Gerald B. Folland

Introduction to Partial Differential Equations

SECOND EDITION

偏微分方程导论 第2版

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GERALD B. FOLLAND

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PREFACE

In 1975 I gave a course in partial differential equations (PDE) at the University of Washington to an audience consisting of graduate students who had taken the standard first-year analysis courses but who had little background in PDE. Accordingly, it focused on basic classical results in PDE but aimed in the direction of the recent developments and made fairly free use of the techniques of real and complex analysis. The roughly polished notes for that course constituted the first edition of this book, which has enjoyed some success for the past two decades as a "modern" introduction to PDE. From time to time, however, my conscience has nagged me to make some revisions — to clean some things up, add more exercises, and include some material on pseudodifferential operators.

Meanwhile, in 1981 I gave another course in Fourier methods in PDE for the Programme in Applications of Mathematics at the Tata Institute for Fundamental Research in Bangalore, the notes for which were published in the Tata Lectures series under the title *Lectures on Partial Differential Equations*. They included applications of Fourier analysis to the study of constant coefficient equations (especially the Laplace, heat, and wave equations) and an introduction to pseudodifferential operators and Calderón-Zygmund singular integral operators. These notes were found useful by a number of people, but they went out of print after a few years.

Out of all this has emerged the present book. Its intended audience is the same as that of the first edition: students who are conversant with real analysis (the Lebesgue integral, L^p spaces, rudiments of Banach and Hilbert space theory), basic complex analysis (power series and contour integrals), and the big theorems of advanced calculus (the divergence theorem, the implicit function theorem, etc.). Its aim is also the same as that of the first edition: to present some basic classical results in a modern setting and to develop some aspects of the newer theory to a point where the student will be equipped to read more advanced treatises. It consists essentially of the union of the first edition and the Tata notes, with the omission of the L^p theory of singular integrals (for which the reader is referred to Stein's classic book [45]) and the addition of quite a few exercises.

Apart from the exercises, the main substantive changes from the first edition to this one are as follows.

- §1F has been expanded to include the full Malgrange-Ehrenpreis theorem and the relation between smoothness of fundamental solutions and hypoellipticity, which simplifies the discussion at a few later points.
- Chapter 2 now begins with a brief new section on symmetry properties of the Laplacian.
- The discussion of the equation $\Delta u = f$ in §2C (formerly §2B) has been expanded to include the full Hölder regularity theorem (and, as a by-product, the continuity of singular integrals on Hölder spaces).
- The solution of the Dirichlet problem in a half-space (§2G) is now done in a way more closely related to the preceding sections, and the Fourier-analytic derivation has been moved to §4B.
- I have corrected a serious error in the treatment of the two-dimensional case in §3E. I am indebted to Leon Greenberg for sending me an analysis of the error and suggesting Proposition (3.36b) as a way to fix it.
- The discussion of functions of the Laplacian in the old §4A has been expanded and given its own section, §4B.
- Chapter 5 contains a new section (§5D) on the Fourier analysis of the wave equation.
- The first section of Chapter 6 has been split in two and expanded to include the interpolation theorem for operators on Sobolev spaces and the local coordinate invariance of Sobolev spaces.
- A new section (§6D) has been added to present Hörmander's characterization of hypoelliptic operators with constant coefficients.
- Chapter 8, on pseudodifferential operators, is entirely new.

In addition to these items, I have done a fair amount of rewriting in order to improve the exposition. I have also made a few changes in notation — most notably, the substitution of $\langle f \mid g \rangle$ for (f, g) to denote the Hermitian inner product $\int f\overline{g}$, as distinguished from the bilinear pairing $\langle f, g \rangle = \int fg$. (I have sworn off using parentheses, perhaps the most overworked symbols in mathematics, to denote inner products.) I call the reader's attention to the existence of an index of symbols as well as a regular index at the back of the book.

The bias toward elliptic equations in the first edition is equally evident here. I feel a little guilty about not including more on hyperbolic equations, but that is a subject for another book by another author.

The discussions of elliptic regularity in §6C and §7F and of Gårding's inequality in §7D may look a little old-fashioned now, as the machinery of pseudodifferential operators has come to be accepted as the "right" way to obtain these results. Indeed, I rederive (and generalize) Gårding's inequality and the local regularity theorem by this method in §8F. However, I think the "low-tech" arguments in the earlier sections are also worth retaining. They provide the quickest proofs when one starts from scratch, and they show that the results are really of a fairly elementary nature.

I have revised and updated the bibliography, but it remains rather short and quite unscholarly. Wherever possible, I have preferred to give references to expository books and articles rather than to research papers, of which only a few are cited.

In the preface to the first edition I expressed my gratitude to my teachers J. J. Kohn and E. M. Stein, who influenced my point of view on much of the material contained therein. The same sentiment applies equally to the present work.

Gerald B. Folland
Seattle, March 1995

CONTENTS

Chapter 0
PRELIMINARIES

The purpose of this chapter is to fix some terminology that will be used throughout the book, and to present a few analytical tools which are not included in the prerequisites. It is intended mainly as a reference rather than as a systematic text.

A. Notations and Definitions

Points and sets in Euclidean space

$\mathbb{R}$ will denote the real numbers, $\mathbb{C}$ the complex numbers. We will be working in $\mathbb{R}^n$, and n will always denote the dimension. Points in $\mathbb{R}^n$ will generally be denoted by x, y, ξ, η; the coordinates of x are $(x_1, \ldots, x_n)$. Occasionally $x_1, x_2, \ldots$ will denote a sequence of points in $\mathbb{R}^n$ rather than coordinates, but this will always be clear from the context. Once in a while there will be some confusion as to whether $(x_1, \ldots, x_n)$ denotes a point in $\mathbb{R}^n$ or the n-tuple of coordinate functions on $\mathbb{R}^n$. However, it would be too troublesome to adopt systematically a more precise notation; readers should consider themselves warned that this ambiguity will arise when we consider coordinate systems other than the standard one.

If U is a subset of $\mathbb{R}^n$, $\overline{U}$ will denote its closure and ∂U its boundary. The word **domain** will be used to mean an open set $\Omega \subset \mathbb{R}^n$, not necessarily connected, such that $\partial\Omega = \partial(\mathbb{R}^n \setminus \overline{\Omega})$. (That is, all the boundary points of Ω are "accessible from the outside.")

If x and y are points of $\mathbb{R}^n$ or $\mathbb{C}^n$, we set

$$x \cdot y = \sum_1^n x_j y_j ,$$

so the Euclidean norm of x is given by

$$|x| = (x \cdot \overline{x})^{1/2} \quad (= (x \cdot x)^{1/2} \text{ if } x \text{ is real.})$$

We use the following notation for spheres and (open) balls: if $x \in \mathbb{R}^n$ and $r > 0$,

$$S_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\},$$
$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

Measures and integrals

The integral of a function f over a subset Ω of $\mathbb{R}^n$ with respect to Lebesgue measure will be denoted by $\int_\Omega f(x)\,dx$ or simply by $\int_\Omega f$. If no subscript occurs on the integral sign, the region of integration is understood to be $\mathbb{R}^n$. If S is a smooth hypersurface (see the next section), the natural Euclidean surface measure on S will be denoted by $d\sigma$; thus the integral of f over S is $\int_S f(x)\,d\sigma(x)$, or $\int_S f\,d\sigma$, or just $\int_S f$. The meaning of $d\sigma$ thus depends on S, but this will cause no confusion.

If f and g are functions whose product is integrable on $\mathbb{R}^n$, we shall sometimes write

$$\langle f, g\rangle = \int fg, \qquad \langle f \,|\, g\rangle = \int f\overline{g},$$

where $\overline{g}$ is the complex conjugate of g. The Hermitian pairing $\langle f|g\rangle$ will be used only when we are working with the Hilbert space L^2 or a variant of it, whereas the bilinear pairing $\langle f, g\rangle$ will be used more generally.

Multi-indices and derivatives

An n-tuple $\alpha = (\alpha_1, \dots a_n)$ of nonnegative integers will be called a **multi-index**. We define

$$|\alpha| = \sum_1^n \alpha_j, \qquad \alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!,$$

and for $x \in \mathbb{R}^n$,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

We will generally use the shorthand

$$\partial_j = \frac{\partial}{\partial x_j}$$

for derivatives on $\mathbb{R}^n$. Higher-order derivatives are then conveniently expressed by multi-indices:

$$\partial^\alpha = \prod_1^n \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Note in particular that if $\alpha = 0$, ∂^α is the identity operator. With this notation, it would be natural to denote by ∂u the n-tuple of functions $(\partial_1 u, \ldots, \partial_n u)$ when u is a differentiable function; however, we shall use instead the more common notation

$$\nabla u = (\partial_1 u, \ldots, \partial_n u).$$

For our purposes, a **vector field** on a set $\Omega \in \mathbb{R}^n$ is simply an $\mathbb{R}^n$-valued function on Ω. If F is a vector field on an open set Ω, we define the directional derivative ∂_F by

$$\partial_F = F \cdot \nabla,$$

that is, if u is a differentiable function on Ω,

$$\partial_F u(x) = F(x) \cdot \nabla u(x) = \sum_1^n F_j(x) \partial_j u(x).$$

Function spaces

If Ω is a subset of $\mathbb{R}^n$, $C(\Omega)$ will dente the space of continuous complex-valued functions on Ω (with respect to the relative topology on Ω). If Ω is open and k is a positive integer, $C^k(\Omega)$ will denote the space of functions possessing continuous derivatives up to order k on Ω, and $C^k(\overline{\Omega})$ will denote the space of all $u \in C^k(\Omega)$ such that $\partial^\alpha u$ extends continuously to the closure $\overline{\Omega}$ for $0 \leq |\alpha| \leq k$. Also, we set $C^\infty(\Omega) = \bigcap_1^\infty C^k(\Omega)$ and $C^\infty(\overline{\Omega}) = \bigcap_1^\infty C^k(\overline{\Omega})$.

We next define the Hölder or Lipschitz spaces $C^\alpha(\Omega)$, where Ω is an open set and $0 < \alpha < 1$. (Here α is a real number, not a multi-index; the use of the letter "α" in both these contexts is standard.) $C^\alpha(\Omega)$ is the space of continuous functions on Ω that satisfy a locally uniform Hölder condition with exponent α. That is, $u \in C^\alpha(\Omega)$ if and only if for any compact $V \subset \Omega$ there is a constant $c > 0$ such that for all $y \in \mathbb{R}^n$ sufficiently close to 0,

$$\sup_{x \in V} |u(x+y) - u(x)| \leq c|y|^\alpha.$$

(Note that $C^1(\Omega) \subset C^\alpha(\Omega)$ for all $\alpha < 1$, by the mean value theorem.) If k is a positive integer, $C^{k+\alpha}(\Omega)$ will denote the set of all $u \in C^k(\Omega)$ such that $\partial^\beta u \in C^\alpha(\Omega)$ for all muli-indices β with $|\beta| = k$ (or equivalently, with $|\beta| \le k$; the lower-order derivatives are automatically in $C^1(\Omega) \subset C^\alpha(\Omega)$.).

The **support** of a function u, denoted by $\operatorname{supp} u$, is the complement of the largest open set on which $u = 0$. If $\Omega \subset \mathbb{R}^n$, we denote by $C_c^\infty(\Omega)$ the space of all C^∞ functions on $\mathbb{R}^n$ whose support is compact and contained in Ω. (In particular, if Ω is open such functions vanish near $\partial\Omega$.)

The space $C^k(\mathbb{R}^n)$ will be denoted simply by C^k. Likewise for C^∞, $C^{k+\alpha}$, and C_c^∞.

If $\Omega \subset \mathbb{R}^n$ is open, a function $u \in C^\infty(\Omega)$ is said to be **analytic** in Ω if it can be expanded in a power series about every point of Ω. That is, u is analytic on Ω if for each $x \in \Omega$ there exists $r > 0$ such that for all $y \in B_r(x)$,

$$u(y) = \sum_{|\alpha| \ge 0} \frac{\partial^\alpha u(x)}{\alpha!}(y - x)^\alpha,$$

the series being absolutely and uniformly convergent on $B_r(x)$. When referring to complex-analytic functions, we shall always use the word **holomorphic**.

The **Schwartz class** $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the space of all C^∞ functions on $\mathbb{R}^n$ which, together with all their derivatives, die out faster than any power of x at infinity. That is, $u \in \mathcal{S}$ if and only if $u \in C^\infty$ and for all multi-indices α and β,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty.$$

Big O and little o

We occasionally employ the big and little o notation for orders of magnitude. Namely, when we are considering the behavior of functions in a neighborhood of a point a (which may be ∞), $O(f(x))$ denotes any function $g(x)$ such that $|g(x)| \le C|f(x)|$ for x near a, and $o(f(x))$ denotes any function $h(x)$ such that $h(x)/f(x) \to 0$ as $x \to a$.

B. Results from Advanced Calculus

A subset S of $\mathbb{R}^n$ is called a **hypersurface of class C^k** $(1 \le k \le \infty)$ if for every $x_0 \in S$ there is an open set $V \subset \mathbb{R}^n$ containing x_0 and a real-valued function $\phi \in C^k(V)$ such that $\nabla\phi$ is nonvanishing on $S \cap V$ and

$$S \cap V = \{x \in V : \phi(x) = 0\}.$$

In this case, by the implicit function theorem we can solve the equation $\phi(x) = 0$ near x_0 for some coordinate x_i — for convenience, say $i = n$ — to obtain

$$x_n = \psi(x_1, \ldots, x_{n-1})$$

for some C^k function ψ. A neighborhood of x_0 in S can then be mapped to a piece of the hyperplane $x_n = 0$ by the C^k transformation

$$x \longrightarrow (x', x_n - \psi(x')) \qquad (x' = (x_1, \ldots, x_{n-1})).$$

This same neighborhood can also be represented in **parametric form** as the image of an open set in $\mathbb{R}^{n-1}$ (with coordinate x') under the map

$$x' \longrightarrow (x', \psi(x')).$$

x' may be thought of as giving local coordinates on S near x_0.

Similar considerations apply if "C^k" is replaced by "analytic."

With S, V, ϕ as above, the vector $\nabla\phi(x)$ is perpendicular to S at x for every $x \in S \cap V$. We shall always suppose that S is **oriented**, that is, that we have made a choice of unit vector $\nu(x)$ for each $x \in S$, varying continuously with x, which is perpendicular to S at x. $\nu(x)$ will be called the **normal** to S at x; clearly on $S \cap V$ we have

$$\nu(x) = \pm\frac{\nabla\phi(x)}{|\nabla\phi(x)|}.$$

Thus ν is a C^{k-1} function on S. If S is the boundary of a domain Ω, we always choose the orientation so that ν points out of Ω.

If u is a differentiable function defined near S, we can then define the **normal derivative** of u on S by

$$\partial_\nu u = \nu \cdot \nabla u.$$

We pause to compute the normal derivative on the sphere $S_r(y)$. Since lines through the center of a sphere are perpendicular to the sphere, we have

$$\text{(0.1)} \qquad \nu(x) = \frac{x-y}{r}, \quad \partial_\nu = \frac{1}{r}\sum_1^n (x_j - y_j)\partial_j \qquad \text{on } S_r(y).$$

We will use the following proposition several times in the sequel:

(0.2) Proposition.
Let S be a compact oriented hypersurface of class C^k, $k \geq 2$. There is a neighborhood V of S in $\mathbb{R}^n$ and a number $\epsilon > 0$ such that the map

$$F(x,t) = x + t\nu(x)$$

is a C^{k-1} diffeomorphism of $S \times (-\epsilon, \epsilon)$ onto V.

Proof (sketch): F is clearly C^{k-1}. Moreover, for each $x \in S$ its Jacobian matrix (with respect to local coordinates on $S \times \mathbb{R}$) at $(x, 0)$ is nonsingular since ν is normal to S. Hence by the inverse mapping theorem, F can be inverted on a neighborhood W_x of each $(x, 0)$ to yield a C^{k-1} map

$$F_x^{-1} : W_x \to (S \cap W_x) \times (-\epsilon_x, \epsilon_x)$$

for some $\epsilon_x > 0$. Since S is compact, we can choose $\{x_j\}_1^N \subset S$ such that the W_{x_j} cover S, and the maps $F_{x_j}^{-1}$ patch together to yield a C^{k-1} inverse of F from a neighborhood V of S to $S \times (-\epsilon, \epsilon)$ where $\epsilon = \min_j \epsilon_{x_j}$. ∎

The neighborhood V in Proposition (0.2) is called a **tubular neighborhood** of S. It will be convenient to extend the definition of the normal derivative to the whole tubular neighborhood. Namely, if u is a differentiable function on V, for $x \in S$ and $-\epsilon < t < \epsilon$ we set

$$\partial_\nu u(x + t\nu(x)) = \nu(x) \cdot \nabla u(x + t\nu(x)). \tag{0.3}$$

If $F = (F_1, \ldots, F_n)$ is a differentiable vector field on a subset of $\mathbb{R}^n$, its **divergence** is the function

$$\nabla \cdot F = \sum_1^n \partial_j F_j.$$

With this terminology, we can state the form of the general Stokes formula that we shall need.

(0.4) The Divergence Theorem.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary $S = \partial\Omega$, and let F be a C^1 vector field on $\overline{\Omega}$. Then

$$\int_S F(y) \cdot \nu(y)\, d\sigma(y) = \int_\Omega \nabla \cdot F(x)\, dx.$$

The proof can be found, for example, in Treves [52, §10].

Every $x \in \mathbb{R}^n \setminus \{0\}$ can be written uniquely as $x = ry$ with $r > 0$ and $y \in S_1(0)$ — namely, $r = |x|$ and $y = x/|x|$. The formula $x = ry$ is called the **polar coordinate** representation of x. Lebesgue measure is given in polar coordinates by

$$dx = r^{n-1}\, dr\, d\sigma(y),$$

where $d\sigma$ is surface measure on $S_1(0)$. (See Folland [14, Theorem (2.49)].)

For example, if $0 < a < b < \infty$ and $\lambda \in \mathbb{R}$, we have

$$\int_{a<|x|<b} |x|^\lambda\, dx = \int_{S_1(0)} \int_a^b r^{n-1+\lambda}\, dr = \begin{cases} \omega_n \frac{b^{n+\lambda} - a^{n+\lambda}}{n+\lambda} & \text{if } \lambda \neq -n, \\ \omega_n \log(b/a) & \text{if } \lambda = -n, \end{cases}$$

where ω_n is the area of $S_1(0)$ (which we shall compute shortly). As an immediate consequence, we have:

(0.5) Proposition.
The function $x \to |x|^\lambda$ is integrable on a neighborhood of 0 if and only if $\lambda > -n$, and it is integrable outside a neighborhood of 0 if and only if $\lambda < -n$.

As another application of polar coordinates, we can compute what is probably the most important definite integral in mathematics:

(0.6) Proposition.
$\int e^{-\pi|x|^2}\, dx = 1$.

Proof: Let $I_n = \int_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx$. Since $e^{-\pi|x|^2} = \prod_1^n e^{-\pi x_j^2}$, Fubini's theorem shows that $I_n = (I_1)^n$, or equivalently that $I_n = (I_2)^{n/2}$. But in polar coordinates,

$$I_2 = \int_0^{2\pi} \int_0^\infty e^{-\pi r^2} r\, dr\, d\theta = 2\pi \int_0^\infty r e^{-r^2}\, dr = \pi \int_0^\infty e^{-\pi s}\, ds = 1. \quad ∎$$

This trick works because we know that the measure of $S_1(0)$ in $\mathbb{R}^2$ is 2π. But now we can turn it around to compute the area ω_n of $S_1(0)$ in $\mathbb{R}^n$ for any n. Recall that the **gamma function** $\Gamma(s)$ is defined for $\mathrm{Re}\, s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt.$$