高等学校计算机系列教材

图论基础

张海良 苏岐芳 林荣斐 编著

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内 容 简 介

本书除了介绍图论的基本概念和简单结构理论外，主要研究总结了近些年来比较热门的关于图的邻接谱、图的匹配多项式、图的着色、图的色多项式、图的拉普拉斯多项式和图的准拉普拉斯多项式的一些主要成果。为了方便读者进行研究和参考，我们也采编了一些和图论问题相关的线性代数、矩阵论的理论知识。在书的后面给出了基本符号及中英文对照表，以方便读者查阅参考。

本书内容详细，证明简洁，并且在每章后面提供了一些新的研究内容与素材并配以一定量的习题，可作为数学与应用数学专业高年级的专业选修课和图论方向的一年级的硕士研究生课程的教材，也可作为广大图论研究工作者的参考用书。

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序　　言

图论不仅在算法设计、运筹学、网络设计，甚至在生物、化学等学科中都有广泛的应用，而且经过几十年的发展现已成为和离散数学、组合数学、代数学、拓扑等数学分支紧密联系的一个重要数学分支，因此，越来越受学术界的关注，并在实际问题中得到广泛的应用。2006 年我校开设“图论”课程的双语教学，旨在扩大理工科高年级本科生的知识面，提高学生对离散现象的分析能力，经过长时间教学实践并结合自己的研究成果，我们决定编写一本具有特色的双语教材。

本书是在作者几年来开设“图论”双语教学所使用的自编讲义的基础上整理而成的。我们借鉴国内外同类教材的精华，详细介绍了图的基本概念、图的谱、树、连通性、欧拉与哈密尔顿图以及图的匹配多项式、图的着色、色多项式和图的准拉普拉斯谱等近几年的一些主要成果，同时也采编了一些解决图论问题的有关线性代数，矩阵论的理论知识。并且，在每章后面提供了一定量的习题，这些习题对于掌握图的结构性质有很大的帮助，部分有难度的习题标以 ♣，此外，作为对图论知识的应用在每章给出一个 GROUP PROJECT 供读者练习。总体上来讲，本教材内容丰富并且由浅入深，是一本图论入门书，也是对图的匹配多项式理论感兴趣的读者的一本较全面的参考书。但由于作者水平有限，经验不足，难免有不少错误，恳请读者批评指正。

在本书即将出版的时候，感谢我的硕士导师刘儒英、冶成福教授，以及各位师兄、师姐的帮助；特别感谢博士导师束金龙导师对本书的编排及修改意见；最后感谢浙江省科协的资助和清华大学出版社的大力支持!

编　者

2010 年 3 月

Contents

Chapter 1 Basic concepts

内容提要

- 图与简单图的基本概念。
- 图的关联矩阵、邻接矩阵。
- 图的简单变换，包括剖分、并、连图、线图、路树等概念。
- 图的谱的概念和几类简单图的谱的计算。
- 代数中关于矩阵的特征多项式、瑞利商等相关结论。

1.1 Graph and simple graph

Examples of graph are not difficult to find. For one, a road map can be interpreted as a graph, the vertex are the junctions and the edges are the stretch of road from one junctions to another, similarly an electrical circuit may give us a graph in which the vertex are the terminals and the edges are the wires. This graph is different from the lines and triangles, cycles in the geometry, and the painting either. Here, the graph we talk about is present a kind of relation on a set. For more the exact definition readers may read *Discrete Mathematics*. It is customary to represent a graph G by drawing on paper. A graph G is an ordered pair of disjoint sets $(V(G), E(G), \psi_G)$, here the set $V(G)$, $E(G)$ are the vertex and the edge set, ψ_G is the incident functions on $V(G)$ and $E(G)$, that is, if $\psi_G(e) = uv$ we say e *incident* with u and v. The vertices u and v are the end vertices of edge e, in other words, uv is an edge of G, we say u and v are *adjacent*. Two edges are *adjacent* if they have exactly one common end vertex.

We give an example to familiar the reader with the graph and associated terminologies.

$G = (V(G), E(G), \psi_G)$, here $V(G) = (v_1, v_2, v_3, v_4)$, $E(G) = (e_1, e_2, e_3, e_4, e_5)$ and ψ_G is defined as $\psi_G(e_1) = v_1v_2$, $\psi_G(e_2) = v_2v_3$, $\psi_G(e_3) = v_3v_4$, $\psi_G(e_4) = v_4v_1$, $\psi_G(e_5) = v_2v_4$, then this graph is showed in Fig 1.1.

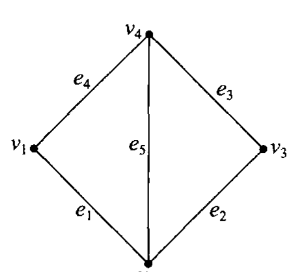

Fig 1.1 a simple graph

If more than one edge incident the same vertex, then we call graph has *multi-edges*, and if the end vertices are same of an edge, then we call the edge is a *loop*. In this book, we only think about the edges that do not have a direction. If a undirected graph without loops and multi-edges we call this graph is a *simple graph.* The number of vertices of a graph G we denoted as *the order* of G; the number of edges of a graph G we denoted as *the size* of G. For convince, we take n as the order of a graph and m the size of a graph in this book. Usually we denote $n = |V(G)|$ and $m = |E(G)|$.

Now, we denote several kind of graphs that has very interesting properties:

If a graph of order n without edge we call it *an empty graph* write as E^n.

A graph of order n and size C_n^2 or $C(n, 2)$ in some books is called a complete graph. This graph is denoted by K^n. In K^n, every two vertices are adjacent, the graph $K^1 = E^1$ is said to be *trivial graph.* A graph G is called a bipartite graph with vertex class V_1, V_2, if, and each edge joins a vertex of V_1 to V_2. K_{mn} is a complete bipartite graph on $n + m$ vertices, in fact, it is a special case of general bipartite graph. The set of vertices adjacent to a vertex $u \in G$ is denoted by $\Gamma(u)$. The *degree* of a vertex u is denoted as $d(u) = |\Gamma(u)|$. *The minimum degree* of a graph is denoted by $\delta(G)$ and δ for short; *the maximum degree* by $\Delta(G)$ and Δ for short, if $\Delta = \delta = k$ we call this graph is k regular.

Example

In fig 1.1 minimal degree $\delta = 2$, maximal degree $\Delta = 3$.

We say that $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In this case, we write $H \subseteq G$. If H contains all edges of G that join two vertices in V' then H is said to be the subgraph induced by V' and is denoted by $G[V']$. If H contains all the vertices that incident with the edges E' then we say H is a sub-graph induced by E' and is denoted by $G[E']$. If $V' = V$ then H is said to be *a spanning subgraph* of G. To example, we give several subgraphs.

Example

A subgraph, an induced subgraph by edges, an induced subgraph by vertex and a spanning subgraph. $V_1 = \{v_1, v_2, v_4\}, E_1 = \{e_1, e_3, e_5\}$ in graph 1.1 $H, G[V1], G[E1]$ are Fig 1.2, Fig 1.3 and Fig 1.4 respectively.

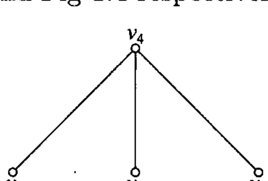

Fig 1.2 subgraph of G

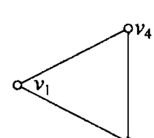

Fig 1.3 vertex induced graph $G[V1]$

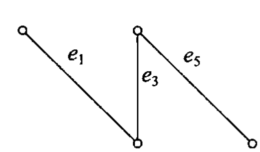

Fig 1.4 edge induced graph $G[E1]$

In order to give readers a wide bases, we give more terminologies, we call a subgraph A is *a clique* if $A \subseteq V(G)$ and every pairs of vertices are adjacent. In the sequent sections, we will know it is a *complete graph.* On the other hand, if non vertices are adjacent in A we call A is *an independent set.* We denote $c(G)$ is the clique number of a graph witch is the maximal number of vertices of all cliques of G, and $\alpha(G)$ is the maximal independent number of a graph.

Similar to the clique and independent set on vertex, we can extent these definitions to the edge set, we call *complete matching* and *an edge covering.* We will study these in chapter 6 for more information.

1.2 Graph operations

Sometimes, we study the properties of a graph by studying another graph get by transforming the original graph. We will study the spectrum of graphs in chapter 1. calculate the number of its *spanning tree* of a graph in chapter 3, calculate the matching number of graphs, study the relation between *the matching polynomial* of a graph and *the characteristic polynomial* of its *path tree* and study the coloring number of a graph in chapter 6. Here, we first present some operations on graphs.

1. deleting an edge or a vertex from G, denoted as G-e or G-v.
2. subdivision an edge or split a vertex.
3. put two graphs together, write as $G_1 \cup G_2$.
4. contracting graph by an edge, delete an edge and put two end vertex together all other vertex and edges keep same.
5. complete product(some books call it joint)$G_1 \bigtriangledown G_2$ of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

In the following chapter, we may study the different polynomials defined on these transformations. In this section, we study the properties on following transformations.

Definition 1.1 *The complement of G, denoted by G^c, is the graph with $V(G) = V(G^c)$ such that two vertices are adjacent in G^c if and only if their are not adjacent in G.*

Obviously, $|V(G)| = |V(G^c)|$ and $|E(G)| + |E(G^c)| = C(n,2) = n(n-1)/2$. We will have more interesting results in the later chapters about the complement graph and with itself.

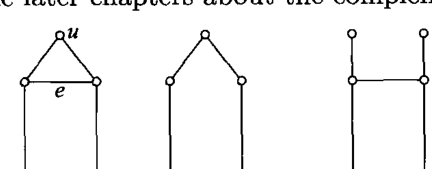

Fig 1.5 graph G and the graph delete e and split from u

Example

We give another example for subdivision and contract by an edge of graph.

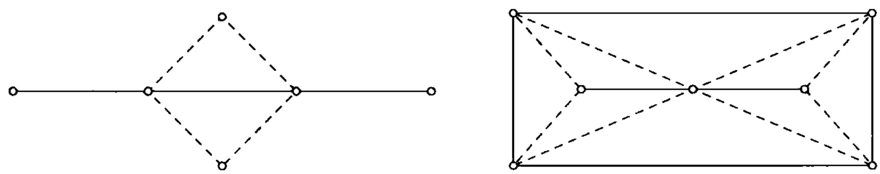

Fig 1.6 a simple graph and its line graph

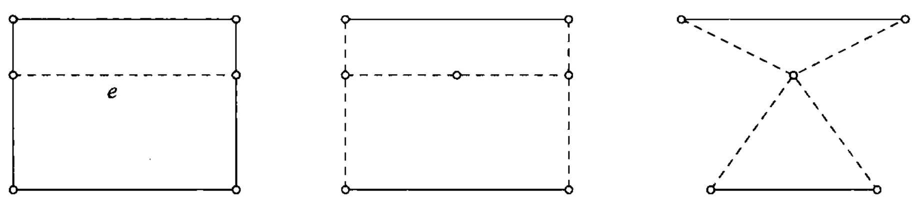

Fig 1.7 the graph obtained by subdivision e and split from e

Example

We construct a new graph from the original one by a simple transformations. Besides these, we also have several special operations these are very important in studying the graph properties. The *line graph* $L(G)$ of an undirected graph G is another graph $L(G)$ that represents the adjacency between edges of G. The line graph is also sometimes called the edge graph, the adjoint graph, the interchange graph, or the derived graph of G. In *Spectra of Graphs*, readers may find more graph transformations like *the direct sun*, *the complete product*, *the product* and *the total graph*, etc.

Definition 1.2 *Given a graph G, its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G; and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are adjacent") in G.*

We give an example for general simple graph G and its line graph $L(G)$. We can easily find an edge of G correspond to an vertex of its line graph $L(G)$. In the graph above edge e correspond to the vertex v of $L(G)$. The degree of v of $L(G)$ satisfies below formula

$$d(v) = d(v_i) + d(v_j) - 2, e = (uv)$$

Obviously, the edge set of $L(G)$ is the edge set of G. The size of G became the order of its line graph. The size of $L(G)$ satisfy following equation.

$$E(L(G)) = \sum_{i=1}^{m} d(v_i)$$

We may give this formula and let the readers proof this as an exercise in the end of this chapter.

$$E(L(G)) = \frac{1}{2}\sum_{i=1}^{m} d(e_i) = \frac{1}{2}(\sum_{i=1}^{n} d_i^2 - 2m)$$

In Fig 1.8, the number of edges of the line graph is 14. If G is *k-regular*, then $L(G)$ is $2k-2$ regular. Besides this, the maximal matching, the independent vertex set, the color number, the connectivity and the character polynomial of G and $L(G)$ are studied by many mathematicians. In section 1.6, we will prove that the eigenvalues of a line graph $L(G)$ are not less than -2. Here, we cite several results about the characteristic polynomials of regular graphs. In the next section, we will give the proof of this theorem.

Theorem 1.1([36]) *If G is a k-regular graph with n vertices and m edges, $P(G,\lambda)$ is the characteristic polynomial of its adjacent matrix, then*

$$\rho(L(G),\lambda) = (\lambda+2)^{m-n}\rho(G,\lambda-k+2)$$

It is interesting that the number of triangles in graph G and its line graph $L(G)$ has below relationship. Let us denote the triangle number of G and $L(G)$ as $\Delta(G)$ and $\Delta(L(G))$, respectively, then

$$\Delta(L(G)) = \Delta(G) + \sum_{i=1}^{n} C(d_i, 3)$$

where d_i is the degree of vertex v_i in G.

We give an example of this formula here.

A semi-regular bipartite graph is a bipartite graph, Let V_1, V_2 be two parts of $V(G)$, $d(v) = s$ if $v \in V_1$; $d(v) = t$ if $v \in V_2$, then Shu jinlong has following theorem:

Theorem 1.2([37]) *$L(G)$ is a connected regular graph if and only if G is a connected graph or semi-regular graph.*

Let $G_1 \bigtriangledown G_2$ (the complete product) denote the joint of G_1 and G_2 obtained by adding all possible edges uv,$u \in G_1$ and $v \in G_2$.

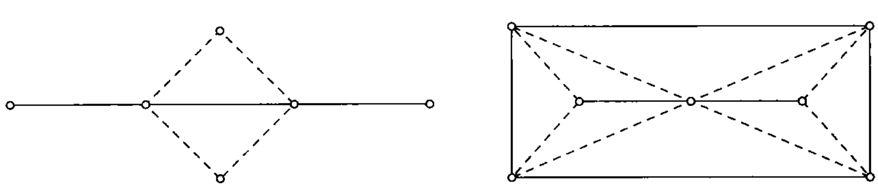

Fig 1.8 The number of triangles in G and its line graph

Theorem 1.3([38]) *Let G_1 and G_2 are k_1-regular graph and k_2-regular graph, respectively, and $k_1 - k_2 = n_1 - n_2$, where n_1 and n_2 are the order of G_1 and G_2, respectively,* *then the quai-Laplacian polynomial of* $G_1 \vee G_2$ *is* $L((G_1 \vee G_2), x) = \frac{(x-n_1-n_2-k_1-k_2)(n_2-n_1-x+2k_1)(n_1-n_2-x+2k_2)}{(x-k_1-k_2-2)(n_2-x+2k_2)(n_1-x+2k_2)} L(G_1, x-n_2)L(G_2, x-n_1)$.

The matching polynomial and characteristic polynomial is connected by the graph and its *path tree.*(see chapter 6) Here, we only give the definition and several simple results.

Definition 1.3 *The path tree* $T(G,u)$ *of* G *take vertex* u *as its root, if:*

1. $V(T(G,u))$*=all the paths start from* u *include* u *itself;*

2. $E(T(G,u))=(P_i,P_j)$ *if one path is contained in the other maximally.*

We give an example in Fig 1.9. Obviously, a path tree of P_n is P_n, Zhang hailiang in [39] gave the path tree of several type of graph and studied the relation of *the largest zero of matching polynomial.* In [4] and in [21], Ma haicheng proved that the largest zero of a graph's matching polynomial equals the largest zero of characteristic polynomial of its path tree. Zhang hailiang gave following properties of several path tree of certain graphs.

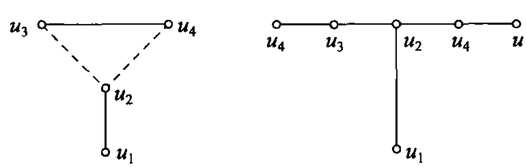

Fig 1.9 A simple graph and its path tree

Theorem 1.4 *1. The path tree of* C_n *is* P_{2n-1}*;*

2. The path tree of $Q(s,t)$ *is* $T_{s-1,s-1,t-1}$, *or* $T_{s-2,t-1,s+t-1}$ *or* $T_{i,t-1,s-1,t-1,j}$, *where* $i+j = s-1$.

Let the n vertices of the given graph G be $v_0, v_1, \ldots, v_n$. The Mycielski graph of G contains G itself as an isomorphic subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and another vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge v_iv_j of G, the Mycielski graph includes two edges, u_iv_j and v_iu_j.

Thus, if G has n vertices and m edges, $My(G)$ has $2n+1$ vertices and $3m+n$ edges. Mycielski's construction is applied to a 5-vertex cycle we get a graph which is called the Grotzsch graph. this graph has 11 vertices and 20 edges. The Grotzsch graph is the smallest triangle-free 4-chromatic graph (Chv á tal 1974). Zhang hai liang in [18] studied the matching polynomial and matching equivalent graphs of this graph.

1.3 Isomorphism

Two graph are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus $G = (V, E)$ is isomorphic to $G' = (V', E')$, we denoted by $G \cong G'$, or simply $G = G'$. If there is a bijection $\theta : V \to V'$ and $\phi : E(G) \to E(G')$ such that $\psi_G(e) = uv$ if and only if $\psi_{G'}(\phi(e)) = \theta(u)\theta(v)$, clearly isomorphic graphs has the same order and size, usually we do not distinguish between isomorphic graphs, unless we consider graphs with a distinguished or labeled set of vertices.

Definition 1.4 *A graph is said to be self-complementary if $G \cong G^c$.*

We have below properties about self-complement graphs.

Theorem 1.5 *A graph is self-complementary then $v \equiv 0, 1 mod(4)$.*

Fig 1.10 gives two isomorphic graph.

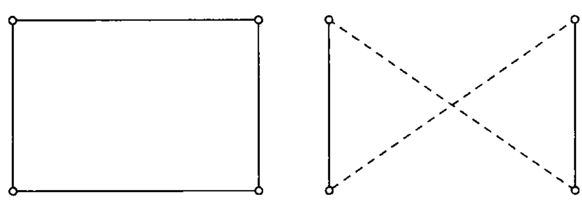

Fig 1.10 two isomorphic graph

1.4 Incident and adjacent matrix

A graph can be represent as a matrix in the computer science. This section we will give matrix theory used in graph theory and build a strong connection between matrix and a graph, first, we start this section with define the adjacency matrix of a graph:

Definition 1.5 *The adjacency matrix $A(G)$ of a simple graph G whose vertex set is $\{v_1, v_2, \ldots, v_n\}$is a square matrix of order n .Whose entry a_{ij} at the place (i, j) is equal to the numbers of edges incident with the v_i, v_j, for simple graph that is 0 or 1. We shall write $A = (a_{ij})$.*

Since this matrix is a symmetric matrix, then it has several properties as below:

Theorem 1.6 *All eigenvalues of A are real numbers.*

Proof. Let λ be an eigenvalue of A and P is the associated eigenvectors of λ. $\bar{\lambda}$ and $\bar{p}$ be the conjugate of λ and p, respectively, then

$$\lambda \bar{p}^t . p = \bar{p}^t (\lambda p) = \bar{p}^t A p$$

since A is symmetric then

$$(\bar{A}\bar{p})^t p = (\bar{\lambda}\bar{p})^t p = \bar{\lambda}\bar{p}^t p$$

$$\lambda\bar{p}^t p = \bar{\lambda}\bar{p}^t p$$

and $\bar{p}^t p \geqslant 0$ so λ is real number.

We can also use the associate law of matrix multiplication and the equation

$$\bar{p}^t Ap = \lambda\bar{p}^t p$$

to proof this theorem.

Theorem 1.7 *For every symmetric matrix A there is an orthogonal matrix P such that* $P^t AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$*,* λ_i *are the eigenvalues of A.*

Proof. According to theorem 1.5, we know that A has an eigenvector v_1, we can assume $\|v_1\| = 1$, and by using the Gram-Schmidt procedure we can find an orthogonal basis $\mathcal{B}$=$\{v_1, v_2, \ldots, v_n\}$ with the eigenvector v_1 as the first element. Let $P_1 = \{v_2, v_3, \ldots, v_n\}$ the $dim(P_1) = n - 1$ Since v_1 is an eigenvector of T_A with the eigenvalue λ_1, then AP_1 also a symmetric transformation, by the introduction, $(v_1, v_2, \cdots, v_n)$ is a orthogonal basis for A. Then by the well know theorem of diagonalizable theorem we finish proving our proof.

The diagonalizable theorem is that if a matrix of order n has n different eigenvectors then this matrix can be diagonalizable.

Definition 1.6 *The incident matrix $M(G)$ of graph G is a $n \times m$ matrix $M = M(G)$, its row is the set of vertices and the columns is the set of edges, and whose entries are given by*

$$m_{ij} = \begin{cases} 1, & if\ v_i\ and\ e_j\ are\ incident \\ 0, & otherwise \end{cases}$$

Example

The adjacency matrix and the incident matrix of graph 1.1 are

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \ and \ \ M(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

respectively.

Definition 1.7 *A matrix is said to be totally unimodular if every minors of order k is* $0, -1, 1$.

In fact, we can easily proofed the incident matrix of a simple graph is totally unimodular by the induction on the order of the minors of the matrix.

Theorem 1.8 (Egervary 1931) *G is bipartite if and only if M is totally unimodular.*

The characteristic polynomial of adjacent matrix of a graph G is defined as the characteristic polynomial of G, write as $\rho(G,\lambda)$, sometimes $\rho(G)$ for short.

Definition 1.8 *The spectrum of a graph G is the set of numbers which are eigenvalues of $A(G)$, together with their multiplicities.If the distinct eigenvalues of $A(G)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{n-1}$,and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{n-1})$,then we shall write:*

$$\mathrm{Spec}G = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{n-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{n-1}) \end{pmatrix}$$

The spectrum of graph 1.1 is

$$\mathrm{Spect}(G) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Suppose that λ is an eigenvalue of A, then since A is real and symmetric, it follows that λ is real,and the multiplicity of λ as a root of the equation $\det(\lambda I - A) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ. The main question arising is this:how much information concerning the structure of G is contained in its spectrum, and how can this information be retrieved from the spectrum?

Theorem 1.9 (Hand-shaking lemma) *For a graph $\sum_{i=1}^{n} d(v_i) = 2\varepsilon$, where ε is size of a graph.*

Proof. Since every edge gives two degrees to a pair of adjacent vertices of a graph, so the sum of degree is twice of the numbers of E of G.

Corollary 1.1 (Hand-shaking theorem) *In any graph the number of odd degree vertices is even.*

Proof. Assume V_1, V_2 represent the odd degree vertices set and the even degree vertices set, respectively, by the Theorem 1.9 we have:

$$\sum_{v\in V_1} d(v) + \sum_{v\in V_2} d(v) = 2\varepsilon$$

The right side of this equation is even, as to the left side $\sum_{v\in V_2} d(v)$ is even , so $\sum_{v\in V_1} d(v)$ must be a even number, but in which every degree of vertex is odd, so in order to grantee the summation is even, the number of vertices must be a even number.

If $V(G) = (v_1, v_2, \ldots, v_n)$, then we say $d(v_1), d(v_2), \ldots, d(v_n)$ is the degree sequence of G. This sequence must have below property.

Theorem 1.10 *For positive integer sequence $d(v_1), d(v_2), \ldots, d(v_n)$ is a degree sequence of a graph if and only if $\sum_{i=1}^{n} d(v_i)$ is even.*

Proof. Necessity is obvious by the theorem 1.9. Now we prove the sufficient condition, for $\sum_{i=1}^{n} d(v_i)$ is even by the hand shaking theorem there must have even number of vertex which has odd degree, then we can construct a graph as below: For the even degree vertex v_i we draw $d(v_i)/2$ loops on v_i; for the odd degree vertices v_j we draw $(d(v_j)-1)/2$ loops and connect every two odd degree vertices with an edge, by the hand shaking theorem there are even number of odd degree vertices, hence, this graph satisfy the condition.

1.5 The spectrum of graph

In this section, we give an expression of characteristic polynomial. We explain the connection of graph structure and the coefficients of characteristic polynomial.Some of results come from the matrix theory directly.

Lemma 1.1([7]) *Let $A = (a_{ij}) \in R^{n\times n}$, then*

$$|\lambda I - A| = \lambda^n + \sum_{k=1}^{k} (-1)^k b_k \lambda^{n-k}$$

where $b_k (k = 1, 2, \ldots, n)$ is the summation over all principle minors of order k, especially,

$$b_1 = a_{11} + a_{22} + \ldots + a_{nn}, b_n = |A|$$

Proof. Let $E = (e_1, e_2, \ldots, e_n), A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$,where e_i and α_i are the $i-th$ columns of unity matrix E and matrix A, respectively, then

$$|\lambda I - A| = |(\lambda e_1 - \alpha_1, \lambda e_2 - \alpha_2, \ldots, \lambda e_n - \alpha_n)|$$

expand this determinant we have:

$$|\lambda I - A| = \lambda^n |(e_1, e_2, \ldots, e_n)| - \lambda_{n-1} \sum |e_1, \ldots, e_{i-1}, \alpha_i, e_{i+1}, \ldots, e_n| + \ldots$$

$$+ (-1)^k \lambda^{n-k} \sum_{1\leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} |(\ldots, a_{i_1}, \ldots, a_{i_k}, \ldots)| + (-1)^n |A|$$

Where

$$|(\ldots, a_{i_1}, \ldots, a_{i_k}, \ldots)|$$

represent the two column of adjacent matrix of A, the others are columns of unity matrix I.

Theorem 1.11([4]) *Let $\rho(G,\lambda)=|\lambda I-A|=\lambda^n+a_1\lambda^{n-1}+\ldots+a_n$ be the characteristic polynomial of an arbitrary undirected multi-graph G. then*

$$a_i^i=\sum_{U\in U_i}(-1)^{p(U)}\cdot 2^{c(U)}\quad (i=1,2,\ldots,n)$$

We call following graphs "elementary figure"

1. the graph K_2,or

2. every graph $C_q(q\geqslant 1)$(loops being included with $q=1$)

call a "basic figure" U every graph all of whose components are elementary figures; let $p(U),c(U)$ be the number of components and the number of circuits contained in U, respectively, and U_i denote the set of all basic figures contained in G having exactly i vertices.

This theorem may be given the following form:

Define the "contribution" b of an elementary figure E by $b(K_2)=-1,b(C_q)=(-1)^{q+1}\cdot 2$ and basic figure U by $b(U)=\prod_{E\in U}b(E)$,then $(-1)^i a_i=\sum_{U\in U_i}b(U)$.

Proof. Let us first consider the absolute term

$$a_n=P_G(0)=(-1)^n|A|=(-1)^n|a_{ik}|$$

According to *Leibniz* definition of the determinants,

$$a_n=\sum_P(-1)^{n+I(P)}a_{1i_1}a_{2i_2}\ldots a_{ni_n}$$

For the sake of simplicity, let us first assume that there are no multiple arcs so that $a_{ik}=0$ or 1 for all i,k. A term

$$S_P=(-1)^{n+I(P)}a_{1i_1}a_{2i_2}\ldots a_{ni_n}$$

of the sum is different from zero if and only if all of the arcs $(1,i_1),(2,i_2),\ldots,(n,i_n)$ are contained in G, P may be represented as a product:

$$P=(1i_1)(\ldots)(\ldots)(\ldots)$$

of disjoint cycles. Evidently, if $S_P\neq 0$, then to each of cycle of P there are corresponds a cycle in G: thus to P, there corresponds a direct sum of (non-intersecting) cycles containing all vertices of G,i.e., a linear directed sub-graph $L\in L_n$. Conversely: to each linear directed subgraph $L\in L_n$ there corresponds a permutation P and a term $S_P=\pm 1$, the sign depending only on the $e(L)$ of even cycles among all cycles of L:

$$S_P=(-1)^{n+e(L)}$$