hence $A_n \leq K_2(\sigma_0)\mathrm{e}^{\lambda_n\sigma_0}$, where $K_2(\sigma_0) = 4M\mathrm{e}^{K\sigma_0}$ when $\sigma_0 \geq 0$, $K_2(\sigma_0) = 4M$ when $\sigma_0 < 0$. So

$$\overline{\lim_{n\to\infty}} \frac{\ln A_n}{\lambda_n} \leq \sigma_0.$$

Hence the front half of formula (1.4) gets proved.

Theorem 1.2 *Suppose that the L-S transform* (1.1) *satisfies* (1.2) *and* (1.3), *then*

$$\overline{\lim_{n\to\infty}} \frac{\ln \widetilde{A}_n}{\lambda_n} \leq \sigma_a^F \leq \overline{\lim_{n\to\infty}} \frac{\ln \widetilde{A}_n}{\lambda_n} + D,$$

where $\widetilde{A}_n = \int_{\lambda_n}^{\lambda_{n+1}} |\mathrm{d}\alpha(x)| = V(\lambda_{n+1}) - V(\lambda_n)$.

Theorem 1.3 *Suppose that the L-S transform* (1.1) *satisfies* (1.2) *and* (1.3), *then*

$$\overline{\lim_{n\to\infty}} \frac{\ln A_n^*}{\lambda_n} \leq \sigma_u^F \leq \overline{\lim_{n\to\infty}} \frac{\ln A_n^*}{\lambda_n} + D, \tag{1.6}$$

where

$$A_n^* = \sup_{\lambda_n < x \leq \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{-\mathrm{i}ty}\mathrm{d}\alpha(y) \right|. \tag{1.7}$$

The proof of the two theorems above can be found in paper [152]. By the three theorems above, we can introduce the corresponding formula about Dirichlet series and the Laplace transform with Valiron formula as a special case. Knopp's formula [59] is the special case of Theorem 1.1 and Theorem 1.2, when $\lambda_n = n - 1$. The formula (1.6) is called *Valiron-Knopp-Bohr formula.*

1.1.3 The maximum modulus and the maximum term of entire functions defined by the L-S transform

The relation between the maximum modulus of entire functions and the maximum term of Taylor series plays an important part when studying integral functions defined by Taylor series. If we want to study such properties of entire functions defined by Dirichlet series or the L-S transform, firstly we should set up a corresponding relation. But it exists certain difficulty, so Ritt [110] and Valiron [23] introduced the condition (1.0) when they studied integral functions defined by Dirichlet series. Tanaka [11] gave suitable concepts of "the maximum modulus" and "the maximum term" to Dirichlet series, and deduced the relation between them. In the following, we will study "the maximum modulus" and "the maximum term" of integral functions by the L-S transform, the form of which is more universal than that introduced by Tanaka. We will also deduce the theorem of Liouville type to entire functions defined by the L-S transform, by applying the getting relation.

Suppose that the transform (1.1) satisfies $\sigma_u^F = -\infty$, then $F(s)$ is an entire function. Choose the sequence (1.2) which satisfies the conditions (1.3). Set

$$M(\sigma, F) = \sup_{-\infty<t<+\infty} |F(\sigma + \mathrm{i}t)|,$$

$$M_u(\sigma, F) = \sup_{0<x<+\infty, -\infty<t<+\infty} \left| \int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y} \mathrm{d}\alpha(y) \right|,$$

$$\mu(\sigma, F) = \max_{1\le n<+\infty} \{A_n^* \mathrm{e}^{-\lambda_n \sigma}\},$$

where A_n^* is defined by (1.7). We call $M_u(\sigma, F)$ the *maximum modulus* of $F(s)$, $\mu(\sigma, F)$ the *maximum term* of $F(s)$.

Because $\sigma_u^F = -\infty$, by (1.6) it is not difficult to see that $\mu(\sigma, F)$ is meaningful. $\{P_n\} = \{(-\lambda_n, -\ln A_n^*)\} (n = 1, 2, \cdots)$ denotes a sequence points on the xOy plane. Make a convex *Newton polygon* $\Pi(F)$ based on these points, such that its vertexes are points in $\{P_n\}$ and the other points in $\{P_n\}$ are on or above the edge of $\Pi(F)$. Construct it like this: fetch a downward perpendicular half line beginning with P_1 and let it twist anticlockwise until crossing a point in $\{P_n\}$. If P_3 is the farthest point in $\{P_n\}$ on the line to P_1, we let the segment $\overline{P_1P_3}$ be an edge of $\Pi(F)$. Extending it, then we have a half line beginning with P_3. Let it twist anticlockwise until crossing a point in $\{P_n\}$. If P_6 is the farthest point in $\{P_n\}$ on the line to P_3, we let the segment $\overline{P_3P_6}$ be the other edge of $\Pi(F) \cdots$ Then there is a Newton polygon which satisfies the assumption above. Let $n(\sigma)$ denote the largest one of indexes of all points $\{P_n\}$ on such lines, which go through P_n with their slope of σ but do not traverse the polygon $\Pi(F)$. So

$$\mu(\sigma, F) = A_{n(\sigma)}^* \mathrm{e}^{-\lambda_{n(\sigma)}\sigma}.$$

Let G_n denote the y-axis of the points on the edge of $\Pi(F)$, the abscissa of which is $-\lambda_n$. Therefore

$$G_n \leq -\ln A_n, \qquad G_{n(\sigma)} = -\ln A_{n(\sigma)}.$$

It is not difficult to have

$$\ln \mu(\sigma, F) = \begin{cases} -G_1, & when \quad \sigma \geq -\frac{G_2-G_1}{\lambda_2-\lambda_1}, \\ -G_1 + \int_\sigma^{-\frac{G_2-G_1}{\lambda_2-\lambda_1}} \lambda_{n(x)} \mathrm{d}x, & when \quad \sigma < -\frac{G_2-G_1}{\lambda_2-\lambda_1}. \end{cases}$$

Thus we can see that $\ln \mu(\sigma, F)$ is a decreasing convex function.

We define

$$\overline{\rho} = \overline{\lim_{\sigma\to-\infty}} \frac{\ln^+ \ln^+ M(\sigma, F)}{-\sigma},$$

$$\rho = \overline{\lim_{\sigma\to-\infty}} \frac{\ln^+ \ln^+ M_u(\sigma, F)}{-\sigma},$$

$$\rho_\mu = \overline{\lim_{\sigma\to-\infty}} \frac{\ln^+ \ln^+ \mu(\sigma, F)}{-\sigma},$$

where $\ln^+ B = \max\{\ln B, 0\}$. $\overline{\rho}$ and ρ are called the *linear order*(*Ritt order*) and the *order* of entire functions $F(s)$ respectively. Obviously, $M(\sigma, F) \leq M_u(\sigma, F)$, thus $\overline{\rho} \leq \rho$. When $\rho \in (0, +\infty), \rho = +\infty$ or $\rho = 0$, we say that $F(s)$ is an analytic function of finite positive order, infinite order or zero order in the complex plane, respectively. We can also define values corresponding to $\overline{\rho}, \rho$ and ρ_μ in a strip region. Now we are going to find the relation between $M_u(\sigma, F)$ and $\mu(\sigma, F)$.

Theorem 1.4 *Suppose that* $\sigma_u^F = -\infty$, *and the sequence* (1.2) *satisfies* (1.3), *then for any given* $\varepsilon > 0$, *when* $\sigma \leq 0$, *we have*

$$\frac{1}{2}\mu(\sigma, F) \leq M_u(\sigma, F) \leq 2\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{-K\sigma} \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n(D+\varepsilon)}, \tag{1.8}$$

where K *is the constant from* (1.5).

Proof. Firstly, we prove the front half of formula (1.8). Set

$$I(x; \sigma + \mathrm{i}t) = \int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y} \mathrm{d}\alpha(y).$$

We have

$$\begin{aligned} \int_{\lambda_n}^x \mathrm{e}^{-\mathrm{i}ty} \mathrm{d}\alpha(y) &= \int_{\lambda_n}^x \mathrm{e}^{\sigma y} \mathrm{d}_y I(y; \sigma + \mathrm{i}t) \\ &= I(y; \sigma + \mathrm{i}t)\mathrm{e}^{\sigma y}|_{\lambda_n}^x - \sigma \int_{\lambda_n}^x \mathrm{e}^{\sigma y} I(y; \sigma + \mathrm{i}t) \mathrm{d}y \quad (x > \lambda_n). \end{aligned}$$

Therefore when $\sigma \leq 0$, it follows that

$$|\int_{\lambda_n}^{x} \mathrm{e}^{-\mathrm{i}ty}\mathrm{d}\alpha(y)| \leq M_u(\sigma, F)[|\mathrm{e}^{\sigma x} + \mathrm{e}^{\sigma\lambda_n}| + |\mathrm{e}^{\sigma x} - \mathrm{e}^{\sigma\lambda_n}|] \leq 2M_u(\sigma, F)\mathrm{e}^{\lambda_n\sigma} \quad (x > \lambda_n).$$

Thus $A_n^* \leq 2M_u(\sigma, F)\mathrm{e}^{\lambda_n\sigma} \quad (\sigma \leq 0)$, then the front half of formula (1.8) is obtained. Secondly, for any $x > 0$, there exists $n \in \mathbf{N}, \lambda_n < x \leq \lambda_{n+1}$, such that

$$\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y) = \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y) + \int_{\lambda_n}^{x} \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y).$$

Set

$$I_k(x; \mathrm{i}t) = \int_{\lambda_k}^{x} \mathrm{e}^{-\mathrm{i}ty}\mathrm{d}\alpha(y) \qquad (\lambda_k < x \leq \lambda_{k+1}). \tag{1.9}$$

For any $t \in \mathbf{R}$, when $\lambda_k < x \leq \lambda_{k+1}$, we obtain

$$|I_k(x; \mathrm{i}t)| \leq A_k^* \leq \mu(\sigma, F)\mathrm{e}^{\lambda_k\sigma} \quad \text{or} \quad |I_k(x; \mathrm{i}t)| \leq \mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_k(\sigma-D-\varepsilon)}.$$

Hence for any $x \in (\lambda_k, \lambda_{k+1}]$ and $\sigma \leq 0$,

$$\begin{aligned}\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y) &= \sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \mathrm{e}^{-\sigma y}\mathrm{d}_y I_k(y; \mathrm{i}t) + \int_{\lambda_n}^{x} \mathrm{e}^{-\sigma y}\mathrm{d}_y I_k(y; \mathrm{i}t)\\ &= \sum_{k=1}^{n-1}[\mathrm{e}^{-\lambda_{k+1}\sigma}I_k(\lambda_{k+1}; \mathrm{i}t) + \sigma\int_{\lambda_k}^{\lambda_{k+1}} \mathrm{e}^{-\sigma y}I_k(y; \mathrm{i}t)\mathrm{d}y]\\ &\quad + \mathrm{e}^{-\sigma x}I_n(x; \mathrm{i}t) + \sigma\int_{\lambda_n}^{x} \mathrm{e}^{-\sigma y}I_n(y; \mathrm{i}t)\mathrm{d}y.\end{aligned}$$

Therefore

$$\begin{aligned}|\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)| &\leq \sum_{k=1}^{n-1}\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_k(\sigma-D-\varepsilon)}(\mathrm{e}^{-\lambda_{k+1}\sigma} + \mathrm{e}^{-\lambda_{k+1}\sigma} - \mathrm{e}^{-\lambda_k\sigma})\\ &\quad + \mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_n(\sigma-D-\varepsilon)}(\mathrm{e}^{-\sigma x} + \mathrm{e}^{-\sigma x} - \mathrm{e}^{-\lambda_n\sigma})\\ &\leq \sum_{k=1}^{n-1} 2\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_k(\sigma-D-\varepsilon)}\mathrm{e}^{-\lambda_{k+1}\sigma}\\ &\quad + 2\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_k(\sigma-D-\varepsilon)}\mathrm{e}^{-\lambda_{n+1}\sigma}\\ &\leq \sum_{k=1}^{n-1} 2\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_k(\sigma-D-\varepsilon)}\mathrm{e}^{-(\lambda_k+K)\sigma}\\ &\quad + 2\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{\lambda_k(\sigma-D-\varepsilon)}\mathrm{e}^{-(\lambda_n+K)\sigma}\\ &= 2\mu(\sigma - D - \varepsilon, F)\mathrm{e}^{-K\sigma}\sum_{k=1}^{n} \mathrm{e}^{-\lambda_k(D+\varepsilon)},\end{aligned}$$

where K is from (1.5). From the second formula of (1.3), for the above $\varepsilon > 0$, when n is sufficiently large, it follows that $\ln n < \lambda_n(D + \frac{\varepsilon}{2})$, then

$$\mathrm{e}^{-\lambda_n(D+\varepsilon)} = [\mathrm{e}^{-\lambda_n(D+\frac{\varepsilon}{2})}]^{\frac{D+\varepsilon}{D+\frac{\varepsilon}{2}}} < (\frac{1}{n})^{\frac{D+\varepsilon}{D+\frac{\varepsilon}{2}}}. \tag{1.10}$$

Consequently $\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n(D+\varepsilon)}$ is convergent. This completes the proof.