# Elements of the Representation Theory of Associative Algebras

## 1: Techniques of Representation Theory

IBRAHIM ASSEM
DANIEL SIMSON
ANDRZEJ SKOWROŃSKI

London Mathematical Society
Student Texts 65

# 结合代数表示论基础 第1卷

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## Volume 1 Techniques of Representation Theory

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CAMBRIDGE
UNIVERSITY PRESS

**图书在版编目（CIP）数据**

结合代数表示论基础．第1卷 = Elements of the Representation Theory of Associative Algebras Vol 1：英文/（加）阿瑟姆（Assem，I.）著．—影印本．—北京：世界图书出版公司北京公司，2011.1

ISBN 978-7-5100-2969-1

Ⅰ．①结…　Ⅱ．①阿…　Ⅲ．①结合代数—英文　Ⅳ．①O153.3

中国版本图书馆 CIP 数据核字（2010）第212149号

| | |
|---|---|
| **书　　名：** | Elements of the Representation Theory of Associative Algebras Vol. 1 |
| **作　　者：** | Ibrahim Assem，Daniel Simson，Andrzej Skowroński |
| **中 译 名：** | 结合代数表示论基础 第1卷 |
| **责任编辑：** | 高蓉　刘慧 |
| **出 版 者：** | 世界图书出版公司北京公司 |
| **印 刷 者：** | 三河国英印务有限公司 |
| **发　　行：** | 世界图书出版公司北京公司（北京朝内大街137号　100010） |
| **联系电话：** | 010-64021602，010-64015659 |
| **电子信箱：** | kjb@wpcbj.com.cn |
| **开　　本：** | 24开 |
| **印　　张：** | 19.5 |
| **版　　次：** | 2011年01月 |
| **版权登记：** | 图字：01-2010-3779 |
| **书　　号：** | 978-7-5100-2969-1/O・849　　**定　　价：** 59.00元 |

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*Elements of the Representation Theory of Associative Algebras Vol.* 1, 1st ed. (978-0-521-58631-3) by Ibrahim Assem, Daniel Simson & Andrzej Skowroński first published by Cambridge University Press 2006

This reprint edition for the People's Republic of China is published by arrangement with the Press Syndicate of the University of Cambridge, Cambridge, United Kingdom.

# Introduction

The idea of representing a complex mathematical object by a simpler one is as old as mathematics itself. It is particularly useful in classification problems. For instance, a single linear transformation on a finite dimensional vector space is very adequately characterised by its reduction to its rational or its Jordan canonical form. It is now generally accepted that the representation theory of associative algebras traces its origin to Hamilton's description of the complex numbers by pairs of real numbers. During the 1930s, E. Noether gave to the theory its modern setting by interpreting representations as modules. That allowed the arsenal of techniques developed for the study of semisimple algebras as well as the language and machinery of homological algebra and category theory to be applied to representation theory. Using these, the theory grew rapidly over the past thirty years.

Nowadays, studying the representations of an algebra (which we always assume to be finite dimensional over an algebraically closed field, associative, and with an identity) is understood as involving the classification of the (finitely generated) indecomposable modules over that algebra and the homomorphisms between them. The rapid growth of the theory and the extent of the published original literature became major obstacles for the beginners seeking to make their way into this area.

We are writing this textbook with these considerations in mind: It is therefore primarily addressed to graduate students starting research in the representation theory of algebras. It should also, we hope, be of interest to mathematicians working in other fields.

At the origin of the present developments of the theory is the almost simultaneous introduction and use on the one hand of quiver-theoretical techniques by P. Gabriel and his school and, on the other hand, of the theory of almost split sequences by M. Auslander, I. Reiten, and their students. An essential rôle in the theory is also played by integral quadratic forms. Our approach in this book consists in developing these theories on an equal footing, using their interplay to obtain our main results. Our strong belief is that this combination is best at yielding both concrete illustrations of the concepts and the theorems and an easier computation of actual examples. We have thus taken particular care in introducing in the text as many as possible of the latter and have included a large number of workable exercises.

With these purposes in mind, we divide our material into two parts.

The first volume serves as a general introduction to some of the techniques most commonly used in representation theory. We start by showing in Chapters II and III how one can represent an algebra by a bound quiver and a module by a linear representation of the bound quiver. We then turn in Chapter IV to the Auslander–Reiten theory of almost split sequences, giving various characterisations of these, showing their existence in module categories, and introducing one of our main working tools, the so-called Auslander–Reiten quiver. As a first and easy application of these concepts, we show in Chapter V how one can obtain a complete description of the representation theory of the Nakayama (or generalised uniserial) algebras. We return to theory in Chapter VI, giving an outline of tilting theory, another of our main working tools. A first application of tilting theory is the classification in Chapter VII of those hereditary algebras that are representation–finite (that is, admit only finitely many isomorphism classes of indecomposable modules) by means of the Dynkin diagrams, a result now known as Gabriel's theorem. We then study in Chapter VIII a class of algebras whose representation theory is as "close" as possible to that of hereditary algebras, the class of tilted algebras introduced by D. Happel and C. M. Ringel. Besides the general properties of tilted algebras, we give a very handy criterion, due to S. Liu and A. Skowroński, allowing verification of whether a given algebra is tilted or not. The last chapter in this volume deals with indecomposable modules not lying on an oriented cycle of nonzero nonisomorphisms between indecomposable modules.

Throughout this volume, we essentially use integral quadratic form techniques. We present them here in the spirit of Ringel [144].

The first volume ends with an appendix collecting, for the convenience of the reader, the notations and terminology on categories, functors, and homology and recalling some of the basic facts from category theory and homological algebra needed in the book. In Chapter I, we introduce the notation and terminology we use on algebras and modules, and we briefly recall some of the basic facts from module theory. We introduce the notions of the radical of an algebra and of a module; the notions of semisimple module, projective cover, injective envelope, the socle, and the top of a module, local algebra, primitive idempotent. We also collect basic facts from the module theory of finite dimensional $K$-algebras.

The reader interested mainly in linear representations of quivers and path algebras or familiar with elementary facts on rings and modules can skip Chapter I.

It is our experience that the contents of the first volume of this book can be covered during one (eight-month) course.

The main aim of the second volume, "Representation–Infinite Tilted Algebras," is to study some interesting classes of representation-infinite algebras $A$ and, in particular, to give a fairly complete description of the representation theory of representation-infinite tilted algebras. If the algebra $A$ is tame hereditary, that is, if the underlying graph of its quiver is a Euclidean diagram, we show explicitly how to compute the regular indecomposable modules over $A$, and then over any tame concealed algebra.

It was not possible to be encyclopedic in this work. Therefore many important topics from the theory have been left out. Among the most notable omissions are covering techniques, the use of derived categories and partially ordered sets. Some other aspects of the theory presented here are discussed in the books [21], [31], [76], [98], [84], [151], and especially [144].

Throughout this book, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ mean the sets of natural numbers, integers, rational, real, and complex numbers, and $\mathbb{M}_n(K)$ means the set of all square $n \times n$ matrices over $K$. The cardinality of a set $X$ is denoted by $|X|$.

We take pleasure in thanking all our colleagues and students who helped us with their comments and suggestions. We wish particularly to express our appreciation to Sheila Brenner, Otto Kerner, and Kunio Yamagata for their helpful discussions and suggestions. Particular thanks are due to François Huard and Jessica Lévesque, and to Mrs. Jolanta Szelatyńska for her help in preparing a print-ready copy of the manuscript.

# Contents

# Chapter I

# Algebras and modules

We introduce here the notations and terminology we use on algebras and modules, and we briefly recall some of the basic facts from module theory. Examples of algebras, modules, and functors are presented. We introduce the notions of the (Jacobson) radical of an algebra and of a module; the notions of semisimple module, projective cover, injective envelope, the socle and the top of a module, local algebra, and primitive idempotent. We also collect basic facts from the module theory of finite dimensional $K$-algebras. In this chapter we present complete proofs of most of the results, except for a few classical theorems. In these cases the reader is referred to the following textbooks on this subject [2], [6], [49], [61], [131], and [165].

Throughout, we freely use the basic notation and facts on categories and functors introduced in the Appendix.

The reader interested mainly in linear representations of quivers and path algebras or familiar with elementary facts on rings and modules can skip this chapter and begin with Chapter II.

For the sake of simplicity of presentation, we always suppose that $K$ is an algebraically closed field and that an algebra means a finite dimensional $K$-algebra, unless otherwise specified.

## I.1 Algebras

By a **ring**, we mean a triple $(A, +, \cdot)$ consisting of a set $A$, two binary operations: addition $+ : A \times A \to A$, $(a,b) \mapsto a+b$; multiplication $\cdot : A \times A \to A$, $(a,b) \mapsto ab$, such that $(A, +)$ is an abelian group, with zero element $0 \in A$, and the following conditions are satisfied:

(i) $(ab)c = a(bc)$,

(ii) $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$

for all $a, b, c \in A$. In other words, the multiplication is associative and both left and right distributive over the addition. A ring $A$ is **commutative** if $ab = ba$ for all $a, b \in A$.

We only consider rings such that there is an element $1 \in A$ where $1 \neq 0$ and $1a = a1 = a$ for all $a \in A$. Such an element is unique with respect to this property; we call it the **identity** of the ring $A$. In this case the ring is a quadruple $(A, +, \cdot, 1)$. Throughout, we identify the ring $(A, +, \cdot, 1)$ with its underlying set $A$.

A ring $K$ is a **skew field** (or division ring) if every nonzero element $a$ in $K$ is invertible, that is, there exists $b \in K$ such that $ab = 1$ and $ba = 1$. A skew field $K$ is said to be a **field** if $K$ is commutative.

A field $K$ is **algebraically closed** if any nonconstant polynomial $h(t)$ in one indeterminate $t$ with coefficients in $K$ has a root in $K$.

If $A$ and $B$ are rings, a map $f : A \to B$ is called a **ring homomorphism** if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in A$. If, in addition, $A$ and $B$ are rings with identity elements we assume that the ring homomorphism $f$ preserves the identities, that is, that $f(1) = 1$.

Let $K$ be a field. A $K$-**algebra** is a ring $A$ with an identity element (denoted by 1) such that $A$ has a $K$-vector space structure compatible with the multiplication of the ring, that is, such that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in K$ and all $a, b \in A$. A $K$-algebra $A$ is said to be **finite dimensional** if the dimension $\dim_K A$ of the $K$-vector space $A$ is finite.

A $K$-vector subspace $B$ of a $K$-algebra $A$ is a $K$-**subalgebra** of $A$ if the identity of $A$ belongs to $B$ and $bb' \in B$ for all $b, b' \in B$. A $K$-vector subspace $I$ of a $K$-algebra $A$ is a **right ideal** of $A$ (or **left ideal** of $A$) if $xa \in I$ (or $ax \in I$, respectively) for all $x \in I$ and $a \in A$. A two-sided ideal of $A$ (or simply an ideal of $A$) is a $K$-vector subspace $I$ of $A$ that is both a left ideal and a right ideal of $A$.

It is easy to see that if $I$ is a two-sided ideal of a $K$-algebra $A$, then the quotient $K$-vector space $A/I$ has a unique $K$-algebra structure such that the canonical surjective linear map $\pi : A \to A/I$, $a \mapsto \overline{a} = a + I$, becomes a $K$-algebra homomorphism.

If $I$ is a two-sided ideal of $A$ and $m \geq 1$ is an integer, we denote by $I^m$ the two-sided ideal of $A$ generated by all elements $x_1 x_2 \ldots x_m$, where $x_1, x_2, \ldots, x_m \in I$, that is, $I^m$ consists of all finite sums of elements of the form $x_1 x_2 \ldots x_m$, where $x_1, x_2, \ldots, x_m \in I$. We set $I^0 = A$. The ideal $I$ is said to be **nilpotent** if $I^m = 0$ for some $m \geq 1$.

If $A$ and $B$ are $K$-algebras, then a ring homomorphism $f : A \to B$ is called a $K$-**algebra homomorphism** if $f$ is a $K$-linear map. Two $K$-algebras $A$ and $B$ are called isomorphic if there is a $K$-algebra isomorphism $f : A \to B$, that is, a bijective $K$-algebra homomorphism. In this case we write $A \cong B$.

Throughout this book, $K$ denotes an algebraically closed field.

**1.1. Examples.** (a) The ring $K[t]$ of all polynomials in the indeterminate $t$ with coefficients in $K$ and the ring $K[t_1, \ldots, t_n]$ of all polynomials in commuting indeterminates $t_1, \ldots, t_n$ with coefficients in $K$ are infinite dimensional $K$-algebras.

(b) If $A$ is a $K$-algebra and $n \in \mathbb{N}$, then the set $\mathbb{M}_n(A)$ of all $n \times n$ square matrices with coefficients in $A$ is a $K$-algebra with respect to the usual matrix addition and multiplication. The identity of $\mathbb{M}_n(A)$ is the matrix $E = \mathrm{diag}(1, \ldots, 1) \in \mathbb{M}_n(A)$ with 1 on the main diagonal and zeros elsewhere. In particular $\mathbb{M}_n(K)$ is a $K$-algebra of dimension $n^2$. A $K$-basis of $\mathbb{M}_n(K)$ is the set of matrices $e_{ij}$, $1 \leq i, j \leq n$, where $e_{ij}$ has the coefficient 1 in the position $(i, j)$ and the coefficient 0 elsewhere.

(c) The subset

$$\mathbb{T}_n(K) = \begin{bmatrix} K & 0 & \ldots & 0 \\ K & K & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K & K & \ldots & K \end{bmatrix}$$

of $\mathbb{M}_n(K)$ consisting of all triangular matrices $[a_{ij}]$ in $\mathbb{M}_n(K)$ with zeros over the main diagonal is a $K$-subalgebra of $\mathbb{M}_n(K)$. If $n = 3$ then the subset

$$A = \begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{bmatrix}$$

of $\mathbb{M}_3(K)$ consisting of all lower triangular matrices $\lambda = [\lambda_{ij}] \in \mathbb{T}_3(K)$ with $\lambda_{21} = 0$ is a $K$-subalgebra of $\mathbb{M}_3(K)$, and also of $\mathbb{T}_3(K)$.

(d) Suppose that $(I; \preceq)$ is a finite **poset** (partially ordered set), where $I = \{a_1, \ldots, a_n\}$ and $\preceq$ is a partial order relation on $I$. The subset

$$KI = \{\, \lambda = [\lambda_{ij}] \in \mathbb{M}_n(K);\ \lambda_{st} = 0 \text{ if } a_s \not\preceq a_t \}$$

of $\mathbb{M}_n(K)$ consisting of all matrices $\lambda = [\lambda_{ij}]$ such that $\lambda_{ij} = 0$ if the relation $a_i \preceq a_j$ does not hold in $I$ is a $K$-subalgebra of $\mathbb{M}_n(K)$. We call $KI$ the **incidence algebra** of the poset $(I; \preceq)$ with coefficients in $K$. The matrices $\{e_{ij}\}$ with $a_i \preceq a_j$ form a basis of the $K$-vector space $KI$.

Without loss of generality, we may suppose that $I = \{1, \ldots, n\}$ and that $i \preceq j$ implies that $i \geq j$ in the natural order. This can easily be done by a suitable renumbering of the elements in $I$. In this case, $KI$ takes the form of the lower triangular matrix algebra

$$KI = \begin{bmatrix} K & 0 & \ldots & 0 \\ K_{21} & K & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \ldots & K \end{bmatrix},$$

where $K_{ij} = K$ if $i \preceq j$ and $K_{ij} = 0$ otherwise. For example, if $(I; \preceq)$ is the poset $\{1 \succ 2 \succ 3 \succ \cdots \succ n\}$ then the algebra $KI$ is isomorphic to the algebra $\mathbb{T}_n(K)$ in Example 1.1 (c). If $(I; \preceq)$ is the poset $\{1 \succ 3 \prec 2\}$ then the incidence algebra $KI$ is isomorphic to the five-dimensional algebra $A$ in Example 1.1 (c). If the poset $(I; \preceq)$ is given by $I = \{1,2,3,4\}$ and the relations $\{3 \succ 4 \prec 2 \prec 1 \succ 3\}$ then

$$KI = \begin{bmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & K & K \end{bmatrix}.$$

(e) The associative ring $K\langle t_1, t_2\rangle$ of all polynomials in two noncommuting indeterminates $t_1$ and $t_2$ with coefficients in $K$ is an infinite dimensional $K$-algebra. Note that, if $I$ is the two-sided ideal in $K\langle t_1, t_2\rangle$ generated by the element $t_1t_2 - t_2t_1$, then the $K$-algebra $K\langle t_1, t_2\rangle/I$ is isomorphic to $K[t_1, t_2]$.

(f) Let $(G, \cdot)$ be a finite group with identity element $e$ and let $A$ be a $K$-algebra. The **group algebra** of $G$ with coefficients in $A$ is the $K$-vector space $AG$ consisting of all the formal sums $\sum_{g\in G} g\,\lambda_g$, where $\lambda_g \in A$ and $g \in G$, with the multiplication defined by the formula

$$(\sum_{g\in G} g\,\lambda_g)\cdot(\sum_{h\in G} h\,\mu_h) = \sum_{f=gh\in G} f\lambda_g\mu_h.$$

Then $AG$ is a $K$-algebra of dimension $|G| \cdot \dim_K A$ (here $|G|$ denotes the order of $G$) and the element $e = e1$ is the identity of $AG$. If $A = K$, then the elements $g \in G$ form a basis of $KG$ over $K$.

For example, if $G$ is a cyclic group of order $m$, then $KG \cong K[t]/(t^m - 1)$.

(g) Assume that $A_1$ and $A_2$ are $K$-algebras. The **product of the algebras** $A_1$ and $A_2$ is the algebra $A = A_1 \times A_2$ with the addition and the multiplication given by the formulas $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$, where $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. The identity of $A$ is the element $1 = (1,1) = e_1 + e_2 \in A_1 \times A_2$, where $e_1 = (1,0)$ and $e_2 = (0,1)$.

(h) For any $K$-algebra $A$ we define the **opposite algebra** $A^{\mathrm{op}}$ of $A$ to be the $K$-algebra whose underlying set and vector space structure are just those of $A$, but the multiplication $*$ in $A^{\mathrm{op}}$ is defined by formula $a * b = ba$.

**1.2. Definition.** The (Jacobson) **radical** $\operatorname{rad} A$ of a $K$-algebra $A$ is the intersection of all the maximal right ideals in $A$.

It follows from (1.3) that $\operatorname{rad} A$ is the intersection of all the maximal left ideals in $A$. In particular, $\operatorname{rad} A$ is a two-sided ideal.

**1.3. Lemma.** *Let $A$ be a $K$-algebra and let $a \in A$. The following conditions are equivalent:*

(a) $a \in \operatorname{rad} A$;

(a′) *$a$ belongs to the intersection of all maximal left ideals of $A$;*
(b) *for any $b \in A$, the element $1 - ab$ has a two-sided inverse;*
(b′) *for any $b \in A$, the element $1 - ab$ has a right inverse;*
(c) *for any $b \in A$, the element $1 - ba$ has a two-sided inverse;*
(c′) *for any $b \in A$, the element $1 - ba$ has a left inverse.*

**Proof.** (a) implies (b′). Let $b \in A$ and assume to the contrary that $1 - ab$ has no right inverse. Then there exists a maximal right ideal $I$ of $A$ such that $1 - ab \in I$. Because $a \in \operatorname{rad} A \subseteq I$, $ab \in I$ and $1 \in I$; this is a contradiction. This shows that $1 - ab$ has a right inverse.

(b′) implies (a). Assume to the contrary that $a \notin \operatorname{rad} A$ and let $I$ be a maximal right ideal of $A$ such that $a \notin I$. Then $A = I + aA$ and therefore there exist $x \in I$ and $b \in A$ such that $1 = x + ab$. It follows that $x = 1 - ab \in I$ has no right inverse, contrary to our assumption. The equivalence of (a′) and (c′) can be proved in a similar way.

The equivalence of (b) and (c) is a consequence of the following two simple implications:

(i) If $(1 - cd)x = 1$, then $(1 - dc)(1 + dxc) = 1$.
(ii) If $y(1 - cd) = 1$, then $(1 + dyc)(1 - dc) = 1$.

(b′) implies (b). Fix an element $b \in A$. By (b′), there exists an element $c \in A$ such that $(1 - ab)c = 1$. Hence $c = 1 - a(-bc)$ and, according to (b′), there exists $d \in A$ such that $1 = cd = d + abcd = d + ab$. It follows that $d = 1 - ab$, $c$ is the left inverse of $1 - ab$ and (b) follows. That (c′) implies (c) follows in a similar way. Because (b) implies (b′) and (c) implies (c′) obviously, the lemma is proved. □

**1.4. Corollary.** *Let* $\operatorname{rad} A$ *be the radical of an algebra $A$.*

(a) $\operatorname{rad} A$ *is the intersection of all the maximal left ideals of $A$.*
(b) $\operatorname{rad} A$ *is a two-sided ideal and* $\operatorname{rad}(A/\operatorname{rad} A) = 0$.
(c) *If $I$ is a two-sided nilpotent ideal of $A$, then $I \subseteq \operatorname{rad} A$. If, in addition, the algebra $A/I$ is isomorphic to a product $K \times \cdots \times K$ of copies of $K$, then $I = \operatorname{rad} A$.*

**Proof.** The statements (a) and (b) easily follow from (1.3).

(c) Assume that $I^m = 0$ for some $m > 0$. Let $x \in I$ and let $a$ be an element of $A$. Then $ax \in I$ and therefore $(ax)^r = 0$ for some $r > 0$. It follows that the equality $(1 + ax + (ax)^2 + \cdots + (ax)^{r-1})(1 - ax) = 1$ holds for any element $a \in A$, and, according to (1.3), the element $x$ belongs to $\operatorname{rad} A$. Consequently, $I \subseteq \operatorname{rad} A$. To prove the reverse inclusion, assume that the algebra $A/I$ is isomorphic to a product of copies of $K$. It follows that $\operatorname{rad}(A/I) = 0$. Next, the canonical surjective algebra homomorphism $\pi : A \to A/I$ carries $\operatorname{rad} A$ to $\operatorname{rad}(A/I) = 0$. Indeed, if $a \in \operatorname{rad} A$ and $\pi(b) = b + I$, with $b \in A$, is any element of $A/I$ then, by (1.3), $1 - ba$ is invertible in $A$ and therefore the element $\pi(1-ba) = 1-\pi(b)\pi(a)$ is invertible in $A/I$; thus $\pi(a) \in \operatorname{rad} A/I = 0$, by (1.3). This yields $\operatorname{rad} A \subseteq \operatorname{Ker} \pi = I$ and finishes the proof. □

**1.5. Examples.** (a) Let $s_1, \dots, s_n$ be positive integers and let $A = K[t_1, \dots, t_n]/(t_1^{s_1}, \dots, t_n^{s_n})$. Because the ideal $I = (\bar{t}_1, \dots, \bar{t}_n)$ of $A$ generated by the cosets $\bar{t}_1, \dots, \bar{t}_n$ of the indeterminates $t_1, \dots, t_n$ modulo the ideal $(t_1^{s_1}, \dots, t_n^{s_n})$ is nilpotent, then (1.4) yields $I \subseteq \operatorname{rad} A$. On the other hand, there is a $K$-algebra isomorphism $A/I \cong K$. It follows that $I$ is a maximal ideal and therefore $\operatorname{rad} A = I$.

(b) Let $I$ be a finite poset and $A = KI$ be its incidence $K$-algebra viewed, as in (1.1)(d), as a subalgebra of the full matrix algebra $\mathbb{M}_n(K)$. Then $\operatorname{rad} A$ is the set $U$ of all matrices $\lambda = [\lambda_{ij}] \in KI$ with $\lambda_{ii} = 0$ for $i = 1, 2, \dots, n$, and the algebra $A/\operatorname{rad} A$ is isomorphic to the product $K \times \cdots \times K$ of $n$ copies of $K$. Indeed, we note that the set $U$ is clearly a two-sided ideal of $KI$, it is easily seen that $U^n = 0$ and finally the algebra $A/U$ is isomorphic to the product of $n$ copies of $K$, thus (1.4)(c) applies.

(c) By applying the preceding arguments, one also shows that the radical $\operatorname{rad} A$ of the lower triangular matrix algebra $A = \mathbb{T}_n(K)$ of (1.1)(c) consists of all matrices in $A$ with zeros on the main diagonal. It follows that $(\operatorname{rad} A)^n = 0$.

In the study of modules over finite dimensional $K$-algebras over an algebraically closed field $K$ an important rôle is played by the following theorem, known as the Wedderburn–Malcev theorem.

**1.6. Theorem.** *Let $A$ be a finite dimensional $K$-algebra. If the field $K$ is algebraically closed, then there exists a $K$-subalgebra $B$ of $A$ such that there is a $K$-vector space decomposition $A = B \oplus \operatorname{rad} A$ and the restriction of the canonical surjective algebra homomorphism $\pi : A \to A/\operatorname{rad} A$ to $B$ is a $K$-algebra isomorphism.*

**Proof.** See [61, section VI.2] and [131, section 11.6]. □

## I.2 Modules

**2.1. Definition.** Let $A$ be a $K$-algebra. A **right $A$-module** (or a right module over $A$) is a pair $(M, \cdot)$, where $M$ is a $K$-vector space and $\cdot : M \times A \to M$, $(m, a) \mapsto ma$, is a binary operation satisfying the following conditions:

(a) $(x + y)a = xa + ya$;
(b) $x(a + b) = xa + xb$;
(c) $x(ab) = (xa)b$;
(d) $x1 = x$;

(e) $(x\lambda)a = x(a\lambda) = (xa)\lambda$
for all $x, y \in M$, $a, b \in A$ and $\lambda \in K$.

The definition of a left $A$-module is analogous. Throughout, we write $M$ or $M_A$ instead of $(M, \cdot)$. We write $A_A$ and ${}_AA$ whenever we view the algebra $A$ as a right or left $A$-module, respectively.

A module $M$ is said to be **finite dimensional** if the dimension $\dim_K M$ of the underlying $K$-vector space of $M$ is finite.

A $K$-subspace $M'$ of a right $A$-module $M$ is said to be an **$A$-submodule** of $M$ if $ma \in M'$ for all $m \in M'$ and all $a \in A$. In this case the $K$-vector space $M/M'$ has a natural $A$-module structure such that the canonical epimorphism $\pi : M \to M/M'$ is an $A$-module homomorphism.

Let $M$ be a right $A$-module and let $I$ be a right ideal of $A$. It is easy to see that the set $MI$ consisting of all sums $m_1a_1 + \ldots + m_sa_s$, where $s \geq 1$, $m_1, \ldots, m_s \in M$ and $a_1, \ldots, a_s \in I$, is a submodule of $M$.

A right $A$-module $M$ is said to be **generated** by the elements $m_1, \ldots, m_s$ of $M$ if any element $m \in M$ has the form $m = m_1a_1 + \cdots + m_sa_s$ for some $a_1, \ldots, a_s \in A$. In this case, we write $M = m_1A + \ldots + m_sA$. A module $M$ is said to be **finitely generated** if it is generated by a finite subset of elements of $M$.

Let $M_1, \ldots, M_s$ be submodules of a right $A$-module $M$. We define $M_1 + \ldots + M_s$ to be the submodule of $M$ consisting of all sums $m_1 + \cdots + m_s$, where $m_1 \in M_1, \cdots, m_s \in M_s$, and we call it the submodule generated by $M_1, \ldots, M_s$, or the sum of $M_1, \ldots, M_s$.

Note that a right module $M$ over a finite dimensional $K$-algebra $A$ is finitely generated if and only if $M$ is finite dimensional. Indeed, if $x_1, \ldots, x_m$ is a $K$-basis of $M$, then it is obviously a set of $A$-generators of $M$. Conversely, if the $A$-module $M$ is generated by the elements $m_1, \ldots, m_n$ over $A$ and $\xi_1, \ldots \xi_s$ is a $K$-basis of $A$ then the set $\{m_j\xi_i; j = 1, \ldots, n, i = 1, \ldots, s\}$ generates the $K$-vector space $M$.

Throughout, we frequently use the following lemma, known as Nakayama's lemma.

**2.2. Lemma.** *Let $A$ be a $K$-algebra, $M$ be a finitely generated right $A$-module, and $I \subseteq \operatorname{rad} A$ be a two-sided ideal of $A$. If $MI = M$, then $M = 0$.*

**Proof.** Suppose that $M = MI$ and $M = m_1A + \cdots + m_sA$, that is, $M$ is generated by the elements $m_1, \ldots, m_s$. We proceed by induction on $s$. If $s = 1$, then the equality $m_1A = m_1I$ implies that $m_1 = m_1x_1$ for some $x_1 \in I$. Hence $m_1(1 - x_1) = 0$ and therefore $m_1 = 0$, because $1 - x_1$ is invertible. Consequently $M = 0$, as required.

Assume that $s \geq 2$. The equality $M = MI$ implies that there are