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Relevant Negation and Classical Negation

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Abstract. Negation in situaitons acts relevantly, but in worlds it acts classically. In this paper I set out a basic relevant logic of information, L_1, and a very closely related logic L_2. L_2 is the logic that is characterized by the class models that includes both worlds and situations. The situations of one of these models, taken by itself (with its accessibility relation, incompatibility relation, and set of distingushed situations) is an L_1 model structure. Given some intuitive conditions relating L_1 model structures to possible worlds, we end up with models for a logic that relates relevant negation to classical negation in a reasonable way and allows us to say that the two are compatible.

1. Introduction

The classical truth condition for negation is intuitive. A negation, $\neg A$, is true if and only if A fails to be true. Issues concerning vagueness of course cause problems for this intuitive condition, but I set those aside for the purposes of this paper. I treat negation, in relation to truth, as fully classical.

Relevant logic was created to give more intuitive treatments of implication and validity. The creators of relevant logic did not want to overthrow the classical treatment of conjunction, disjunction, or negation. The idea behind relevant logic is to reject the so-called paradoxes of material and strict implication and to avoid the fallacies of relevance. The paradoxes of implication in which we are interested here are:

$$p \to (q \vee \neg q)$$

$$(q \wedge \neg q) \to r$$

And here are the fallacies of relevance that we will discuss:

$$\frac{A}{\therefore B \vee \neg B}$$

$$\frac{A \wedge \neg A}{\therefore B}$$

These fallacies of relevance are the inferential counterparts of the paradoxes given above. According to relevant logicians, the problem with these fallacies is that their

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premises seem to have nothing to do with their conclusions. The premises are irrelevant to their conclusions. Similarly, in the paradoxes their antecedents have nothing to do with their consequents.

The formal semantics for relevant logic is an indexical semantics (see section 3 below). This means that in models for relevant logic formulas are satisfied or fail to be satisfied at points. In order to make invalid an inference or paradox listed above, we need to have models in which there is a point at which the premise or antecedent is satisfied and the conclusion or consequent fails to be satisfied. Traditionally, the satisfaction of a formula at a point in relevant models has been understood as meaning that the formula is true at that point. Thus, in order to falsify the paradoxes, on this traditional reading, we need points at which the law of excluded middle, i.e.,

$$B \vee \neg B$$

fails to be true and we need points at which a contradiction, viz.,

$$A \wedge \neg A$$

is true. Both of these are incompatible with a classical understanding of negation (or perhaps conjunction and disjunction). Thus, given the traditional reading of the semantics, we are forced into a non-classical understanding of at least some of the connectives other than implication. In fact, it is negation that is usually understood in a non-classical fashion. Thus, relevant logicians felt themselves driven to adopt a non-classical treatment of negation in order to have a relevant understanding of implication and validity.

In my opinion, relevant logicians do not need to adopt a non-classical truth condition for negation or any of the other connectives except implication. The problem is in the interpretation of the formal semantics for relevant logic. Instead of viewing it as incorporating a non-classical theory of truth for formulas, we should think of it as formalizing a theory of information. The difference between truth and information can be illustrated clearly now with a brief example. The table on which my computer sits is not green. The sentence 'this table is not green' as said by me as I write this is true because my table fails to be green. But the information that this table is not green is available to me in my current environment because the table is light brown, a colour which is incompatible with its being green. Thus, we can distinguish between the truth condition for negation – which is understood in terms of the negated sentence's failing to be true – and its information condition – which is understood in terms of something's having properties incompatible with the negated sentence.

We can see also that there are important and more general differences between information conditions and truth conditions. First, we relativize information conditions to environments. My current environment contains the information that my table is not brown, but it does not contain any information about the weather in

Guangzhou at the present moment. It makes sense to think of information as a *local phenomenon*, that is, to treat information as being related to particular environments. But it does not make sense to think of truth as relativized to environments. In this environment, it is not true that it is not sunny in Guangzhou. Rather, we merely have no information about the matter. Truth on the other hand is determined by the world as a whole.[1]

A second important general difference between truth and information is that information is in a certain sense always positive in nature, whereas truth may not be. To return to the example of the colour of my desk, there is some positive information that precludes its being green. What determines that the sentence 'this desk is not green' is true on the other hand is merely the failure of 'this desk is green' to be true. Information is by its nature something that is available to us (or other sentient beings). What is given to us somehow depends of the positive facts of the environments in which we find ourselves.

In this paper I set out a simple logic of information – a very weak relevant logic. I then relate it to a classical treatment of the truth conditions of the propositional connectives. I do so by first presenting the Routley-Meyer semantics and its informational interpretation. Then, I develop a second logic that incorporates the relationship between environments (or "situations") and possible worlds. This relationship the metaphysical correlate of the relationship between information and truth. I then end the paper by pointing out a virtue of the ability of relevant logic to incorporate the classical view of negation in a relevant treatment of denial.

2. The Semantic Framework, Part I: Situations and Worlds

The philosophical framework that I use to interpret relevant logic is based on the situation semantics of Jon Barwise and John Perry [3]. Barwise and Perry hold that we should the notion of a *situation* should play the role usually occupied by possible worlds in semantics, that is, the role of an index which satisfies or fails to satisfy particular formulas. A concrete situation is just a part of a world. For example, my study as I write this sentence contains certain information about the colour and shape of my computer, my present actions, my dog's current actions (she is asleep), and so on. A concrete situation does not have to be spatio-temporally continuous. For example, the situation that provides the background to a phone conversation might include parts of the surroundings of both of the participants of that conversation. We do not use concrete situations in the semantics of relevant logic, but some of their close relatives – abstract situations. An abstract situation is an abstract object that

[1] Actually this is a more difficult matter than we can go into here. Sentences with indexical expersions (such as "here" and "now") and ones with contextually restricted quantification ("everyone had fun") require some sort of restriction of context of evaluation or of domain. But if we think about "fully articulated" senteces, then we do not have this problem.

purports to characterize a concrete situation. In [9], I take abstract situations to be set theoretic constructions. An abstract situation that accurately characterizes the information contained in a concrete situation in some possible world is a *possible situation* and an abstract situation that does not accurately represent any concrete situation is called an *impossible situation*.

The notion of a situation's containing certain information is crucial to my current project, so I should explain it a little further. Let's return to the concrete situation that is my study at this time. Actually, there are various different situations here, but we will ignore that for the moment and pretend that there is just one. There is certain information "available" to me in this situation. The colours and shapes of the objects in the room, my own position at my desk, the sound of my dog's breathing, the noise of the cicadas outside, the smell of the tree outside my window, and so on. If I turn on the television or the radio, more information will be available to me. Information does not have to be perceptually present in order to count as information. As long as there is a reliable connection between me and a particular fact, the information that this is a fact is available information from my point of view (see [4]).

Situation semantics provides one theory on which to base an informational approach to the semantics for a logical system. I have argued elsewhere that an informational perspective on semantics is more appropriate than a truth conditional approach for relevant logic (see [10] and [11]). Here I will briefly repeat one of these arguments to motivate and help explain the view.

As we said, the reason why relevant logic was originally developed was to avoid the so-called paradoxes of material and strict implication and to avoid certain inference forms thought to be "fallacies of relevance". Here again is one such fallacious inference scheme:

$$\frac{A}{\therefore B \vee \neg B}$$

In this scheme a tautology – the law of excluded middle – follows from an arbitrary proposition. On the classical definition of validity, an inference is valid if in every case in which the premises are all true, the conclusion is also true. One way to avoid making this inference form valid is to accept the classical definition of validity while rejecting the classical logicians notion of what counts as an admissible case – that is, one might accept "cases" in which the law of excluded middle fails to be true. This is the approach of Beall and Restall's truth conditional form of logical pluralism [5].[2]

If we accept the Beall-Restall line we have to accept pluralism about the truth conditions of the logical connectives. In order to construct cases in which the law of excluded middle fails to be true we must reject the classical truth conditions for either negation or disjunction. The point of relevant logic, however, was not to force a rethinking of the nature of truth. Rather, as I have said, the point was to alter

[2]Truth conditional logical pluralism is not the only type of logical pluralism found in the literature. There is also informational logical pluralism (see [1]), which is a view closer to the one of this paper.

the classical notion of inference. Thus, it would seem that modifying the notion of validity, which is the semantic counterpart of inference, makes more sense than to change the classical notion of truth. I suggest that understanding validity as information preservation rather than as truth preservation fits better with the motivation for relevant logic.

The concept of information preservation is similar to truth preservation. An inference rule preserves information if and only if every abstract situation which contains the information represented by the premises of the rule, then it also contains the information expressed by the conclusion. In order to make the notion of information preservation useful as a basis for a formal semantics, we need an inductive definition of the conditions under which situations contain the information expressed by formulas. We do this in the next section.

3. The Semantic Framework, Part II: Routley-Meyer Models

In the early 1970s, Richard Routley and Robert Meyer first developed their model theory for relevant logic. We will discuss a slightly modified version of this semantics which treats negation using an incompatibility relation originally employed by Robert Goldblatt in his semantics for orthologic (a generalization of quantum logic) [8] and adapted to relevant logic by J.M. Dunn [6].

A relevant model structure is a quadruple $\mathbf{S} =< S, 0, R, \perp>$, where S is a non-empty set (of situations), 0 is a non-empty subset of S (the "logical situations"), R is a ternary relation on S (which is used to give an information condition for implication), and $\perp$ ("perp") is a binary relation on S (used to give an information condition for negation). We define a partial order $\leq$ on S such that $s \leq t$ if and only if there is a situation o in 0 for which $Rost$. We can think of $\leq$ as an informational version of a hereditariness relation, like that found in Kripke's semantics for intuitionist logic. A set of situations closed upwards under $\leq$ is a set X such that if for any situations s and t if $s \in X$ and $s \leq t$, then $t \in X$.

All relevant model structures satisfy the following conditions:

1. if $Rstu$ and $s' \leq s$, then $Rs'tu$;
2. $\leq$ is transitive and reflexive;
3. $\perp$ is symmetrical.

A value assignment v on $\mathbf{S}$ is a function from the propositional variables to the set of subsets of S closed upwards under $\leq$. Each value assignment determines a satisfaction relation, $\models$, that obeys the following information conditions:

- $s \models p$ iff $s \in v(p)$
- $s \models A \wedge B$ iff $s \models A$ and $s \models B$
- $s \models A \vee B$ iff $s \models A$ or $s \models B$

- $s \models A \to B$ iff $\forall x \forall y((Rsxy \wedge x \models A) \supset y \models B)$
- $s \models \neg A$ iff $\forall x(x \models A \supset s \perp x)$

A formula A is valid in $\mathcal{M}$ if and only if for all $o \in 0$, $o \models A$.[3] The logic that is characterized by this class of model structures is a very weak system. It is an version of the base relevant system B with a minimal (weak intuitionist) negation. The negation is minimal in the sense that it does not satisfy double negation elimination ($\neg\neg A \to A$) nor does it satisfy ex falso quodlibet ($(A \wedge \neg A) \to B$), like the negation of Johannson's minimal logic. We call this logic L_1. It is axomatized in the section 4.

The following are two lemmas that we use quite often. They are proven in [13].

Lemma 1 (Hereditariness) If $s \models A$ and $s \leq t$, then $t \models A$.

Lemma 2 (Semantic Entailment) For any model $\mathcal{M} =< S, 0, R, \perp, v >$, $\mathcal{M} \models A \to B$ iff for all $s \in S$ if $s \models A$, then $s \models B$.

4. L_1 and Soundness

This class of models characterizes a logic that we call L_1.

Axiom Schemes:

1. $A \to A$
2. $A \to (A \vee B)$; $B \to (A \vee B)$
3. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
4. $(A \to C) \to ((A \wedge B) \to C)$
5. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
6. $A \to \neg\neg A$

Rules:

Modus ponens:

$$\frac{\begin{array}{c} A \to B \\ A \end{array}}{B}$$

Affixing:

$$\frac{\begin{array}{c} A \to B \\ B \to C \end{array}}{A \to C}$$

[3] I have argued elsewhere for a very different treatment of disjunction in informational semantics [11]. But here I am using a simple version of the semantics for relevant logic to experiment with a particular view of the relationship between situations and worlds. At some later date I should try to modify the present theory to accommodate my more complicated view of the logical constants.

Adjunction:

$$\frac{\begin{array}{c}A\\ B\end{array}}{A \wedge B}$$

Contraposition:

$$\frac{A \rightarrow B}{\neg B \rightarrow \neg A}$$

Soundness for the negation-free fragment of the logic is proven in the same way as in the original soundness proofs for relevant logic (see, e.g., [13], [14], [2]).

To prove the validity of axiom 6, we assume that for an arbitrary situation s, $s \models A$. Suppose that $t \models \neg A$. And assume, for the sake of a reductio that s is compatible with t. Since incompatibility is symmetrical, so is compatibility. Thus, t is compatible with s. But then, by the information condition for negation, $s \nvDash A$, contrary to our supposition. Thus, by reductio, $t \nvDash \neg A$. Generalizing, by the information condition for negation, $s \models \neg\neg A$.

Let us now prove the validity of the contraposition rule. Suppose that $\mathcal{M} \models A \rightarrow B$. Also suppose that for an arbitrary situation s that $s \models \neg B$. By the information condition for negation, if $t \models B$, then $s \perp t$. Suppose that $t \models A$. Then, by lemma 2 $t \models B$. So, $s \perp t$. Generalizing, by the information condition for negation, $s \models \neg B$. So, by lemma 2 $\mathcal{M} \models \neg B \rightarrow \neg A$.

5. Canonical Model Construction and Completeness for L_1

In this section I sketch the completeness proof for L_1. This completeness proof has become quite standard in the literature on relevant logic and so there is no need to prove every lemma. But we will need to understand the construction of the canonical model in order to apply it again to L_2 in section 8 below.

First we need a few definitions and then to sketch the relevant form of the Lindenbaum extension lemma. A theory for a logic L is a set of formulas Γ such that if $(A_1 \wedge ... \wedge A_n) \rightarrow B$ is a theorem of L and $A_1, ..., A_n \in \Gamma$ then $B \in \Gamma$. A theory of L is prime if and only if for every disjunction $A \vee B \in \Gamma$ either A or B is in Γ. A theory is L-regular if and only if it contains all the theorems of L.

In order to prove the Lindenbaum theorem, we also need the notion of a consistent L-pair (Γ, Δ): this is a pair of sets of formulas such that there are no formulas $A_1, ..., A_n \in \Gamma$ and $B_1, ..., B_m \in \Delta$ such that $(A_1 \wedge ... \wedge A_n) \rightarrow (B_1 \vee ... \vee B_m)$ is a theorem of L.

Lemma 3 If (Γ, Δ) is a consistent L_1-pair, then there is a prime theory $\Gamma' \supseteq \Gamma$ such that (Γ', Δ) is a consistent L_1-pair.

The proof for this lemma, due to Belnap and Gabbay, incorporates a generalization of the construction used to prove the Lindenbaum lemma for classical and modal logics. We begin as usual with an exhaustive enumeration of the formulas $A_0, A_1, A_2, \ldots, A_n, \ldots$ We then set out the following inductive definition of (Γ_n, Δ_n).

$$\Gamma_0 = \Gamma; \;\; \Delta_0 = \Delta$$

$$\Gamma_{n+1} = \Gamma_n \cup \{A_n\}; \;\; \Delta_{n+1} = \Delta_n$$

if $(\Gamma_n \cup \{A_n\}, \Delta_n)$ is consistent, and

$$\Gamma_{n+1} = \Gamma_n; \;\; \Delta_{n+1} = \Delta_n \cup \{A_n\}$$

otherwise.

$$\Gamma' = \bigcup_{n=0}^{\infty} \Gamma_n; \;\; \Delta' = \bigcup_{n=0}^{\infty} \Delta_n$$

We then go on in the usual way to show that Γ' is a prime regular L_1 theory and that (Γ', Δ) is a consistent L_1-pair (see [13] or [14]). The Lindenbaum lemma shows among other things that the set of theorems of L_1 is the intersection of the set of prime regular L_1 theories.

The canonical model for L_1 is a quintuple $\mathcal{M}_{L_1} =< S_{L_1}, 0_{L_1}, R_{L_1}, \perp_{L_1}, v_{L_1} >$ such that:

- S_{L_1} is the set of prime L_1 theories;
- 0_{L_1} is the set of prime regular L_1 theories;
- $R_{L_1}stu$ iff $\forall A \forall B((A \rightarrow B \in s \wedge A \in t) \supset B \in u)$;
- $s \perp_{L_1} t$ iff $\exists A(A \in t \wedge \neg A \in s)$;
- $v_{L_1}(p) = \{s \in S_{L_1} : p \in s\}$.

For the most part, the canonical model construction for L_1 is fairly standard. We can show in the standard way that R_{L_1} behaves as a relation in a Routley-Meyer model (see, e.g., [12] or [14]). The only features of the canonical model that we have to prove to be correct are some that have to do with the incompatibility relation.

To show that $\perp_{L_1}$ is symmetrical, assume that $s \perp_{L_1} t$. Then there is some wff A such that $A \in t$ and $\neg A \in s$. By the double negation axiom $\neg\neg A \in t$, so there is some formula B (namely $\neg A$) such that $\neg B \in s$ and $B \in t$. Therefore, $t \perp_{L_1} s$.

The following lemma shows that the incompatibility relation captures the information condition for negation.

Lemma 4 $\neg A \in s$ iff $\forall t(A \in t \supset s \perp_{L_1} t)$.

Proof $\Longrightarrow$ Follows from the definition of $\perp_{L_1}$.

$\Longleftarrow$ Suppose that $\forall t(A \in t \supset s \perp_{L_1} t)$. Also suppose that $\neg A \notin s$. By the contraposition rule, $(\{A\}, \{B : \neg B \in s\})$ is a consistent L_1 pair. So by lemma ?, we can extend $\{A\}$ to a prime L_1 theory, u such that $(u, \{B : \neg B \in s\})$ is a consistent L_1 pair. But, by the definition of $\perp_{L_1}$, it is *not* the case that $s \perp_{L_1} u$ and $A \in u$, contrary the assumption of the reductio. Hence, $\neg A \in s$. □

In order to prove completeness, one then uses the standard lemmas together with the ones that we have proven here to show that:

Theorem 5 For all formulas A and all situation $s \in S_{L_1}$, $s \models A$ iff $A \in s$.

Completeness follows from this theorem and the Lindenbaum lemma.

6. Relating Situations to Worlds

Now that we have distinguished between information conditions and truth conditions and similarly between situations and worlds, it is time to relate them to one another in a formal manner. I do so by defining another class of model structures $\mathbf{W} =< W, S, 0, R, \perp, In >$, where $< S, 0, R, \perp>$ is an L_1 model structure, W is a non-empty set (of "possible worlds"), and $In(w)$ is a set of situations downwards closed under $\leq$. An L_2 model is a septuple $< W, S, 0, R, \perp, In, v >$, where $< W, S, 0, R, \perp, In >$ is an L_2 model structure and v is a function from propositional variables into $\wp(S \cup W)$, such for any propositional variable p $v(p) \cap S$ is closed upwards under $\leq$. In all L_2 models, the following conditions obtain:

- W0 For every propositional variable p, $w \in v(p)$ iff there is some situation s in w such that $s \in v(p)$.
- W1 For every world w there is some $o \in 0$ such that o is in w.
- W2 For every world w and all situations s and t in w there is some situation u in w such that $s \leq u$ and $t \leq u$.
- W3 For every world w and all situations s and t in w, s is compatible with t.
- W4 For every world w there is some situation s in w such that for all situations t, $s \perp t$ iff t is not in w.

We introduce a second satisfaction relation, $\Vdash$, between worlds and formulas. This relation represents truth rather than information containment. It is defined by the following clauses:

- $w \Vdash p$ iff $w \in v(p)$
- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash \neg A$ iff $w \nVdash A$
- $w \Vdash A \rightarrow B$ iff $\exists s(s \in In(w) \ \wedge \ s \models A \rightarrow B)$

A formula A is valid in an L_2 model if and only if for every $w \in W$, $w \Vdash A$.

These conditions allow us to make the connection between worlds and situations precise. In particular, we can now prove the following:

Proposition 6 For any world w and formula A, $w \Vdash A$ iff there is a situation s in w such that $s \models A$.

Proof By induction on the complexity of A.

Case 1. $A = p$. $w \Vdash A$ iff there is a situation s in w such that $s \models A$ by W0.

Case 2. $A = B \wedge C$. Suppose first that $w \Vdash B \wedge C$. Then, by the truth condition for conjunction, $w \Vdash B$ and $w \Vdash C$. By the inductive hypothesis, $w \Vdash B$ iff there is a situation s in w such that $s \models B$ and $w \Vdash C$ iff there is a situation t in w such that $t \models C$. By W2 there is some situation u in w such that $s \leq u$ and $t \leq u$. By lemma 1 $u \models B$ and $u \models C$ and so $u \models B \wedge C$.

Now suppose that there is a situation s in w such that $s \models B \wedge C$. By the information condition for conjunction, $s \models B$ and $s \models C$. So, by the inductive hypothesis, $w \Vdash B$ and $w \Vdash C$, hence $w \Vdash B \wedge C$.

Case 3. $A = B \vee C$. Follows straightforwardly from the inductive hypothesis and the information and truth conditions for disjunction.

Case 4. $A = B \rightarrow C$. Follows directly from the truth condition for implication.

Case 5. $A = \neg B$. Suppose first that $w \Vdash \neg B$. By W4 there is some situation s in w such that if s and t are compatible, then t is also in w. By the inductive hypothesis, for all such ts, $t \not\models B$. So, by the information condition for negation, $s \models \neg B$.

Now suppose that there is some situation s in w such that $s \models \neg B$. Let t be some other situation in w. Then, by W3 s and t are compatible. By the information condition for negation $t \not\models B$. So, by the inductive hypothesis $w \nVdash B$ and by the truth condition for negation $w \Vdash \neg B$. □

7. The Logic L_2

The logic L_2 is axiomatized by including

All the theorems of L_1

and

$$A \vee \neg A$$

together with the following rules:

Modus Ponens

$$\frac{\begin{array}{c}A \rightarrow B \\ A\end{array}}{B}$$

Affixing:

$$\frac{\begin{array}{c}A \rightarrow B \\ B \rightarrow C\end{array}}{A \rightarrow C}$$

Adjunction

$$\frac{\begin{array}{c}A\\B\end{array}}{A \wedge B}$$

Contraposition:

$$\frac{A \rightarrow B}{\neg B \rightarrow \neg A}$$

Disjunctive Syllogism:

$$\frac{\begin{array}{c}A \vee B\\ \neg A\end{array}}{B}$$

One might be surprised to see disjunctive syllogism in the list of rules of a *relevant logic*. For it has long been a notorious feature of relevant logics that they eschew disjunctive syllogism. But if we are to adopt a classical view of the truth conditions of negation, then the truths are closed under disjunctive syllogism. So we need it either as a primitive rule or at least as an admissible rule in our logic. Whereas we can show that disjunctive syllogism is admissible in many other relevant logics, I am not sure that this is true of L_1 – this is an open question.

Soundness is easily proven.

8. Completeness of L_2

In order to show that L_2 is complete over this semantics we first formulate a logic L_3. The set of truths at worlds is closed under this logic and we will need it in order to construct our canonical model. The logic is stated as a sequent system. Sequents have the form $\Gamma \vdash \Delta$, where neither side of the turnstile can be empty and both sets are finite. The system has one axiom

$$A \vdash_{L_3} B \text{ if } A \rightarrow B \text{ is a theorem of } L_2$$

and the following rules. A rule of adjunction:

$$A_1, \ldots, A_n \vdash_{L_3} A_1 \wedge \ldots \wedge A_n$$

A modus ponens rule:

$$A \rightarrow B, A \vdash_{L_3} B$$

A transitivity rule for implication:

$$A \rightarrow B, B \rightarrow C \vdash_{L_3} A \rightarrow C$$

And a disjunctive syllogism rule:

$$A \vee B, \neg A \vdash_{L_3} B$$

We also need a weakening rule

$$\frac{\Gamma \vdash_{L_3} \Delta}{\Gamma, \Gamma' \vdash_{L_3} \Delta, \Delta'}, \text{ where } \Gamma', \Delta' \text{ may be empty.}$$

two interchange rules

$$\frac{\Gamma, A, B, \Gamma' \vdash_{L_3} \Delta}{\Gamma, B, A, \Gamma' \vdash_{L_3} \Delta}, \text{ where } \Gamma, \Gamma' \text{ may be empty}$$

$$\frac{\Gamma \vdash_{L_3} \Delta, A, B, \Delta'}{\Gamma \vdash_{L_3} \Delta, B, A, \Delta'}, \text{ where } \Delta, \Delta' \text{ may be empty}$$

and two versions of Gentzen's cut rule:

$$\frac{\Gamma, A \vdash_{L_3} \Delta \quad \Gamma' \vdash_{L_3} A}{\Gamma, \Gamma' \vdash_{L_3} \Delta} \qquad \frac{\Gamma \vdash_{L_3} B_1, ..., B_n, \Delta \quad \vdash_{L_2} (B_1 \vee ... \vee B_n) \to A}{\Gamma \vdash_{L_3} A, \Delta}$$

We also need a rule connecting disjunction with the comma on the right-hand side of the turnstile:

$$A_1 \vee ... \vee A_n \vdash_{L_3} A_1, ..., A_n$$

The first four rules are just the rules of L_2. The use of theories closed under the rules of the logic in order to prove completeness is taken from [15] and [7]. We use some additional rules in this case in order to facilitate the particular version of the completeness proof that we give below.

Lemma 7 The following are provable:

$$\text{(i) } \frac{\Gamma, A, A, \Gamma' \vdash_{L_3} \Delta}{\Gamma, A, \Gamma' \vdash_{L_3} \Delta}.$$

$$\text{(ii) } \frac{\Gamma \vdash_{L_3} \Delta, A, A, \Delta'}{\Gamma \vdash_{L_3} \Delta, A, \Delta'}$$

Proof (i)

$$\dfrac{\dfrac{\Gamma, A, A, \Gamma' \vdash_{L_3} \Delta \quad \overline{A \wedge A \vdash_{L_3} A}^{\,Ax}}{\Gamma, A \wedge A, \Gamma' \vdash_{L_3} \Delta}cut \qquad \overline{A \vdash_{L_3} A \wedge A}^{\,Ax}}{\Gamma, A, \Gamma' \vdash_{L_3} \Delta}cut$$

(ii)

$$\frac{\Gamma \vdash_{L_3} \Delta, A, A, \Delta' \quad \vdash_{L_2} (A \vee A) \to A}{\Gamma \vdash_{L_3} \Delta, A, \Delta'}cut$$

□

Lemma 8 For all sets of wffs Γ and Δ, $\Gamma \vdash_{L3} \Delta$ iff $\wedge\Gamma \vdash_{L3} \vee\Delta$, where $\wedge\Gamma$ is the conjunction of elements of Γ and $\vee\Delta$ is the disjunction of elements of Δ.

Proof Let $\Gamma = \{G_1, ..., G_n\}$ and $\Delta = \{D_1, ..., D_m\}$.

$\Longrightarrow$ From

$$\dfrac{G_1, ..., G_n \vdash_{L3} D_1, ..., D_m \quad \overline{G_1 \wedge ... \wedge G_n \vdash_{L3} G_1}^{Ax} \quad \cdots \quad \overline{G_1 \wedge ... \wedge G_n \vdash_{L3} G_n}^{Ax}}{G_1 \wedge ... \wedge G_n, ..., G_1 \wedge ... \wedge G_n \vdash_{L3} D_1, ..., D_m}\,cut$$

and

$$\vdash_{L_2} D_1 \rightarrow (D_1 \vee ... \vee D_m) \quad \cdots \quad \vdash_{L_2} D_m \rightarrow (D_1 \vee ... \vee D_m)$$

together with the cut rule, we can derive

$$G_1 \wedge ... \wedge G_n, ..., G_1 \wedge ... \wedge G_n \vdash_{L3} D_1 \vee ... \vee D_m, ..., D_1 \vee ... \vee D_m$$

and from this and lemma 7 we obtain

$$G_1 \wedge ... \wedge G_n \vdash_{L3} D_1 \vee ... \vee D_m$$

which is what we want.

$\Longleftarrow$

$$\dfrac{\dfrac{G_1 \wedge ... \wedge G_n \vdash_{L3} D_1 \vee ... \vee D_m \quad G_1, ..., G_n \vdash_{L3} G_1 \wedge ... \wedge G_n}{G_1, ..., G_n \vdash_{L3} D_1 \vee ... \vee D_m}\,cut \qquad D_1 \vee ... \vee D_m \vdash_{L3} D_1, ..., D_m}{G_1, ..., G_n \vdash_{L3} D_1, ..., D_m}\,cut$$

□

We now define a consequence operator C_{L_3} on sets of formulas such that $C_{L_3}(\Gamma) = \{A : \Gamma' \vdash_{L3} A$, where Γ' is a finite subset of $\Gamma\}$. Consider the set L_2 of theorems of L_2. We can show that this set is closed under C_{L_3}, i.e.

$$C_{L_3}(L_2) = L_2.$$

To prove this, we first show the following lemma.

Lemma 9 If $\Gamma \vdash_{L_3} \Delta$, L_2 is closed under $\wedge\Gamma \vdash_{L_3} \vee\Delta$.

Proof For the axiom, adjunction, modus ponens, the disjunction rule, and the interchange rules this is obvious. So we prove it for weakening and the cut rule.

Weakening. Suppose that L_2 is closed under $\wedge\Gamma \vdash_{L_3} \vee\Delta$. This means that we have a truth preserving inference from $\vdash_{L_2} \wedge\Gamma$ to $\vdash_{L_2} \vee\Delta$. We can now show that