R.Tyrrell Rockafellar

Princeton Landmarks in Mathematics

Convex Analysis

凸分析

PRINCETON UNIVERSITY PRESS

世界图书出版公司
www.wpcbj.com.cn

R. Tyrrell Rockafellar

Princeton Landmarks in Mathematics

Convex Analysis

Convex Analysis

BY

R. TYRRELL ROCKAFELLAR

PRINCETON, NEW JERSEY
PRINCETON UNIVERSITY PRESS

图书在版编目（CIP）数据

凸分析 = Convex Analysis：英文/（德）洛克菲拉（Rockafellar，R. T.）著. —影印本. —北京：世界图书出版公司北京公司，2011. 1

ISBN 978-7-5100-2960-8

Ⅰ. ①凸…　Ⅱ. ①洛…　Ⅲ. ①凸分析—教材—英文
Ⅳ. ①O174. 13

中国版本图书馆 CIP 数据核字（2010）第 212306 号

书　　名：	Convex Analysis
作　　者：	R. Tyrrell Rockafellar
中 译 名：	凸分析
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24 开
印　　张：	20
版　　次：	2011 年 01 月
版权登记：	图字：01-2010-4876
书　　号：	978-7-5100-2960-8/O · 858　　**定　　价：** 59. 00 元

LONDON MATHEMATICAL SOCIETY STUDENT TEXTS

Managing editor: Professor D. J. Benson,
Department of Mathematics, University of Aberdeen, UK

17 Aspects of quantum field theory in curved spacetime, S. A. FULLING
18 Braids and coverings: selected topics, VAGN LUNDSGAARD HANSEN
20 Communication theory, C. M. GOLDIE & R. G. E. PINCH
21 Representations of finite groups of Lie type, FRANCOIS DIGNE & JEAN MICHEL
22 Designs, graphs, codes and their links, P. J. CAMERON & J. H. VAN LINT
23 Complex algebraic curves, FRANCES KIRWAN
24 Lectures on elliptic curves, J. W. S. CASSELS
26 An introduction to the theory of L-functions and Eisenstein series, H. HIDA
27 Hilbert Space: compact operators and the trace theorem, J. R. RETHERFORD
28 Potential theory in the complex plane, T. RANSFORD
29 Undergraduate commutative algebra, M. REID
31 The Laplacian on a Riemannian manifold, S. ROSENBERG
32 Lectures on Lie groups and Lie algebras, R. CARTER, G. SEGAL & I. MACDONALD
33 A primer of algebraic *D*-modules, S. C. COUTINHO
34 Complex algebraic surfaces, A. BEAUVILLE
35 Young tableaux, W. FULTON
37 A mathematical introduction to wavelets, P. WOJTASZCZYK
38 Harmonic maps, loop groups, and integrable systems, M. GUEST
39 Set theory for the working mathematician, K. CIESIELSKI
40 Ergodic theory and dynamical systems, M. POLLICOTT & M. YURI
41 The algorithmic resolution of diophantine equations, N. P. SMART
42 Equilibrium states in ergodic theory, G. KELLER
43 Fourier analysis on finite groups and applications, AUDREY TERRAS
44 Classical invariant theory, PETER J. OLVER
45 Permutation groups, P. J. CAMERON
46 Riemann surfaces: A primer, A. BEARDON
47 Introductory lectures on rings and modules, J. BEACHY
48 Set theory, A HAJNAL & P. HAMBURGER
49 K-theory for C*-algebras, M. RØRDAM, F. LARSEN & N. LAUSTSEN
50 A brief guide to algebraic number theory, H. P. F. SWINNERTON-DYER
51 Steps in commutative algebra: Second edition, R. Y. SHARP
52 Finite Markov chains and algorithmic applications, O. HÄGGSTRÖM
53 The prime number theorem, G. J. O. JAMESON
54 Topics in graph automorphisms and reconstruction, J. LAURI & R. SCAPELLATO
55 Elementary number theory, group theory, and Ramanujan graphs, G. DAVIDOFF, P. SARNAK & A. VALETTE
56 Logic, induction and sets, T. FORSTER
57 Introduction to Banach algebras and harmonic analysis, H. G. DALES *et al.*
58 Computational algebraic geometry, HAL SCHENCK
59 Frobenius algebras and 2-D topological quantum field theories, J. KOCK
60 Linear operators and linear systems, J. R. PARTINGTON
61 An introduction to noncommutative Noetherian rings, K. R. GOODEARL & R. B. WARFIELD
62 Topics from one dimensional dynamics, K. M. BRUCKS & H. BRUIN
63 Singularities of plane curves, C. T. C. WALL
64 A short course on Banach space theory, N. L. CAROTHERS
65 Elements of the representation theory of associative algebras Volume I, I. ASSEM, A. SKOWROŃSKI & D. SIMSON
66 An introduction to sieve methods and their applications, A. C. COJOCARU & M. R. MURTY
67 Elliptic functions, V. ARMITAGE & W. F. EBERLEIN
68 Hyperbolic geometry from a local viewpoint, L. KEEN & N. LAKIC
69 Lectures on Kähler geometry, A. MOROIANU
70 Dependence Logic, J. VÄÄNÄNEN

CONVEX ANALYSIS

This book is dedicated to
WERNER FENCHEL

Preface

Convexity has been increasingly important in recent years in the study of extremum problems in many areas of applied mathematics. The purpose of this book is to provide an exposition of the theory of convex sets and functions in which applications to extremum problems play the central role.

Systems of inequalities, the minimum or maximum of a convex function over a convex set, Lagrange multipliers, and minimax theorems are among the topics treated, as well as basic results about the structure of convex sets and the continuity and differentiability of convex functions and saddle-functions. Duality is emphasized throughout, particularly in the form of Fenchel's conjugacy correspondence for convex functions.

Much new material is presented. For example, a generalization of linear algebra is developed in which "convex bifunctions" are the analogues of linear transformations, and "inner products" of convex sets and functions are defined in terms of the extremal values in Fenchel's Duality Theorem. Each convex bifunction is associated with a generalized convex program, and an adjoint operation for bifunctions that leads to a theory of dual programs is introduced. The classical correspondence between linear transformations and bilinear functionals is extended to a correspondence between convex bifunctions and saddle-functions, and this is used as the main tool in the analysis of saddle-functions and minimax problems.

Certain topics which might properly be regarded as part of "convex analysis," such as fixed-point theorems, have been omitted, not because they lack charm or applications, but because they would have required technical developments somewhat outside the mainstream of the rest of the book.

In view of the fact that economists, engineers, and others besides pure mathematicians have become interested in convex analysis, an attempt has been made to keep the exposition on a relatively elementary technical level, and details have been supplied which, in a work aimed only at a mathematical in-group, might merely have been alluded to as "exercises." Everything has been limited to R^n, the space of all n-tuples of real numbers, even though many of the results can easily be formulated in a broader setting of functional analysis. References to generalizations and extensions are collected along with historical and bibliographical comments in a special section at the end of the book, preceding the bibliography itself.

As far as technical prerequisites are concerned, the reader should be able to get by, for the most part, with a sound knowledge of linear algebra and elementary real analysis (convergent sequences, continuous functions, open and closed sets, compactness, etc.) as pertains to the space R^n. Nevertheless, while no actual familiarity with any deeper branch of abstract mathematics is required, the style does presuppose a certain "mathematical maturity" on the part of the reader.

A section of remarks at the beginning of the book describes the contents of each part and outlines a selection of material which would be appropriate for an introduction to the subject.

This book grew out of lecture notes from a course I gave at Princeton University in the spring of 1966. In a larger sense, however, it grew out of lecture notes from a similar course given at Princeton fifteen years earlier by Professor Werner Fenchel of the University of Copenhagen. Fenchel's notes were never published, but they were distributed in mimeographed form, and they have served many researchers long and well as the main, and virtually the only, reference for much of the theory of convex functions. They have profoundly influenced my own thinking, as evidenced, to cite just one aspect, by the way conjugate convex functions dominate much of this book. It is highly fitting, therefore, that this book be dedicated to Fenchel, as honorary co-author.

I would like to express my deep thanks to Professor A. W. Tucker of Princeton University, whose encouragement and support has been a mainstay since student days. It was Tucker in fact who suggested the title of this book. Further thanks are due to Dr. Torrence D. Parsons, Dr. Norman Z. Shapiro, and Mr. Lynn McLinden, who looked over the manuscript and gave some very helpful suggestions. I am also grateful to my students at Princeton and the University of Washington, whose comments on the material as it was taught led to many improvements of the presentation, and to Mrs. Janet Parker for her patient and very competent secretarial assistance.

Preparation of the 1966 Princeton lecture notes which preceded this book was supported by the Office of Naval Research under grant NONR 1858(21), project NR-047-002. The Air Force Office of Scientific Research subsequently provided welcome aid at the University of Washington in the form of grant AF-AFOSR-1202-67, without which the job of writing the book itself might have dragged on a long time, beset by interruptions.

R. T. R.

Introductory Remarks: A Guide for the Reader

This book is not really meant to be read from cover to cover, even if there were anyone ambitious enough to do so. Instead, the material is organized as far as possible by subject matter; for example, all the pertinent facts about relative interiors of convex sets, whether of major or minor importance, are collected in one place (§6) rather than derived here and there in the course of other developments. This type of organization may make it easier to refer to basic results, at least after one has some acquaintance with the subject, yet it can get in the way of a beginner using the text as an introduction. Logical development is maintained as the book proceeds, but in many of the earlier sections there is a mass of lesser details toward the end in which one could get bogged down.

Nevertheless, this book can very well be used as an introduction if one makes an appropriate selection of material. The guidelines are given below, where it is described just which results in each section are really essential and which can safely be skipped over, at least temporarily, without causing a gap in proof or understanding.

Part I: Basic Concepts

Convex sets and convex functions are defined here, and relationships between the two concepts are discussed. The emphasis is on establishing criteria for convexity. Various useful examples are given, and it is shown how further examples can be generated from these by means of operations such as addition or taking convex hulls.

The fundamental idea to be understood is that the convex functions on R^n can be identified with certain convex subsets of R^{n+1} (their epigraphs), while the convex sets in R^n can be identified with certain convex functions on R^n (their indicators). These identifications make it easy to pass back and forth between a geometric approach and an analytic approach. Ordinarily, in dealing with functions one thinks geometrically in terms of the graphs of the functions, but in the case of convex functions pictures of epigraphs should be kept in mind instead.

Most of the material, though elementary, is basic to the rest of the book, but some parts should be left out by a reader who is encountering the subject for the first time. Although only linear algebra is involved in §1

(Affine Sets), the concepts may not be entirely familiar; §1 should therefore be perused up through the definition of barycentric coordinate systems (preceding Theorem 1.6) as background for the introduction of convexity. The remainder of §1, concerning affine transformations, is not crucial to a beginner's understanding. All of §2 (Convex Sets and Cones) is essential and the first half of §3, but the second half of §3, starting with Theorem 3.5, deals with operations of minor significance. Very little should be skipped in §4 (Convex Functions) except some of the examples. However, the end of §5 (Functional Operations), following Theorem 5.7, is not needed in any later section.

Part II: Topological Properties

The properties of convexity considered in Part I are primarily algebraic: it is shown that convex sets and functions form classes of objects which are preserved under numerous operations of combination and generation. In Part II, convexity is considered instead in relation to the topological notions of interior, closure, and continuity.

The remarkably uncomplicated topological nature of convex sets and functions can be traced to one intuitive fact: if a line segment in a convex set C has one endpoint in the interior of C and the other endpoint on the boundary of C, then all the intermediate points of the line segment lie in the interior of C. A concept of "relative" interior can be introduced, so that this fact can be used as a basic tool even in situations where one has to deal with configurations of convex sets whose interiors are empty. This is discussed in §6 (Relative Interiors of Convex Sets). The principal results which every student of convexity should know are embodied in the first four theorems of §6. The rest of §6, starting with Theorem 6.5, is devoted mainly to formulas for the relative interiors of convex sets constructed from other convex sets in various ways. A number of useful results are established (particularly Corollaries 6.5.1 and 6.5.2, which are cited often in the text, and Corollary 6.6.2, which is employed in the proof of an important separation theorem in §11), but these can all be neglected temporarily and referred to as the need arises.

In §7 (Closures of Convex Functions) the main topic is lower semi-continuity. This property is in many ways more important than continuity in the case of convex functions, because it relates directly to epigraphs: a function is lower semi-continuous if and only if its epigraph is closed. A convex function which is not already lower semi-continuous can be made so simply by redefining its values (in a uniquely determined manner) at certain boundary points of its effective domain. This leads to the notion of the closure operation for convex functions, which corresponds to the closure operation for epigraphs (as subsets of R^{n+1}) when the functions are proper. All of §7, with the exception of Theorem 7.6, is essential if one is to understand what follows.

All of §8 (Recession Cones and Unboundedness) is also needed in the long run, although the need is not as ubiquitous as in the case of §6 and §7. The first half of §8 elucidates the idea that unbounded convex sets are just like bounded convex sets, except that they have certain "points at infinity." The second half of §8 applies this idea to epigraphs to obtain results about the growth properties of convex functions. Such properties are important in formulating a number of basic existence theorems scattered throughout the book, the first ones occurring in §9 (Some Closedness Criteria).

The question which §9 attempts to answer is this: when is the image of a closed convex set under a linear transformation closed? It turns out that this question is fundamental in investigations of the existence of solutions to various extremum problems. The principal results of §9 are given in Theorems 9.1 and 9.2 (and their corollaries). The reader would do well, however, to skip §9 entirely on the first encounter and return to it later, if desired, in connection with applications in §16.

Only the first theorem of §10 (Continuity of Convex Functions) is basic to convex analysis as a whole. The fancier continuity and convergence theorems are a culmination in themselves. They are used only in §24 and §25 to derive continuity and convergence theorems for subdifferentials and gradient mappings of convex functions, and in §35 to derive similar results in the case of saddle-functions.

Part III: Duality Correspondences

Duality between points and hyperplanes has an important role to play in much of analysis, but nowhere perhaps is the role more remarkable than in convex analysis. The basis of duality in the theory of convexity is, from a geometric point of view, the fact that a closed convex set is the intersection of all the closed half-spaces which contain it. From the point of view of functions, however, it is the fact that a closed convex function is the pointwise supremum of all the affine functions which minorize it. These two facts are equivalent when regarded in terms of epigraphs, and a geometric formulation is usually preferable for the sake of intuition, but in this case both formulations are important. The second formulation of the basis of duality has the advantage that it leads directly to a symmetric one-to-one duality correspondence among closed convex functions, the conjugacy correpsondence of Fenchel.

Conjugacy contains, as a special case in a certain sense, a symmetric one-to-one correspondence among closed convex cones (polarity), but it has no symmetric counterpart in the class of general closed convex sets. The analogous correspondence in the latter context is between convex sets on the one hand and positively homogeneous convex functions (their support functions) on the other. For this reason it is often better in applications, as far as duality is concerned, to express a given situation in terms of convex functions, rather than convex sets. Once this is done, geometric reasoning can still be applied, of course, to epigraphs.

The foundations for the theory of duality are laid in §11 (Separation Theorems). All of the material in this section, except Theorem 11.7, is essential. In §12 (Conjugates of Convex Functions), the conjugacy correspondence is defined, and a number of examples of corresponding functions are given. Theorems 12.1 and 12.2 are the fundamental results which should be known; the rest of §12 is dispensible.

Conjugacy is applied in §13 (Support Functions) to produce results about the duality between convex sets and positively homogeneous convex functions. The support functions of the effective domain and level sets of a convex function f are calculated in terms of the conjugate function f^* and its recession function. The main facts are stated in Theorems 13.2, 13.3, and 13.5, the last two presupposing familiarity with §8. The other theorems, as well as all the corollaries, can be skipped over and referred to if and when they are needed.

In §14 (Polars of Convex Sets), the conjugacy correspondence for convex functions is specialized to the polarity correspondence for convex cones, whereupon the latter is generalized to the polarity correspondence for arbitrary closed convex sets containing the origin. Polarity of convex cones has several applications elsewhere in this book, but the more general polarity is not mentioned subsequently, except in §15 (Polars of Convex Functions), where its relationship with the theory of norms is discussed. The purpose of §15, besides the development of Minkowski's duality correspondence for norms and certain of its generalizations, is to provide (in Theorem 15.3 and Corollary 15.3.1) further examples of conjugate convex functions. However, of all of §14 and §15, it would suffice, as long as one was not specifically interested in approximation problems, to read merely Theorem 14.1.

The theorems of §16 (Dual Operations) show that the various functional operations in §5 fall into dual pairs with respect to the conjugacy correspondence. The most significant result is Theorem 16.4, which describes the duality between addition and infimal convolution of convex functions. This result has important consequences for systems of inequalities (§21) and the calculus of subgradients (§23), and therefore for the theory of extremum problems in Part VI. The second halves of Theorems 16.3, 16.4, and 16.5 (which give conditions under which the respective minima are attained and the closure operation is not needed in the duality formulas) depend on §9. This much of the material could be omitted on a first reading of §16, along with Lemma 16.2 and all corollaries.

Part IV: Representation and Inequalities

The objective here is to obtain results about the representation of convex sets as convex hulls of sets of points and directions, and to apply these results to the study of systems of linear and nonlinear inequalities. Most of the material concerns refinements of convexity theory which take special advantage of dimensionality or the presence of some degree of linearity. The reader could skip Part IV entirely without jeopardizing his understanding of the remainder of this book. Or, as a compromise, only the more fundamental material in Part IV, as indicated below, could be covered.

The role of dimensionality in the generation of convex hulls is explored in §17 (Carathéodory's Theorem), the principal facts being given in Theorems 17.1 and 17.2. Problems of representing a given convex set in terms of extreme points, exposed points, extreme directions, exposed directions, and tangent hyperplanes are taken up in §18 (Extreme Points and Faces of Convex Sets). All of §18 is put to use in §19 (Polyhedral Convexity); applications also occur in the study of gradients (§25) and in the maximization of convex functions (§32). The most important results in §19 are Theorems 19.1, 19.2, 19.3, and their corollaries.

In §20 (Some Applications of Polyhedral Convexity), it is shown how certain general theorems of convex analysis can be strengthened in the case where some, but not necessarily all, of the convex sets or functions involved are polyhedral. Theorems 20.1 and 20.2 are used in §21 to establish relatively difficult refinements of Helly's Theorem and certain other results which are applicable in §27 and §28 to the existence of Lagrange multipliers and optimal solutions to convex programs. Theorem 20.1 depends on §9, although Theorem 20.2 does not. However, it is possible to understand the fundamental results of §21 (Helly's Theorem and Systems of Inequalities) and their proofs without knowledge of §20, or even of §18 or §19. In this case one should simply omit Theorems 21.2, 21.4, and 21.5.

Finite systems of equations and linear inequalities, weak or strict, are the topic in §22 (Linear Inequalities). The results are special, and they are not invoked anywhere else in the book. At the beginning, various facts are stated as corollaries of fancy theorems in §21, but then it is demonstrated that the same special facts can be derived, along with some improvements, by an elementary and completely independent method which uses only linear algebra and no convexity theory.

Part V: Differential Theory

Supporting hyperplanes to convex sets can be employed in situations where tangent hyperplanes, in the sense of the classical theory of smooth surfaces, do not exist. Similarly, subgradients of convex functions, which correspond to supporting hyperplanes to epigraphs rather than tangent hyperplanes to graphs, are often useful where ordinary gradients do not exist.

The theory of subdifferentiation of convex functions, expounded in §23 (Directional Derivatives and Subgradients), is a fundamental tool in the analysis of extremum problems, and it should be mastered before proceeding. Theorems 23.6, 23.7, 23.9, and 23.10 may be omitted, but one should definitely be aware of Theorem 23.8, at least in the non-polyhedral case for which an alternative and more elementary proof is given. Most of §23 is independent of Part IV.

The main result about the relationship between subgradients and ordinary gradients of convex functions is established in Theorem 25.1, which can be read immediately following §23. No other result from §24, §25, or §26 is specifically required elsewhere in the book, except in §35, where analogous theorems are proved for saddle-functions. The remainder of Part V thus serves its own purpose.

In §24 (Differential Continuity and Monotonicity), the elementary theory of left and right derivatives of closed proper convex functions of a single variable is developed. It is shown that the graphs of the subdifferentials of such functions may be characterized as "complete non-decreasing curves." Continuity and monoticity properties in the one-dimensional case are then generalized to the n-dimensional case.

Aside from the theorem already referred to above, §25 (Differentiability of Convex Functions) is devoted mainly to proving that, for a finite convex function on an open set, the ordinary gradient mapping exists almost everywhere and is continuous. The question of when the gradient mapping comprises the entire subdifferential mapping, and when it is actually one-to-one, is taken up in §26 (The Legendre Transformation). The central purpose of §26 is to explain the extent to which conjugate convex functions can, in principle, be calculated in a classical manner by inverting a gradient mapping. The duality between smoothness and strict convexity is also discussed. The development in §25 and §26 depends to some extent on §18, but not on any sections of Part IV following §18.

Part VI: Constrained Extremum Problems

The theory of extremum problems is, of course, the source of motivation for many of the results in this book. It is in §27 (The Minimum of a Convex

Function) that applications to this theory are begun in a systematic way. The stage is set by Theorem 27.1, which summarizes some pertinent facts proved in earlier sections. All the theorems of §27 concern the manner in which a convex function attains its minimum relative to a given convex set, and all should be included in a first reading, except perhaps for refinements which take advantage of polyhedral convexity.

Problems in which a convex function is minimized subject to a finite system of convex inequalities are studied in §28 (Ordinary Convex Programs and Lagrange Multipliers). The emphasis is on the existence, interpretation, and characterization of certain vectors of Lagrange multipliers, called Kuhn-Tucker vectors. The text may be simplified somewhat by deleting the provisions for linear equation constraints, and Theorem 28.2 may be replaced by its special case Corollary 28.2.1 (which has a much easier proof), but beyond this nothing other than examples ought to be omitted.

The theory of Lagrange multipliers is broadened and in some ways sharpened in §29 (Bifunctions and Generalized Convex Programs). The concept of a convex bifunction, which can be regarded as an extension of that of a linear transformation, is used to construct a theory of perturbations of minimization problems. Generalized Kuhn-Tucker vectors measure the effects of the perturbations. Theorems 29.1, 29.3, and their corollaries contain all the facts needed in the sequel.

In §30 (Adjoint Bifunctions and Dual Programs) the duality theory of extremum problems is set forth. Practically everything up through Theorem 30.5 is fundamental, but the remainder of §30 consists of examples and may be truncated as desired. Duality theory is continued in §31 (Fenchel's Duality Theorem). The primary purpose of §31 is to furnish additional examples interesting for their applications. Later sections do not depend on the material in §31, except for §38.

Results of a rather different character are described in §32 (The Maximum of a Convex Function). The proofs of these results involve none of the preceding sections of Part VI, but familiarity with §18 and §19 is required. No subsequent reference is made to §32.

Part VII: Saddle-functions and Minimax Theory

Saddle-functions are functions which are convex in some variables and concave in others, and the extremum problems naturally associated with them involve "minimaximization," rather than simple minimization or maximization. The theory of such minimax problems can be developed by much the same approach as in the case of minimization of convex functions. It turns out that the general minimax problems for (suitably regularized) saddle-functions are precisely the Lagrangian saddle-point problems associated with generalized (closed) convex programs. Understandably, therefore, convex bifunctions are central to the discussion of saddle-functions, and the reader should not proceed without already being familiar with the basic ideas in §29 and §30.

Saddle-functions on $R^m \times R^n$ correspond to convex bifunctions from R^m to R^n in much the same way that bilinear functions on $R^m \times R^n$ correspond to linear transformations from R^m to R^n. This is the substance of §33 (Saddle-functions). In §34 (Closures and Equivalence Classes), certain closure operations for saddle-functions similar to the one for convex functions are studied. It is shown that each finite saddle-function defined on a product of convex sets in $R^m \times R^n$ determines a unique equivalence class of closed saddle-functions defined on all of $R^m \times R^n$, but one does not actually have to read up on the latter fact (embodied in Theorems 34.4 and 34.5) before passing to minimax theory itself.

The results about saddle-functions proved in §35 (Continuity and Differentiability) are mainly analogues or extensions of results about convex functions in §10, §24, and §25, and they are not a prerequisite for what follows.

Saddle-points and saddle-values are discussed in §36 (Minimax Problems). It is then explained how the study of these can be reduced to the study of convex and concave programs dual to each other. Existence theorems for saddle-points and saddle-values are then derived in §37 (Conjugate Saddle-functions and Minimax Theorems) in terms of a conjugacy correspondence for saddle-functions and the "inverse" operation for bifunctions.

Part VIII: Convex Algebra

The analogy between convex bifunctions and linear transformations, which features so prominently in Parts VI and VII, is pursued further in §38 (The Algebra of Bifunctions). "Addition" and "multiplication" of bifunctions are studied in terms of a generalized notion of inner product based on Fenchel's Duality Theorem. It is a remarkable and non-trivial fact that such natural operations for bifunctions are preserved, as in linear algebra, when adjoints are taken.

The results about bifunctions in §38 are specialized in §39 (Convex Processes) to a class of convex-set-valued mappings which are even more analogous to linear transformations.