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Advanced Lectures in Mathematics

Arithmetic Geometry and Automorphic Forms

算术几何与自守形式

Editors: James Cogdell • Jens Funke
Michael Rapoport • Tonghai Yang

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SUANSHU JIHE YU ZISHOU XINGSHI

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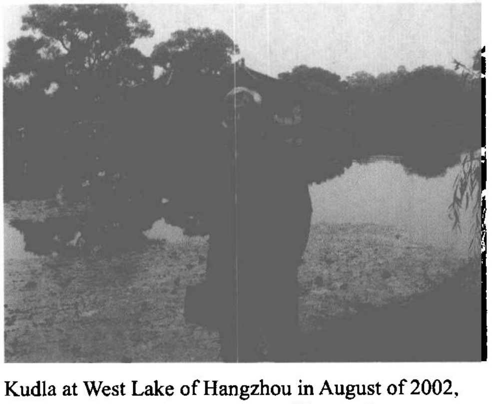
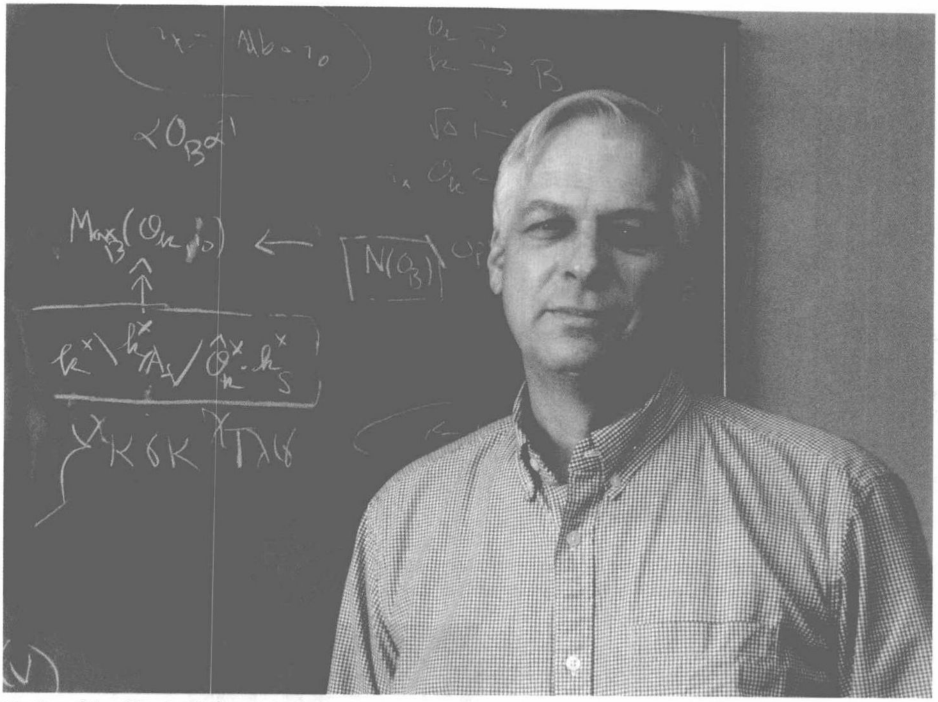
ADVANCED LECTURES IN MATHEMATICS

Kudla's official department photo at the University of Maryland,
taken by Bill Adams in the fall of 1976 when he first arrived at Maryland

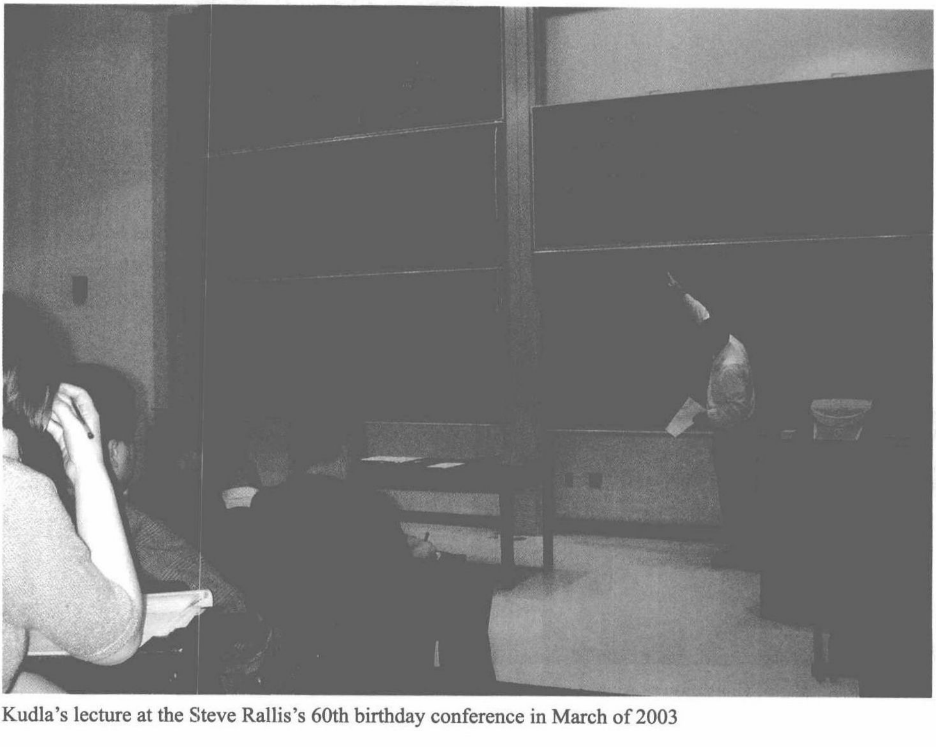
Kudla at his office in University of Toronto



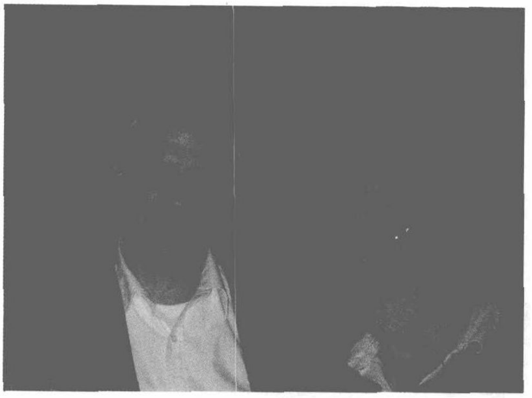
Kudla at West Lake of Hangzhou in August of 2002,
during a number theory ICM satellite conference



Kudla at his office in University of Toronto

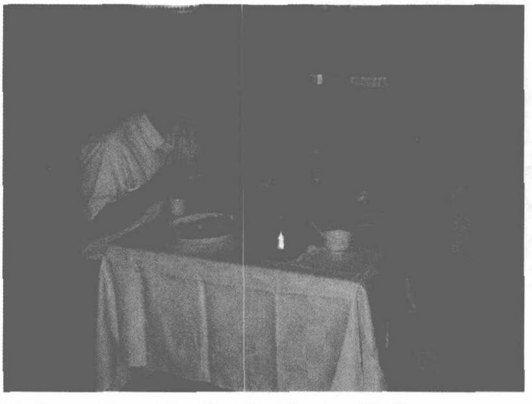


Kudla's lecture at the Steve Rallis's 60th birthday conference in March of 2003



Kudla with Rallis at Rallis's
60th birthday conference in March of 2003

Kudla and Rapoport at Kudla's office in
University of Maryland in May of 2001



Kudla and Yang at lunch at the Morningside Center
of Mathematics, Beijing in Summer of 2002

Four Steves (Steve Gelbart, Steve Miller, Steve Rallis and Steve Kudla) at Tel Aviv University in March of 2001

Kudla at Rallis's 60th birthday conference in March of 2003

Preface

Stephen S. Kudla turned 60 at the end of August of 2010. We had hoped to mark the occasion with a conference highlighting Kudla's contributions to arithmetic, geometry and automorphic forms. However, Kudla declined the honor ... he says he much prefers the back row to the front row. In its stead, we decided to produce a volume in his honor marking the occasion. We invited 17 mathematicians with personal and/or mathematical connections with Kudla to contribute. The result is the volume you are reading now, the proceedings of a conference that didn't happen.

Kudla has contributed significantly to the themes of arithmetic, geometry and automorphic forms. After writing a thesis on "Real points on algebraic varieties defined by quaternion algebras" under the direction of M. Kuga at Stonybrook, Kudla's interest immediately turned to the arithmetic theory of theta series and through it the Weil representation and the arithmetic theory of automorphic forms. His work to date is bookended by his interest in the geometry and arithmetic of the special cycles occurring in the Fourier coefficients of automorphic forms. He began with the geometry of special cycles that arise as the Fourier coefficients of theta series, particularly the papers with John Millson in the 1970–1980's, and recently (at least since his 1997 *Annals* paper) he has turned to arithmetic algebraic geometry and arithmetic intersection numbers of the cycles that occur in the coefficients of Eisenstein series and their derivatives. In between are the series of work with Rallis on Siegel-Weil formulas and its applications, the work with Harris and Gross on special values of L-functions, as well as the formulation of see-saw dual pairs and theta and epsilon dichotomy in the theory of the theta correspondence. His work on the arithmetic Siegel-Weil formula, and particularly the outline and conjectures that he made in his survey article "Special cycles and the derivatives of Eisenstein series" from the 2001 MSRI workshop Heegner Points and Rankin L-Series, has led to what is now called "the Kudla Program".

Kudla has also contributed significantly to the mathematical community as a teacher and mentor. In his thirty-year career at the University of Maryland he directed 15 PhD theses and is continuing to direct theses at the University of Toronto. He has helped to organize many workshops and conferences over his career, from the 1990 Maryland Conference on the Representation Theory of p-adic Groups to the recent 2008 Thematic Program on Arithmetic Geometry, Hyperbolic Geometry, and Related Topics at the Fields Institute. Notable were the regular series of Oberwolfach meetings on automorphic forms and connections with arithmetic and geometry that he organized with Schwermer and several similar programs at the

ESI in Vienna. He has been a regular invited speaker at instructional workshops for graduate students and young researchers. His "Schloß Hirschberg" Notes on the Local Theta Correspondence, from his instructional lectures at the European School on Group Theory in 1996, continue to serve as a welcome introduction to the subject. Kudla is a very conscientious professional, as an editor, as a referee, and as a panel member; he has very solid judgments which the community highly values.

As might be expected, the contributions to this volume echo the mathematical interests of Kudla and provide a current snapshot of developments in these areas. The paper of Funke and Millson presents the current state of affairs in the study of special cycles that originated in the late 1970's, while the contributions of Bruinier, Howard and Yang reflect Kudla's more recent impact on arithmetic algebraic geometry and intersection theory. Gan's paper is a recent development on the Siegel-Weil formula with its origins in Kudla's work with Rallis, whereas the paper of Moeglin takes off from the local Siegel-Weil formula. The paper of Harris, Li and Sun is partially an outgrowth of Harris' earlier work with Kudla on theta dichotomy. The other papers, those by Cogdell, Ginzburg, Grbac, Gross, Jiang, Jorgenson, Kramer, Rallis, Roberts, Schmidt, Soudry, Schwermer, Vigneras and Wallach, all reflect Kudla's general interest in the arithmetic of automorphic forms and their L-functions.

We would like to thank all those who made this volume possible. Firstly we would like to thank all the contributors for allowing us to publish their papers. We also thank the (anonymous) referees who are necessary to guarantee the quality of such a volume. We would also like to thank Peng Li at Higher Education Press and Lizhen Ji, who is on the Editorial Board for the *Advanced Lectures in Mathematics* series, for their help with the production of this volume.

Most importantly, we would like to thank Stephen S. Kudla for his contributions to mathematics and the mathematical community. We wish him continued success in the future.

James Cogdell
Jens Funke
Michael Rapoport
Tonghai Yang

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CM Values of Automorphic Green Functions on Orthogonal Groups over Totally Real Fields

To Stephen S. Kudla, on the occasion of his 60th birthday

Jan H. Bruinier[*†] and Tonghai Yang[‡§]

Abstract

To Generalize the work of Gross-Zagier and Schofer on singular moduli, we study the CM values of regularized theta lifts of harmonic Whittaker forms. We compute the archimedian part of the height pairing of arithmetic special divisors and CM cycles on Shimura varieties associated to quadratic spaces over an arbitrary totally real base field. As a special case, we obtain an explicit formula for the norms of the CM values of those meromorphic modular forms arising as regularized theta lifts of holomorphic Whittaker forms.

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1 Introduction

The values of the classical j-function at complex multiplication points (CM points) are known as singular moduli. They play an important role in number theory, for instance, they are algebraic integers that generate Hilbert class fields of imaginary quadratic fields, and they parametrize elliptic curves with complex multiplication. Gross and Zagier found a beautiful explicit formula for the prime factorization of the norm of (the difference of two) singular moduli [GZ1]. A striking consequence is that the prime factors are small: If E is an elliptic curve with complex multiplication by the maximal order of an imaginary quadratic field of discriminant

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$\Delta < 0$, then any prime dividing the norm of $j(E)$ must divide $\frac{1}{4}(-3\Delta - x^2)$ for some integer x with $|x| < \sqrt{3|\Delta|}$ and $x^2 \equiv \Delta \pmod 4$.

The work of Gross and Zagier has inspired a lot of subsequent research in different directions. For instance, Dorman relaxed some technical assumptions [Do1], and obtained an analogue for rank 2 Drinfeld modules [Do2]. Elkies considered Hauptmodules on certain genus zero compact Shimura curves and computed (partly numerically) some of their CM values [El1], [El2]. Lauter found a conjecture on the primes occuring in the denominators of the CM values of Igusa's j-invariants on the moduli space of principally polarized abelian surfaces. Bounds for the denominators of these invariants have interesting applications for the construction of CM genus 2 curves in cryptography. See [GL1], [GL2], [Ya3] for results in this context.

The j-function is an example of a Borcherds product for the group $\mathrm{SL}_2(\mathbb{Z})$, and the difference $j(z_1) - j(z_2)$ of two j-functions is an example of a Borcherds product for $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. Therefore it is natural to ask whether similar factorization formulas can be obtained for such modular forms in greater generality. An affirmative answer was given by the authors of the present paper for the values of Borcherds products on Hilbert modular surfaces at "big" CM cycles [BY1], and by Schofer for the values of Borcherds products at "small" CM cycles [Scho].

The argument of Schofer is very natural and is based on a method introduced in [Ku5]. It employs the construction of Borcherds products as regularized theta lifts [Bo1], the idea of see-saw dual pairs [Ku1], and the Siegel-Weil formula [We2], [KR]. As an application, Errthum derived factorization formulas for the norms of CM values of Hauptmodules on Shimura curves of genus zero associated to quaternion algebras over $\mathbb{Q}$ [Err]. In particular, he verified the numerical values computed by Elkies for such Shimura curves. The authors of the present paper extended Schofer's approach to compute the CM values of automorphic Green functions [Br1], [BF], yielding a direct link between certain height pairings and central derivatives of Rankin-Selberg L-functions, see [BY2]. As an application, they obtained a new proof of the Gross-Zagier formula for the canonical heights of Heegner points on modular curves [GZ2].

Borcherds products are certain meromorphic modular forms for the orthogonal group of a quadratic space *over* $\mathbb{Q}$ of signature $(n, 2)$, which are constructed as regularized theta lifts of weakly holomorphic elliptic modular forms of weight $1 - n/2$. It was already pointed out by Borcherds in [Bo1] that a direct analogue of his construction for totally real number fields other than $\mathbb{Q}$ cannot exist. This would be a lifting from weakly holomorphic Hilbert modular forms of typically negative weight to meromorphic modular forms for the orthogonal group of a quadratic space over a totally real number field F. But according to the Koecher principle, any weakly holomorphic Hilbert modular form is automatically holomorphic at the cusps as well.

As a solution to this problem the first author proposed in [Br2] to use harmonic "Whittaker forms" (see Section 3.2) as input data for a regularized theta lift over an arbitrary totally real field F. He constructed a map which produces meromorphic modular forms and automorphic Green functions on orthogonal groups and which reduces to the Borcherds lift in the special case where the ground field is $\mathbb{Q}$. For instance, for a Shimura curve associated to a quaternion algebra over a totally real field, one obtains meromorphic modular forms whose whose zeros and poles lie on CM points.

In the present paper we find an explicit formula for the CM values of regularized theta lifts of harmonic Whittaker forms, thereby extending the results of [Scho] and the results of [BY2] on archimedian height pairings to quadratic spaces over arbitrary totally real base fields F. Although the basic idea is the same as in Schofer's work, several new aspects arise. For instance, the regularized integral is *not* defined as an integral over a (truncated) fundamental domain for the Hilbert modular group, but as an integral over a fundamental domain for the subgroup of translations. Moreover, since the Shimura variety we work with is defined over F, the archimedian height pairing consists of several local contributions. For the individual pieces we obtain an integral representation (Theorem 6.3), which cannot be evaluated in finite form. Only if we piece together the local heights at all archimedian places, we obtain a finite expression, which yields (for weakly holomorphic input) the prime factorization of the norm of a CM value (Theorem 7.2).

As an example we compute the CM values of the Hauptmodule on a Shimura curve considered by Elkies associated to a quaternion algebra over the maximal totally real field subfield of $\mathbb{Q}(\zeta_7)$.

We now describe the main results of this paper in more detail. Let F be a totally real number field of degree d and discriminant D. Let $\sigma_1, \dots, \sigma_d$ be the different embeddings of F into $\mathbb{R}$. We write $\mathcal{O}_F$ for the ring of integers and ∂_F for the different of F. Let (V, Q) be a quadratic space over F of dimension $\ell = n + 2$. We assume that V has signature $(n, 2)$ at the place σ_1 of F, and that V is positive definite at all other archimedian places. Throughout we assume that V is anisotropic over F or has Witt rank over F less than n. By the assumption on the signature this is always the case if $d > 1$ or $n > 2$.

We consider the algebraic group $H = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GSpin}(V)$ over $\mathbb{Q}$ given by Weil restriction of scalars. We realize the hermitian symmetric space associated to $H(\mathbb{R})$ as the Grassmannian $\mathbb{D}$ of oriented negative 2-planes in $V \otimes_{F,\sigma_1} \mathbb{R}$. For a compact open subgroup $K \subset H(\hat{\mathbb{Q}})$ we consider the Shimura variety

$$X_{K,1} = H(\mathbb{Q})\backslash(\mathbb{D} \times H(\hat{\mathbb{Q}}))/K.$$

It is a complex quasi-projective variety of dimension n, which has a canonical model over $\sigma_1(F)$, see [Shih]. To simplify the exposition we assume throughout this introduction that K stabilizes an even unimodular $\mathcal{O}_F$-lattice $L \subset V$, that n is even, and that the Shimura variety $X_{K,1}$ is connected. These assumptions are not required in the body of the paper.

It is a special feature of such Shimura varieties that they come with a large supply of algebraic cycles given by quadratic subspaces of V, see [Ku4]. For instance, let $W \subset V$ be a totally positive definite subspace of dimension n defined over F. Then the orthogonal complement $W^\perp$ is definite of dimension 2, and we obtain two points $z_W^\pm \in \mathbb{D}$ given by $W^\perp \otimes_{F,\sigma_1} \mathbb{R}$ with the two possible choices of an orientation. Let H_W be the pointwise stabilizer of W in H, so that $H_W \cong$

$\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSpin}(W^\perp)$. The natural map

$$H_W(\mathbb{Q})\backslash\{z_W^\pm\}\times H_W(\hat{\mathbb{Q}})/(H_W(\hat{\mathbb{Q}})\cap K)\longrightarrow X_{K,1}$$

defines a dimension 0-cycle $Z_1(W)$ on $X_{K,1}$, which is rational over $\sigma_1(F)$, see [Ku4]. Since $\mathrm{GSpin}(W^\perp)$ can be identified with $k^\times$ for a totally imaginary quadratic extension k of F, the cycle $Z_1(W)$ is called the CM cycle associated to W.

A principal part polynomial is a Fourier polynomial of the Form

$$\mathcal{P}=\sum_{\substack{m\in\partial_F^{-1}\\ m\gg 0}} c(m)q^{-m},$$

where $q^m=e^{2\pi i\,\mathrm{tr}(m\tau)}$ for $\tau\in\mathbb{H}^d$. We let $Z_1(\mathcal{P})=\sum_{m\gg0}c(m)Z_1(m)$ be the corresponding linear combination of special divisors $Z_1(m)$ on $X_{K,1}$, see Sections 2.1 and 7.1. The principal part polynomial is called weakly holomorphic, if

$$\sum_{m\gg 0} c(m)b(m)=0,$$

for all Hilbert cusp forms $g=\sum_m b(m)q^m$ of parallel weight $1+n/2$ for $\mathrm{SL}_2(\mathcal{O}_F)$.

It is proved in [Br2], that if $\mathcal{P}$ is a weakly holomorphic principal part polynomial with integral coefficients $c(m)$, then there exists a meromorphic modular form $R_\mathcal{P}(z,h)$ on $X_{K,1}$ with divisor $Z_1(\mathcal{P})$, which is defined over F, and whose logarithm is essentially given by a regularized theta lift of a weakly holomorphic Whittaker form corresponding to $\mathcal{P}$, see also Theorem 7.4. The weight of $R_\mathcal{P}$ is given by the degree of the divisor $Z_1(\mathcal{P})$, or equivalently by the coefficients of a certain Hilbert Eisenstein series of weight $1+n/2$. In this introduction we assume for simplicity that $\deg(Z_1(\mathcal{P}))=0$, so that $R_\mathcal{P}$ has weight zero. One of our main results (see Corollaries 7.5 and 7.6) describes the norm of the CM value

$$R_\mathcal{P}(Z_1(W))=\prod_{[z,h]\in Z_1(W)} R_\mathcal{P}(z,h)\in\sigma_1(F).$$

Theorem 1.1. *Let $\mathcal{P}$ and W be as above and assume that $Z_1(W)$ and $Z_1(\mathcal{P})$ do not intersect on $X_{K,1}$. Then the norm of the CM value $R_\mathcal{P}(Z_1(W))$ is given by*

$$\log|\mathrm{N}_{\sigma_1(F)/\mathbb{Q}}R_\mathcal{P}(Z_1(W))|=\deg(Z_1(W))\left(\log(C)-\frac{1}{4}\,\mathrm{CT}\langle\mathcal{P},\Theta_P\otimes\mathcal{E}_N^{(0)}\rangle\right).$$

Here Θ_P is the (vector valued) theta function of the totally positive definite lattice $P=W\cap L$, and $\mathcal{E}_N^{(0)}$ is the "holomorphic part" of a parallel weight 1 incoherent Hilbert Eisenstein series associated to the lattice $N=W^\perp\cap L$, see Section 4. Moreover, $\mathrm{CT}(\cdot)$ denotes the constant term of a q-series, and $C\in\mathbb{R}_{>0}$ is a normalizing constant which only depends on $\mathcal{P}$ (and not on W).

The constant C can often be determined by specifying the value of $R_\mathcal{P}$ at some special CM point (see Section 8.3). As a consequence, we find that the prime factors of the norm of the CM value are small (see Corollary 7.5), in analogy to the situation for the classical j-function.

Corollary 1.2. *Let $S(N)$ be the set of finite primes $\mathfrak{p}$ of F for which $N_\mathfrak{p}$ is not unimodular, and let $S(\mathcal{P})$ be the set of totally positive $m \in \partial_F^{-1}$ such that $c(m) \neq 0$. We have*

$$\log|\mathrm{N}_{\sigma_1(F)/\mathbb{Q}} R_\mathcal{P}(Z_1(W))| = \deg(Z_1(W))\log(C) + \sum_{p \text{ prime}} \alpha_p \log(p)$$

with coefficients $\alpha_p \in \mathbb{Q}$, and $\alpha_p = 0$ unless there is a prime $\mathfrak{p}$ of F above p which belongs to $S(N)$ or $\mathfrak{p}|(m - Q(\nu))\partial_F$ for some $m \in S(\mathcal{P})$ and $\nu \in P'$ with $m - Q(\nu) \gg 0$ (totally positive). In particular, $\alpha_p = 0$ unless $p \leq \max(M(\mathcal{P}), |N'/N|, D)$, where

$$M(\mathcal{P}) = \max\{\mathrm{N}(m)D;\ m \in S(\mathcal{P})\}.$$

In Section 8, we consider the Shimura curve X associated to the triangle group $G_{2,3,7}$ as an example. It is a genus zero curve with a number of striking properties. For instance, the minimal quotient area of a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is $1/42$, and it is only attained by the triangle group $G_{2,3,7}$. Elkies constructed a generator t of the function field of X and computed its values at certain CM points, see [El1, Section 5.3] and [El2, Section 2.3].

Let $F = \mathbb{Q}(\zeta_7)^+$ be the maximal totally real subfield of the cyclotomic field $\mathbb{Q}(\zeta_7)$, where $\zeta_7 = e^{2\pi i/7}$. Then F is a cubic Galois extension of $\mathbb{Q}$ which is generated by $\alpha = \zeta_7 + \zeta_7^{-1}$. Let B be the (up to isomorphism unique) quaternion algebra over F which is split at the first infinite and all finite places and ramified at the second and the third infinite place. Let $\mathcal{O}_B$ be a fixed maximal order of B, and let $\mathcal{O}_B^1$ be the group of norm 1 elements. The Shimura curve X can be described as the quotient $X = \mathcal{O}_B^1\backslash\mathbb{H}$. It can also be described as a Shimura variety $X_{K,1}$ associated to the three-dimensional quadratic space over F given by the trace zero elements of B with the reduced norm as the quadratic form.

The elliptic fixed points P_3, P_4, P_7 of orders 3, 2, and 7 are rational CM points of discriminant -3, -4, and -7, respectively. Elkies considered the rational function t on X that has a double zero at P_4, a septuple pole at P_7, and that takes the value 1 at P_3. Here we show that this function is a regularized theta lift in the sense of Theorem 7.4. Employing Theorem 1.1, we verify some of Elkies' computations and determine some further CM values of t. The results are summarized in Table 1 at the end of this paper. For instance, for the CM point P_{11} of discriminant -11, we obtain that

$$t(P_{11}) = \pm\frac{7^3 \cdot 11 \cdot 43^2 \cdot 127^2 \cdot 139^2 \cdot 307^2 \cdot 659^2}{3^3 \cdot 13^7 \cdot 83^7},$$

agreeing with the computation of Elkies.

Besides CM values of meromorphic modular forms that arise as liftings of weakly holomorphic Whittaker forms, we compute CM values of automorphic Green functions associated to special divisors (see Section 6 and Theorem 7.2 in Section 7). In this more general situation the CM value is essentially given by the sum of two terms. The first term is the right hand side of the formula of Theorem 1.1. The second term is the central derivative $L'(\xi(\mathcal{P}), W, 0)$ of a Rankin convolution L-function of the theta function Θ_P and a Hilbert cusp form $\xi(\mathcal{P})$ of parallel weight $1+n/2$ associated to the principal part polynomial $\mathcal{P}$. Similarly as in [BY2, Section 5], the first term should be the negative of a finite intersection pairing, so that the second term should be the height pairing of an arithmetic special divisor and a CM cycle. Note that our approach also gives a formula for the integrals of automorphic Green functions analogously to [Ku5], [BK], see Remark 7.8.

This paper is organized as follows: In Section 2 we set up the notation and define the Shimura variety and its special cycles. In Section 3 we recall from [Br2] some facts on Whittaker forms, and in Section 4 we collect some material on Siegel theta functions, Eisenstein series, and the Siegel-Weil formula. In particular we carefully analyze incoherent Hilbert Eisenstein series. Section 5 deals with the regularized theta lift of Whittaker forms, and Section 6 contains a preliminary result on CM values of automorphic Green functions (Theorem 6.3). In Section 7 we put together the computations at the different archimedean places to obtain our main result, Theorem 7.2. It is convenient to formulate it using the concept of incoherent adelic quadratic spaces. Finally, in Section 8 we consider the example studied in [El1, Section 5.3].

It is a pleasure to dedicate this paper to Steve Kudla on the occasion of his 60th birthday. It is clear that this paper is greatly influenced by many beautiful ideas appearing in his work. We thank him for his constant support. Moreover, we thank the referee, N. Elkies, and J. Voight for useful comments.

2 Quadratic spaces and Shimura varieties

We use the same setup and the same notation as in [Br2]. Let F be a totally real number field of degree d over $\mathbb{Q}$. We write $\mathcal{O}_F$ for the ring of integers in F, and write $\partial = \partial_F$ for the different ideal of F. The discriminant of F is denoted by $D = \mathrm{N}(\partial) = \#\mathcal{O}_F/\partial$. Let $\sigma_1, \dots, \sigma_d$ be the different embeddings of F into $\mathbb{R}$. We write $\mathbb{A}_F$ for the ring of adeles of F and $\hat{F}$ for the subring of finite adeles.

Let (V, Q) be a non-degenerate quadratic space of dimension $\ell = n+2$ over F. We put $V_{\sigma_i} = V \otimes_{F,\sigma_i} \mathbb{R}$ and identify $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_i V_{\sigma_i}$. We assume that V has signature

$$((n,2),(n+2,0),\dots,(n+2,0)),$$

that is, V_{σ_1} has signature $(n,2)$ and V_{σ_i} has signature $(n+2,0)$ for $i=2,\dots,d$. Throughout we assume that V is anisotropic over F or has Witt rank over F less than n. By the assumption on the signature this is always the case if $d>1$ or $n>2$.

Let $\mathrm{GSpin}(V)$ be the "general" Spin group of V, that is, the group of all invertible elements g in the even Clifford algebra of V such that $gVg^{-1} = V$. It is an algebraic group over F, and the vector representation gives rise to an exact sequence

$$1 \longrightarrow F^\times \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

We consider the algebraic group $H = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSpin}(V)$ over $\mathbb{Q}$ given by Weil restriction of scalars. So for any $\mathbb{Q}$-algebra R, we have $H(R) = \mathrm{GSpin}(V)(R \otimes_{\mathbb{Q}} F)$.