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Ralph L. Cohen　Kathryn Hess　Alexander A. Voronov

String Topology and Cyclic Homology

弦拓扑与环同调

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Foreword

Free loop spaces play a central rôle in two recent advances in algebraic topology. The first one is string topology, a subject born with the seminal work of Chas and Sullivan in 1999, who uncovered new algebraic structure in the homology of free loop spaces on manifolds. The second one is topological cyclic homology, a topological version of Connes' cyclic homology introduced in 1993 by Bökstedt, Hsiang, and Madsen.

A summer school was held in Almería from September 16 to 20, 2003, to cover topics in this new and exciting field.

The first part of this book consists of the joint account of the two lecture series which focused on string topology (Cohen and Voronov). It discusses the loop product from the original point of view of Chas and Sullivan, from the Cohen-Jones stable point of view, as well as Voronov's operadic point of view.

The second part is essentially an account of the course devoted to the construction of algebraic models for computing topological cyclic homology (Hess). Starting with the study of free loop spaces and their algebraic models, it continues with homotopy orbit spaces of circle actions, and culminates in the Hess-Rognes construction of a model for computing spectrum cohomology of topological cyclic homology.

Ralph L. Cohen, Kathryn Hess, and Alexander A. Voronov

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Contents

Part I

Notes on String Topology

Ralph L. Cohen and Alexander A. Voronov

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Introduction

String topology is the study of algebraic and differential topological properties of spaces of paths and loops in manifolds. It was initiated by the beautiful paper of Chas and Sullivan [CS99] in which algebraic structures in both the nonequivariant and equivariant homology (and indeed chains) of the (free) loop space, LM, of a closed, oriented manifold were uncovered. This has lead to considerable work by many authors over the past five years. The goals of this paper are twofold. First, this paper is meant to be an introduction to this new and exciting field. Second, we will attempt to give a "status report". That is, we will describe what has been learned over the last few years, and also give our views about future directions of research. This paper is a joint account of each of the author's lecture series given at the 2003 Summer School on String Topology and Hochschild Homology, in Almería, Spain.

In our view there are two basic reasons for the excitement about the development of string topology. First, it uses most of the modern techniques of algebraic topology, and relates them to several other areas of mathematics. For example, the description of the structure involved in the string topology operations uses such concepts as operads, PROPs, field theories, and Gerstenhaber and Batalin-Vilkovisky algebras. The fundamental role played by moduli spaces of Riemann surfaces in string topology, relates it to basic objects of study in algebraic and symplectic geometry. Techniques in low dimensional topology such as the use of graphs to study these moduli spaces are also used in an essential way. Moreover there are both formal and computational relationships between string topology and Gromov-Witten theory that are only beginning to be uncovered. Gromov-Witten theory is a basic tool in string theory, algebraic geometry, and symplectic geometry, and understanding its relationship to string topology is an exciting area of current and probably future research.

The second reason for the attention the development of string topology has been receiving has to do with the historical significance, in both mathematics and physics, played by spaces of paths and loops in manifolds. The systematic study of the differential topology of path and loop spaces began in the 1930's with Morse, who used his newly developed theory of "calculus of variations in the large" to prove among other things that for any Riemannian metric on the

n-sphere, there are an infinite number of geodesics connecting any two points. In the 1950's, R. Bott studied Morse theory on the loop spaces of Lie groups and symmetric spaces to prove his celebrated periodicity theorem. In the 1970's and 1980's, the K-theoretic tools developed by Waldhausen to study diffeomorphisms of high dimensional manifolds were found to be closely related to the equivariant stable homotopy type of the free loop space. Finally, within the development of string theory in physics, the basic configuration spaces are spaces of paths and loops in a manifold. Some of the topological issues this theory has raised are the following.

1. What mathematical structure should the appropriate notions of field and field strength have in this theory? This has been addressed by the notion of a "B-field", or a "gerbe with connection". These are structures on principal bundles over the loop space.

2. How does one view elliptic operators, such as the Dirac operator, on the loop space of a manifold? The corresponding index theory has been developed in the context of elliptic cohomology theory.

3. How does one understand geometrically and topologically, intersection theory in the infinite dimensional loop space and path space of a manifold?

It is this last question that is the subject of string topology. The goal of these notes is to give an introduction to the exciting developments in this new theory. They are organized as follows. In Chapter 1 we review basic intersection theory, including the Thom-Pontrjagin construction, for compact manifolds. We then develop and review the results and constructions of Chas and Sullivan's original paper. In Chapter 2 we review the concepts of operads and PROPS, discuss many examples, and study in detail the important example of the "cacti operad", which plays a central role in string topology. In Chapter 3 we discuss field theories in general, and the field theoretic properties of string topology. Included are discussions of "fat graphs", and how they give a model for the moduli space of Riemann surfaces, and of "open-closed" string topology, which involves spaces of paths in a manifold with prescribed boundary conditions. In Chapter 4 we discuss a Morse theoretic interpretation of string topology, incorporating the classical energy functional on the loop space, originally studied by Morse himself. In this chapter we also discuss how this perspective suggests a potentially deep relationship with the Gromov-Witten theory of the cotangent bundle. Finally in Chapter 5 we study similar structures on spaces of maps of higher dimensional spheres to manifolds.

Acknowledgments . We are very grateful to David Chataur, José Luis Rodríguez, and Jérôme Scherer for organizing and inviting us to participate in such an active and inspiring summer school. We would also like to thank David Chataur, Jérôme Scherer, and Jim Stasheff for many helpful suggestions regarding an earlier draft of the manuscript.

Chapter 1

Intersection theory in loop spaces

String topology is ultimately about the differential and algebraic topology of spaces of paths and loops in compact, oriented manifolds. The basic spaces of paths that we consider are $C^\infty(\mathbb{R}, M)$, $C^\infty([0,1], M)$, which we denote by $\mathcal{P}(M)$, $C^\infty(S^1, M)$, which we denote by LM, and $\Omega(M, x_0) = \{\alpha \in LM : \alpha(0) = x_0\}$. By the C^∞ notation we actually mean spaces of *piecewise* smooth maps. For example a map $f : [x_0, x_k] \to M$ is *piecewise smooth* if f is continuous and if there exists $x_0 < x_1 < \cdots < x_{k-1} < x_k$ with $f_{|_{(x_i, x_{i+1})}}$ infinitely differentiable for all i. These spaces of paths are infinite dimensional smooth manifolds. See, for example [Kli82].

The most basic algebraic topological property of closed, oriented manifolds is Poincaré duality. This manifests itself in a homological intersection theory. In their seminal paper, [CS99], Chas and Sullivan showed that certain intersection constructions also exist in the chains and homology of loop spaces of closed, oriented manifolds. This endows the homology of the loop space with a rich structure that goes under the heading of "string topology".

In this chapter we review Chas and Sullivan's constructions, as well as certain homotopy theoretic interpretations and generalizations found in [CJ02], [CG04]. In particular we recall from [Coh04b] the ring spectrum structure in the Atiyah dual of a closed manifold, which realizes the intersection pairing in homology, and recall from [CJ02] the existence of a related ring spectrum realizing the Chas-Sullivan intersection product ("loop product") in the homology of a loop space. We also discuss the relationship with Hochschild cohomology proved in [CJ02], and studied further by [Mer03], [FMT02], as well as the homotopy invariance properties proved in [CKS05]. We begin by recalling some basic facts about intersection theory in finite dimensions.

1.1 Intersections in compact manifolds

Let $e : P^p \subset M^d$ be an embedding of closed, oriented manifolds of dimensions p and n respectively. Let k be the codimension, $k = d - p$.

Suppose $\theta \in H_q(M^d)$ is represented by an oriented manifold, $f : Q^q \to M^d$. That is, $\theta = f_*([Q])$, where $[Q] \in H_q(Q)$ is the fundamental class. We may assume that the map f is transverse to the submanifold $P \subset M$, otherwise we perturb f within its homotopy class to achieve transversality. We then consider the "pull-back" manifold

$$Q \cap P = \{x \in Q : f(x) \in P \subset M\}.$$

This is a dimension $q - k$ manifold, and the map f restricts to give a map $f : Q \cap P \to P$. One therefore has the induced homology class,

$$e_!(\theta) = f_*([Q \cap P]) \in H_{q-k}(P).$$

More generally, on the chain level, the idea is to take a q-cycle in M, which is transverse to P in an appropriate sense, and take the intersection to produce a $q - k$-cycle in P. Homologically, one can make this rigorous by using Poincaré duality, to define the intersection or "umkehr" map,

$$e_! : H_q(M) \to H_{q-k}(P)$$

by the composition

$$e_! : H_q(M) \cong H^{d-q}(M) \xrightarrow{e^*} H^{d-q}(P) \cong H_{q-k}(P)$$

where the first and last isomorphisms are given by Poincaré duality.

Perhaps the most important example is the diagonal embedding,

$$\Delta : M \to M \times M.$$

If we take field coefficients, the induced umkehr map is the *intersection pairing*

$$\mu = \Delta_! : H_p(M) \otimes H_q(M) \to H_{p+q-d}(M).$$

Since the diagonal map induces cup product in cohomology, the following diagram commutes:

$$\begin{array}{ccc} H_p(M) \otimes H_q(M) & \xrightarrow{\mu} & H_{p+q-d}(M) \\ {\scriptstyle P.D}\downarrow & & \downarrow{\scriptstyle P.D} \\ H^{d-p}(M) \times H^{d-q}(M) & \xrightarrow{cup} & H^{2d-p-q}(M) \end{array}$$

In order to deal with the shift in grading, we let $\mathbb{H}_*(M) = H_{*+d}(M)$. So $\mathbb{H}_*(M)$ is nonpositively graded.

Proposition 1.1.1. *Let k be a field, and M^d a closed, oriented, connected manifold. Then $\mathbb{H}_*(M^d; k)$ is an associative, commutative graded algebra over k, together with a map $\epsilon : \mathbb{H}_*(M; k) \to k$ such that the composition*

$$\mathbb{H}_*(M) \times \mathbb{H}_*(M) \xrightarrow{\mu} \mathbb{H}_*(M) \xrightarrow{\epsilon} k$$

is a nonsingular bilinear form. If $k = \mathbb{Z}/2$ the orientation assumption can be dropped.

In this proposition the map $\epsilon : \mathbb{H}_q(M) \to k$ is zero unless $q = -d$, in which case it is the isomorphism

$$\mathbb{H}_{-d}(M) = H_0(M) \cong k.$$

Such an algebraic structure, namely a commutative algebra A together with a map $\epsilon : A \to k$ making the pairing $\langle a, b \rangle = \epsilon(a \cdot b)$ a nondegenerate bilinear form, is called a **Frobenius algebra.**

We leave to the reader the exercise of proving the following, (see [Abr96]).

Proposition 1.1.2. *A k-vector space A is a Frobenius algebra if and only if it is a commutative algebra with unit and a cocommutative co-algebra*

$$\Delta : A \to A \otimes A$$

with co-unit $\epsilon : A \to k$, so that Δ is a map of A-bimodules.

Intersection theory can also be realized by the "Thom collapse" map. Consider again the embedding of compact manifolds, $e : P \hookrightarrow M$, and extend e to a tubular neighborhood, $P \subset \eta_e \subset M$. Consider the projection map,

$$\tau_e : M \to M/(M - \eta_e). \tag{1.1}$$

Notice that $M/(M - \eta_e)$ is the one point compactification of the tubular neighborhood, $M/(M - \eta_e) \cong \eta_e \cup \infty$. Furthermore, by the tubular neighborhood theorem, this space is homeomorphic to the Thom space P^{ν_e} of the normal bundle, $\nu_e \to P$,

$$M/(M - \eta_e) \cong \eta_e \cup \infty \cong P^{\nu_e}.$$

So the Thom collapse map can be viewed as a map,

$$\tau_e : M \to P^{\nu_e}. \tag{1.2}$$

Then the homology intersection map $e_!$ is equal to the composition,

$$e_! : H_q(M) \xrightarrow{(\tau_e)_*} H_q(P^{\nu_e}) \cong H_{q-k}(P) \tag{1.3}$$

where the last isomorphism is given by the Thom isomorphism theorem. In fact this description of the umkehr map $e_!$ shows that it can be defined in *any* generalized homology theory, for which there exists a Thom isomorphism for the normal bundle. This is an orientation condition. In these notes we will usually restrict our attention to ordinary homology, but intersection theories in such (co)homology theories as K-theory and cobordism theory are very important as well.

1.2 The Chas-Sullivan loop product

The Chas-Sullivan "loop product" in the homology of the free loop space of a closed oriented d-dimensional manifold,

$$\mu : H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM) \tag{1.4}$$

is defined as follows.

Let $\mathrm{Map}(8, M)$ be the mapping space from the figure 8 (i.e the wedge of two circles) to the manifold M. As mentioned above, the maps are required to be piecewise smooth (see [CJ02]). Notice that $\mathrm{Map}(8, M)$ can be viewed as the subspace of $LM \times LM$ consisting of those pairs of loops that agree at the basepoint $1 \in S^1$. In other words, there is a pullback square

$$\begin{array}{ccc} \mathrm{Map}(8, M) & \xrightarrow{e} & LM \times LM \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev \times ev} \\ M & \xrightarrow[\Delta]{} & M \times M \end{array} \tag{1.5}$$

where $ev : LM \to M$ is the fibration given by evaluating a loop at $1 \in S^1$. In fact, it can be shown that ev is a locally trivial fiber bundle [Kli82]. The map $ev : \mathrm{Map}(8, M) \to M$ evaluates the map at the crossing point of the figure 8. Since $ev \times ev$ is a fibre bundle, $e : \mathrm{Map}(8, M) \hookrightarrow LM \times LM$ can be viewed as a codimension d embedding, with normal bundle $ev^*(\nu_\Delta) \cong ev^*(TM)$.

The basic Chas-Sullivan idea, is to take a chain $c \in C_p(LM \times LM)$ that is transverse to the submanifold $\mathrm{Map}(8, M)$ in an appropriate sense, and take the intersection to define a chain $e_!(c) \in C_{p-d}(\mathrm{Map}(8, M))$. This will allow the definition of a map in homology, $e_! : H_*(LM \times LM) \to H_{*-d}(\mathrm{Map}(8, M))$. The striking thing about the Chas-Sullivan construction is that this umkehr map exists in the absence of Poincaré duality in this infinite dimensional context.

As was done in [CJ02], one can also use the Thom collapse approach to define the umkehr map in this setting. They observed that the existence of this pullback diagram of fiber bundles, means that there is a natural tubular neighborhood of the embedding, $e : \mathrm{Map}(8, M) \to LM \times LM$, namely the inverse image of a tubular neighborhood of the diagonal embedding, $\Delta : M \to M \times M$. That is, $\eta_e = (ev \times ev)^{-1}(\eta_\Delta)$. Because ev is a locally trivial fibration, the tubular neighborhood η_e is homeomorphic to the total space of the normal bundle $ev^*(TM)$. This induces a homeomorphism of the quotient space to the Thom space,

$$(LM \times LM)/((LM \times LM) - \eta_e) \cong (\mathrm{Map}(8, M))^{ev^*(TM)}. \tag{1.6}$$

Combining this homeomorphism with the projection onto this quotient space, defines a Thom-collapse map

$$\tau_e : LM \times LM \to \mathrm{Map}(8, M)^{ev^*(TM)}. \tag{1.7}$$

For ease of notation, we refer to the Thom space of the pullback bundle, $ev^*(TM) \to \mathrm{Map}(8, M)$ as $(\mathrm{Map}(8, M))^{TM}$.

Notice that if h_* is any generalized homology theory that supports an orientation of M (i.e the tangent bundle TM), then one can define an umkehr map,

$$e_! : h_*(LM \times LM) \xrightarrow{\tau_e} h_*((\mathrm{Map}(8, M))^{TM}) \xrightarrow{\cap u} h_{*-d}(\mathrm{Map}(8, M)) \qquad (1.8)$$

where $u \in h^d((\mathrm{Map}(8, M))^{TM})$ is the Thom class given by the orientation.

Chas and Sullivan also observed that given a map from the figure 8 to M then one obtains a loop in M by starting at the intersection point, traversing the top loop of the 8, and then traversing the bottom loop. This defines a map

$$\gamma : \mathrm{Map}(8, M) \to LM.$$

Thought of in a slightly different way, the pullback diagram 1.5 says that we can view $\mathrm{Map}(8, M)$ as the fiber product $\mathrm{Map}(8, M) \cong LM \times_M LM$, as was done in [CJ02]. The map γ defines a multiplication map, which by abuse of notation we also call $\gamma : LM \times_M LM \to LM$. This map extends the usual multiplication in the based loop space, $\gamma : \Omega M \times \Omega M \to \Omega M$. In fact if one may view $ev : LM \to M$ as a fiberwise H-space (actually an H-group), which is to say an H-group in the category of spaces over M. It is actually an A_∞ space in this category, coming from the A_∞ structure of the multiplication in ΩM, which is the fiber of $ev : LM \to M$. This aspect of the theory is studied further in [Gru05].

Chas and Sullivan also observed that the multiplication $\gamma : \mathrm{Map}(8, M) \to LM$ is homotopy commutative, and indeed there is a canonical, explicit homotopy. In the formulas that follow, we identify $S^1 = \mathbb{R}/\mathbb{Z}$. Now as above, consider $\mathrm{Map}(8, M)$ as a subspace of $LM \times LM$, and suppose $(\alpha, \beta) \in \mathrm{Map}(8, M)$. We consider, for each $t \in [0, 1]$ a loop $\gamma_t(\alpha, \beta)$, which starts at $\beta(-t)$, traverses the arc between $\beta(-t)$ and $\beta(0) = \alpha(0)$, then traverses the loop defined by α, and then finally traverses the arc between $\beta(0)$ and $\beta(-t)$. A formula for $\gamma_t(\alpha, \beta)$ is given by

$$\gamma_t(\alpha, \beta)(s) = \begin{cases} \beta(2s - t), & \text{for} \quad 0 \le s \le \frac{t}{2} \\ \alpha(2s - t), & \text{for} \quad \frac{t}{2} \le s \le \frac{t+1}{2} \\ \beta(2s - t), & \text{for} \quad \frac{t+1}{2} \le s \le 1. \end{cases} \qquad (1.9)$$

One sees that $\gamma_0(\alpha, \beta) = \gamma(\alpha, \beta)$, and $\gamma_1(\alpha, \beta) = \gamma(\beta, \alpha)$.

The Chas-Sullivan product in homology is defined by composing the umkehr map $e_!$ with the multiplication map γ.

Definition 1.2.1. Define the loop product in the homology of a loop space to be the composition

$$\mu : H_*(LM) \otimes H_*(LM) \to H_*(LM \times LM) \xrightarrow{e_!} H_{*-d}(\mathrm{Map}(8, M)) \xrightarrow{\gamma_*} H_{*-d}(LM).$$

Recall that the umkehr map $e_!$ is defined for any generalized homology theory h_* supporting an orientation. Now suppose that in addition, h_* is a multiplicative theory. That is, the corresponding cohomology theory h^* has a cup product, or more precisely, h_* is represented by a *ring* spectrum. Then there is a loop product in $h_*(LM)$ as well,

$$\mu : h_*(LM) \otimes h_*(LM) \to h_{*-d}(LM).$$

In order to accommodate the change in grading, one defines

$$\mathbb{H}_*(LM) = H_{*+d}(LM).$$

Using the naturality of the umkehr map (i.e the naturality of the Thom collapse map) as well as the homotopy commutativity of the multiplication map γ, the following is proved in [CS99].

Theorem 1.2.1. *Let M be a compact, closed, oriented manifold. Then the loop product defines a map*

$$\mu_* : \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_*(LM)$$

making $\mathbb{H}_*(LM)$ an associative, commutative algebra. Furthermore, the evaluation map $ev : LM \to M$ defines an algebra homomorphism from the loop algebra to the intersection ring,

$$ev_* : \mathbb{H}_*(LM) \to \mathbb{H}_*(M).$$

As was shown in [CJ02], this structure also applies to $h_*(LM)$, where h_* is any multiplicative generalized homology theory which supports an orientation of M.

1.3 The Batalin-Vilkovisky structure and the string bracket

One aspect of the loop space LM that hasn't yet been exploited is the fact there is an obvious circle action

$$\rho : S^1 \times LM \longrightarrow LM \tag{1.10}$$

defined by $\rho(t, \alpha)(s) = \alpha(t+s)$. The purpose of this section is to describe those constructions of Chas and Sullivan [CS99] that exploit this action.

The existence of the S^1-action defines an operator

$$\begin{aligned} \Delta : H_q(LM) &\to H_{q+1}(LM) \\ \theta &\to \rho_*(e_1 \otimes \theta) \end{aligned}$$