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Second Edition

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series Texts in Applied Mathematics (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the Applied Mathematical Sciences (AMS) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California — J.E. Marsden
Providence, Rhode Island — L. Sirovich
College Park, Maryland — S.S. Antman

Preface

Partial differential equations are fundamental to the modeling of natural phenomena; they arise in every field of science. Consequently, the desire to understand the solutions of these equations has always had a prominent place in the efforts of mathematicians; it has inspired such diverse fields as complex function theory, functional analysis and algebraic topology. Like algebra, topology and rational mechanics, partial differential equations are a core area of mathematics.

Unfortunately, in the standard graduate curriculum, the subject is seldom taught with the same thoroughness as, say, algebra or integration theory. The present book is aimed at rectifying this situation. The goal of this course was to provide the background which is necessary to initiate work on a Ph.D. thesis in PDEs. The level of the book is aimed at beginning graduate students. Prerequisites include a truly advanced calculus course and basic complex variables. Lebesgue integration is needed only in Chapter 10, and the necessary tools from functional analysis are developed within the course.

The book can be used to teach a variety of different courses. Here at Virginia Tech, we have used it to teach a four-semester sequence, but (more often) for shorter courses covering specific topics. Students with some undergraduate exposure to PDEs can probably skip Chapter 1. Chapters 2–4 are essentially independent of the rest and can be omitted or postponed if the goal is to learn functional analytic methods as quickly as possible. Only the basic definitions at the beginning of Chapter 2, the Weierstraß approximation theorem and the Arzela-Ascoli theorem are necessary for subsequent chapters. Chapters 10, 11 and 12 are independent of each other (except that Chapter 12 uses some definitions from the beginning of Chapter 11) and can be covered in any order desired.

We would like to thank the many friends and colleagues who gave us suggestions, advice and support. In particular, we wish to thank Pavel Bochev, Guowei Huang, Wei Huang, Addison Jump, Kyehong Kang, Michael Keane, Hong-Chul Kim, Mark Mundt and Ken Mulzet for their help. Special thanks is due to Bill Hrusa, who read a good deal of the manuscript, some of it with great care and made a number of helpful suggestions for corrections and improvements.

Notes on the second edition

We would like to thank the many readers of the first edition who provided comments and criticism. In writing the second edition we have, of course, taken the opportunity to make many corrections and small additions. We have also made the following more substantial changes.

- We have added new problems and tried to arrange the problems in each section with the easiest problems first.
- We have added several new examples in the sections on distributions and elliptic systems.
- The material on Sobolev spaces has been rearranged, expanded, and placed in a separate chapter. Basic definitions, examples, and theorems appear at the beginning while technical lemmas are put off until the end. New examples and problems have been added.
- We have added a new section on nonlinear variational problems with "Young-measure" solutions.
- We have added an expanded reference section.

Contents

1
Introduction

This book is intended to introduce its readers to the mathematical theory of partial differential equations. But to suggest that there is a "theory" of partial differential equations (in the same sense that there is a theory of ordinary differential equations or a theory of functions of a single complex variable) is misleading. PDEs is a much larger subject than the two mentioned above (it includes both of them as special cases) and a less well developed one. However, although a casual observer may decide the subject is simply a grab bag of unrelated techniques used to handle different types of problems, there are in fact certain themes that run throughout.

In order to illustrate these themes we take two approaches. The first is to pose a group of questions that arise in many problems in PDEs (existence, multiplicity, etc.). As examples of different methods of attacking these problems, we examine some results from the theories of ODEs, advanced calculus and complex variables (with which the reader is assumed to have some familiarity). The second approach is to examine three partial differential equations (Laplace's equation, the heat equation and the wave equation) in a very elementary fashion (again, this will probably be a review for most readers). We will see that even the most elementary methods foreshadow deeper results found in the later chapters of this book.

1.1 Basic Mathematical Questions

1.1.1 Existence

Questions of existence occur naturally throughout mathematics. The question of whether a solution exists *should* pop into a mathematician's head any time he or she writes an equation down. Appropriately, the problem of existence of solutions of partial differential equations occupies a large portion of this text. In this section we consider precursors of the PDE theorems to come.

Initial-value problems in ODEs

The prototype existence result in differential equations is for initial-value problems in ODEs.

Theorem 1.1 (ODE existence, Picard-Lindelöf). *Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set, and let $\mathbf{F} : D \to \mathbb{R}^n$ be continuous in its first variable and* **uniformly Lipschitz** *in its second; i.e., for $(t, \mathbf{y}) \in D$, $\mathbf{F}(t, \mathbf{y})$ is continuous as a function of t, and there exists a constant γ such that for any $(t, \mathbf{y}_1)$ and $(t, \mathbf{y}_2)$ in D we have*

$$|\mathbf{F}(t,\mathbf{y}_1) - \mathbf{F}(t,\mathbf{y}_2)| \le \gamma |\mathbf{y}_1 - \mathbf{y}_2|. \tag{1.1}$$

Then, for any $(t_0, \mathbf{y}_0) \in D$, there exists an interval $I := (t^-, t^+)$ containing t_0, and at least one solution $\mathbf{y} \in C^1(I)$ of the **initial-value problem**

$$\frac{d\mathbf{y}}{dt}(t) = \mathbf{F}(t, \mathbf{y}(t)), \tag{1.2}$$

$$\mathbf{y}(t_0) = \mathbf{y}_0. \tag{1.3}$$

The proof of this can be found in almost any text on ODEs. We make note of one version of the proof that is the source of many techniques in PDEs: the construction of an equivalent integral equation. In this proof, one shows that there is a continuous function $\mathbf{y}$ that satisfies

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{y}(s))\, ds. \tag{1.4}$$

Then the fundamental theorem of calculus implies that $\mathbf{y}$ is differentiable and satisfies (1.2), (1.3) (cf. the results on smoothness below). The solution of (1.4) is obtained from an iterative procedure; i.e., we begin with an initial guess for the solution (usually the constant function $\mathbf{y}_0$) and proceed to calculate

$$\begin{aligned}
\mathbf{y}_1(t) &= \mathbf{y}_0 + \int_{t_0}^t \mathbf{F}(s, \mathbf{y}_0)\, ds, \\
\mathbf{y}_2(t) &= \mathbf{y}_0 + \int_{t_0}^t \mathbf{F}(s, \mathbf{y}_1(s))\, ds, \\
&\vdots \\
\mathbf{y}_{k+1}(t) &= \mathbf{y}_0 + \int_{t_0}^t \mathbf{F}(s, \mathbf{y}_k(s))\, ds, \\
&\vdots
\end{aligned} \tag{1.5}$$

Of course, to complete the proof one must show that this sequence converges to a solution.

We will see generalizations of this procedure used to solve PDEs in later chapters.

Existence theorems of advanced calculus

The following theorems from advanced calculus give information on the solution of algebraic equations. The first, the inverse function theorem, considers the problem of n equations in n unknowns.

Theorem 1.2 (Inverse function theorem). *Suppose the function*

$$\mathbf{F} : \mathbb{R}^n \ni \mathbf{x} := (x_1, \dots, x_n) \mapsto \mathbf{F}(\mathbf{x}) := (F_1(\mathbf{x}), \dots, F_n(\mathbf{x})) \in \mathbb{R}^n$$

is C^1 in a neighborhood of a point $\mathbf{x}_0$. Further assume that

$$\mathbf{F}(\mathbf{x}_0) = \mathbf{p}_0$$

and

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\mathbf{x}_0) := \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}_0) \end{pmatrix}$$

is nonsingular. Then there is a neighborhood N_x of $\mathbf{x}_0$ and a neighborhood N_p of $\mathbf{p}_0$ such that $F : N_x \to N_p$ is one-to-one and onto; i.e., for every $\mathbf{p} \in N_p$ the equation

$$\mathbf{F}(\mathbf{x}) = \mathbf{p}$$

has a unique solution in N_x.

Our second result, the implicit function theorem, concerns solving a system of p equations in $q + p$ unknowns.

Theorem 1.3 (Implicit function theorem). *Suppose the function*

$$\mathbf{F} : \mathbb{R}^q \times \mathbb{R}^p \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{F}(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^p$$

is C^1 in a neighborhood of a point $(\mathbf{x}_0, \mathbf{y}_0)$. Further assume that

$$\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0},$$

and that the $p \times p$ matrix

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}}(\mathbf{x}_0, \mathbf{y}_0) := \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(\mathbf{x}_0, \mathbf{y}_0) & \cdots & \frac{\partial F_1}{\partial y_p}(\mathbf{x}_0, \mathbf{y}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_p}{\partial y_1}(\mathbf{x}_0, \mathbf{y}_0) & \cdots & \frac{\partial F_p}{\partial y_p}(\mathbf{x}_0, \mathbf{y}_0) \end{pmatrix}$$

is nonsingular. Then there is a neighborhood $N_x \subset \mathbb{R}^q$ of $\mathbf{x}_0$ and a function $\hat{\mathbf{y}} : N_x \to \mathbb{R}^p$ such that

$$\hat{\mathbf{y}}(\mathbf{x}_0) = \mathbf{y}_0,$$

and for every $\mathbf{x} \in N_x$

$$\mathbf{F}(\mathbf{x}, \hat{\mathbf{y}}(\mathbf{x})) = \mathbf{0}.$$

The two theorems illustrate the idea that a nonlinear system of equations behaves essentially like its linearization as long as the linear terms dominate the nonlinear ones. Results of this nature are of considerable importance in differential equations.

1.1.2 Multiplicity

Once we have asked the question of whether a solution to a given problem exists, it is natural to consider the question of how many solutions there are.

Uniqueness for initial-value problems in ODEs

The prototype for uniqueness results is for initial-value problems in ODEs.

Theorem 1.4 (ODE uniqueness). *Let the function $\mathbf{F}$ satisfy the hypotheses of Theorem 1.1. Then the initial-value problem (1.2), (1.3) has at most one solution.*

A proof of this based on Gronwall's inequality is given below.

It should be noted that although this result covers a very wide range of initial-value problems, there are some standard, simple examples for which uniqueness fails. For instance, the problem

$$\begin{aligned} \frac{dy}{dt} &= y^{1/3}, \\ y(0) &= 0 \end{aligned}$$

has an entire family of solutions parameterized by $\gamma \in [0, 1]$:

$$y_\gamma(t) := \begin{cases} 0, & 0 \le t \le \gamma \\ \left[\frac{2}{3}(t - \gamma)\right]^{3/2}, & \gamma < t \le 1. \end{cases}$$

Nonuniqueness for linear and nonlinear boundary-value problems

While uniqueness is often a desirable property for a solution of a problem (often for physical reasons), there are situations in which multiple solutions are desirable. A common mathematical problem involving multiple solutions is an eigenvalue problem. The reader should, of course, be familiar with the various existence and multiplicity results from finite-dimensional linear algebra, but let us consider a few problems from ordinary differential equations. We consider the following second-order ODE depending on the parameter λ:

$$u'' + \lambda u = 0. \tag{1.6}$$

Of course, if we imposed two initial conditions (at one point in space) Theorem 1.4 would imply that we would have a unique solution. (To apply the theorem directly we need to convert the problem from a second-order equation to a first-order system.) However, if we impose the two-point boundary conditions

$$u(0) = 0, \tag{1.7}$$

$$u'(1) = 0, \tag{1.8}$$

the uniqueness theorem does not apply. Instead we get the following result.

Theorem 1.5. *There are two alternatives for the solutions of the boundary-value problem (1.6), (1.7), (1.8).*

1. *If $\lambda = \lambda_n := ((2n+1)^2\pi^2)/4$, $n = 0, 1, 2, \ldots$, then the boundary-value problem has a family of solutions parameterized by $A \in (-\infty, \infty)$:*

$$u_n(x) = A \sin \frac{(2n+1)\pi}{2} x.$$

   In this case we say λ is an eigenvalue.

2. *For all other values of λ the only solution of the boundary-value problem is the trivial solution*

$$u(x) \equiv 0.$$

This characteristic of having either a unique (trivial) solution or an infinite linear family of solutions is typical of linear problems. More interesting multiplicity results are available for nonlinear problems and are the main subject of modern *bifurcation theory*. For example, consider the following nonlinear boundary-value problem, which was derived by Euler to describe the deflection of a thin, uniform, inextensible, vertical, elastic beam under a load λ:

$$\theta''(x) + \lambda \sin \theta(x) = 0, \tag{1.9}$$

$$\theta(0) = 0, \tag{1.10}$$

$$\theta'(1) = 0. \tag{1.11}$$