Mathematics Monograph Series **23**

Finsler Geometry

An Approach via Randers Spaces

Xinyue Cheng and Zhongmin Shen

(Finsler几何——从Randers空间进入)

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SCIENCE PRESS
Beijing

Responsible Editor: Liping Wang

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-031765-0

Preface

In 1854, B. Riemann attempted to study metrics in general forms and introduced the notion of curvature for a special class of metrics—Riemann metrics. This infinitesimal quantity faithfully reveals the local geometry of Riemann metrics and becomes the central concept of Riemannian geometry. In 1918, P. Finsler studied the variational problem in manifolds with a generalized Riemann metric. Thereafter, such metrics are called Finsler metrics. Later, L. Berwald extended the notion of Riemann curvature to Finsler metrics by introducing the so-called Berwald connection. Berwald also introduced some non-Riemannian quantities via his connection. Since then, Finsler geometry has been developed gradually. However, Finsler geometry is much more complicated than Riemannian geometry. In order to grasp the geometric meaning of various quantities in Finsler geometry, one can begin the study on the most simple non-Riemannian Finsler metrics——Randers metrics.

Randers metrics are natural and important Finsler metrics which are defined as the sum of a Riemann metric and a 1-form. They were derived from the research on the general relativity and have been widely applied in many areas of natural science, including biology, physics and psychology, etc. In particular, Randers metrics can be naturally deduced as the solution of Zermelo navigation problem. Randers metrics are computable. Thus people can do in-depth computation of various geometric quantities, hence can understand the geometric properties of such metrics. More importantly, Randers metrics have very rich non-Riemann curvature properties. The study of Randers metrics will lead to a better understanding on Finsler metrics.

This book is a monograph about Randers spaces which is written based on the authors' many years of research in studying geometry of Randers spaces. The main purpose of this book is to introduce the basic concepts and important progress in Finsler geometry via Randers metrics, meanwhile to provide many important and interesting examples with special curvature properties. This book contains many important results about Randers metrics obtained in the past decade.

We authors sincerely thank the following institutions for their valuable comments and great help: David Bao, Weiping Zhang, Yibing Shen, Sandor Bácsó, Tran Quoc Binh, Zizhou Tang, Xiaohuan Mo, etc. The authors are very grateful to S. S. Chern for his great encouragement and support before he passed away. The authors would like to thank Yanfang Tian, Jinglong Jiang, Feng Mu, Mingao Yuan, Weiwei Zeng, Chunmei Qin, Lili Zhao, Esra Sengelen and Akbar Tayebi for carefully reading through the first version of this book and correcting several mistakes.

Finally, we want to thank Ms. Liping Wang for her editorial guidance and her much help for our book.

This book is written based on the authors' research supported by National Natural Science Foundation of China (10171117, 10371138, 10671214, 10971239) and Science Foundation (America Natural) (DMS-0810159).

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Contents

Chapter 1

Randers Spaces

Randers spaces are finite dimensional vector spaces equipped with a Randers norm. Euclidean norm is the most special Randers norm. Roughly speaking, a Randers norm is a shifted Euclidean norm. If the unit sphere of a Euclidean norm is called a round sphere, then the unit sphere of a Randers norm is an ellipsoid. Randers norms are special Minkowski norms whose unit sphere is a strong convex hypersurface. More precise definition is given as follows.

Let V be a finite dimensional vector space. A *Minkowski norm* on V is a function $F : V \to [0, +\infty)$ which has the following properties:

(a) F is C^∞ on $V\backslash\{0\}$;

(b) F is positively homogeneous of degree one, that is, $F(\lambda y) = \lambda F(y)$ for any $y \in V$ and $\lambda > 0$;

(c) for any $y \in V\backslash\{0\}$, the *fundamental form* g_y on V is an inner product, where

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s=t=0}.$$

The pair (V, F) is called a *Minkowski space.* A Minkowski norm F is said to be *reversible* if $F(-y) = F(y)$ for $y \in V$.

Let (V, F) be an n-dimensional Minkowski space and $\{e_i\}_{i=1}^n$ be a basis for V. View $F(y) = F(y^i e_i)$ as a function of $(y^i) \in \mathbb{R}^n$. Put

$$g_{ij}(y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(y). \tag{1.1}$$

Then

$$g_y(u,v) = g_{ij}(y)u^i v^j, \quad u = u^i e_i, \quad v = v^j e_j. \tag{1.2}$$

It follows from the homogeneity of F that

$$F(y) = \sqrt{g_{ij}(y)y^i y^j}, \quad y = y^i e_i.$$

Let

$$h_{ij}(y) := F(y)F_{y^i y^j}(y) = g_{ij}(y) - F_{y^i}(y)F_{y^j}(y)$$

and

$$h_y(u,v) := h_{ij}(y)u^iv^j, \quad u = u^ie_i, \quad v = v^je_j.$$

We have

$$h_y(u,v) = g_y(u,v) - F(y)^{-2}g_y(y,u)g_y(y,v).$$

Observe that

$$h_y(u,u) \geqslant g_y(u,u) - F(y)^{-2}g_y(y,y)g_y(u,u) = 0.$$

Thus $h_y(u,u) \geqslant 0$ and equality holds if and only if $u = \lambda y$ for some λ. h_y is called the *angular form.*

1.1 Randers Norms

First, we consider Euclidean norms. Let $\mathbb{R}^n$ denote the standard vector space of dimension n. The *standard Euclidean norm* $|\cdot|$ on $\mathbb{R}^n$ is defined by

$$|y| := \sqrt{\sum_{i=1}^n |y^i|^2}, \quad y = (y^i) \in \mathbb{R}^n.$$

Clearly, it is a special Minkowski norm. The pair $(\mathbb{R}^n, |\cdot|)$ is called the *standard Euclidean space.* More general, let $\langle\ ,\ \rangle$ be an inner product on a vector space V with a basis $\{e_i\}_{i=1}^n$. Define

$$\alpha(y) := \sqrt{\langle y,y\rangle} = \sqrt{a_{ij}y^iy^j}, \quad y = y^ie_i,$$

where $a_{ij} := \langle e_i, e_j\rangle$. Clearly, α is a Minkowski norm with $g_y(u,v) = \langle u,v\rangle$ independent of $y \in V\backslash\{0\}$. α is called a *Euclidean norm* and the pair (V,α) is called a *Euclidean space.* It is well-known that all Euclidean spaces with the same dimension are linearly isometric to each other.

Now, we introduce Randers norms. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Euclidean norm on a vector space V and $\beta = b_iy^i$ be a linear functional on V. Let

$$F(y) := \alpha(y) + \beta(y). \tag{1.3}$$

It is easy to verify that for any pair of vectors $u, v \in V$,

$$\begin{aligned} F(u+v) &= \alpha(u+v) + \beta(u+v) \\ &\leqslant \alpha(u) + \alpha(v) + \beta(u) + \beta(v) \\ &= F(u) + F(v). \end{aligned}$$

Let $\bar{g}(u,v)$ denote the inner product determined by α. Then we have the following inequality for any pair of vectors $y \neq 0, u \in V$:

$$\bar{g}(y,u) \leqslant \alpha(y)\alpha(u),$$

and equality holds if and only if $u = \lambda y$ for some λ. For $F(y) = \alpha(y) + \beta(y)$, we have

$$\begin{aligned} g_y(y,u) &= \frac{1}{2}(\alpha+\beta)^2_{y^i y^j}(y)y^i u^j \\ &= F(y)\Big[\beta(u) + \frac{\bar{g}(y,u)}{\alpha(y)}\Big] \\ &\leqslant F(y)\Big[\beta(u) + \alpha(u)\Big] = F(y)F(u), \end{aligned} \tag{1.4}$$

and equality holds if and only if $u = \lambda y$ for some λ.

Let $b := \|\beta\|_\alpha$ denote the length of β with respect to α. It is given by

$$b = \sqrt{a^{ij}b_i b_j},$$

where $(a^{ij}) = (a_{ij})^{-1}$. To find a condition on β under which $F = \alpha + \beta$ is a Minkowski norm, we compute $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and obtain the following:

$$g_{ij} = \frac{F}{\alpha}\Big[a_{ij} + \frac{\alpha}{F}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big) - \frac{y_i}{\alpha}\frac{y_j}{\alpha}\Big], \tag{1.5}$$

where $y_i := a_{ij}y^j$.

In order to find the formulas for $\det(g_{ij})$ and $(g^{ij}) := (g_{ij})^{-1}$, we need the following lemma:

Lemma 1.1.1 ([BaChSh]) *Let (g_{ij}) and (m_{ij}) be two $n \times n$ symmetric matrices and $c = (c_i)$ be an n-dimensional vector, which satisfy*

$$g_{ij} = m_{ij} + \lambda c_i c_j,$$

where λ is a constant. Then

$$\det(g_{ij}) = (1 + \lambda c^2)\det(m_{ij}). \tag{1.6}$$

Assume that (m_{ij}) is positive definite with $(m_{ij})^{-1} = (m^{ij})$ and $1 + \lambda c^2 \neq 0$. Then (g_{ij}) is invertible and $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$g^{ij} = m^{ij} - \frac{\lambda}{1+\lambda c^2}c^i c^j, \tag{1.7}$$

where $c^i = m^{ij}c_j$ and $c = \sqrt{m^{ij}c_i c_j}$.

Now, let

$$m_{ij} := a_{ij} + \frac{\alpha}{F}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big). \tag{1.8}$$

(m_{ij}) is a positive definite matrix. Letting $\lambda := \alpha/F$ and $c_i := b_i + y_i/\alpha$ in (1.6), we get by (1.8) that

$$\begin{aligned}\det(m_{ij}) &= \det(a_{ij})\Big[1 + \frac{\alpha}{F}a^{ij}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big)\Big]\\ &= \det(a_{ij})\Big[1 + \frac{\alpha}{F}\Big(1 + 2\frac{\beta}{\alpha} + b^2\Big)\Big]\\ &= \det(a_{ij})\frac{2F + \beta + \alpha b^2}{F}.\end{aligned}$$

By (1.7), we get

$$\begin{aligned}m^{ij} &= a^{ij} - \frac{\alpha/F}{2 + (\beta + \alpha b^2)/F}\Big(\frac{y^i}{\alpha} + b^i\Big)\Big(\frac{y^j}{\alpha} + b^j\Big)\\ &= a^{ij} - \frac{\alpha}{2F + \beta + \alpha b^2}\Big(\frac{y^i}{\alpha} + b^i\Big)\Big(\frac{y^j}{\alpha} + b^j\Big),\end{aligned}$$

where $(m^{ij}) := (m_{ij})^{-1}$. Further, let $\lambda := -1$ and $c_i := y_i/\alpha$. We have

$$\begin{aligned}1 + \lambda m^{ij}c_i c_j &= 1 - \Big[a^{ij} - \frac{\alpha}{2F + \beta + \alpha b^2}\Big(\frac{y^i}{\alpha} + b^i\Big)\Big(\frac{y^j}{\alpha} + b^j\Big)\Big]\frac{y_i}{\alpha}\frac{y_j}{\alpha}\\ &= \frac{F^2}{\alpha(2F + \beta + \alpha b^2)}.\end{aligned}$$

By (1.5) $\sim$ (1.7), we obtain the following formulas:

$$\det(g_{ij}) = \Big(\frac{F}{\alpha}\Big)^{n+1}\det(a_{ij}), \tag{1.9}$$

$$g^{ij} = \frac{\alpha}{F}a^{ij} - \frac{\alpha}{F^2}(b^i y^j + b^j y^i) + \frac{b^2\alpha + \beta}{F^3}y^i y^j. \tag{1.10}$$

From the definition of the angular metric tensor, we have the following formula for Randers metrics:

$$h_{ij} = FF_{y^iy^j} = (\alpha + \beta)\cdot\frac{1}{\alpha}(a_{ij} - \alpha_{y^i}\alpha_{y^j}) = \frac{\alpha + \beta}{\alpha}\Big(a_{ij} - \frac{y_i y_j}{\alpha^2}\Big). \tag{1.11}$$

Clearly, $F(y) > 0$ for all $y \neq 0$ if and only if $b < 1$. Further, (g_{ij}) is positive definite if and only if $b < 1$ ([BaChSh],[BaRo],[Ma1]). In fact, when $F(y) > 0$,

$$F_\varepsilon := \alpha + \varepsilon\beta > 0$$

for any $0 \leqslant \varepsilon \leqslant 1$. Let $g^\varepsilon_{ij} := \frac{1}{2}[F^2_\varepsilon]_{y^iy^j}$. By (1.9), we have

$$\det(g^\varepsilon_{ij}) = \Big(\frac{F_\varepsilon}{\alpha}\Big)^{n+1}\det(a_{ij}) > 0.$$

Let $\lambda_1(\varepsilon) \leqslant \lambda_2(\varepsilon) \leqslant \cdots \leqslant \lambda_{n-1}(\varepsilon) \leqslant \lambda_n(\varepsilon)$ denote the eigenvalues of (g^{ε}_{ij}). The multiplicity of the eigenvalues might change as ε changes, but each eigenvalue $\lambda_i(\varepsilon)$ depends on ε continuously. Thus, from $\lambda_i(0) > 0$ and $\det(g^{\varepsilon}_{ij}) > 0$ for $0 \leqslant \varepsilon \leqslant 1$, we have $\lambda_i(1) > 0$. Namely, $(g_{ij}) = (g^1_{ij})$ is positive definite.

A Minkowski norm in the form (1.3) is called the *Randers norm.* Randers norms were first introduced by physicist G. Randers in 1941 from the standpoint of general relativity ([Ra]).

1.2 Distortion and Volume Form

Let (V, F) be an n-dimensional Minkowski space and $\{e_i\}_{i=1}^n$ be an arbitrary basis on V, and $\{\theta^i\}_{i=1}^n$ be the basis for V^* dual to $\{e_i\}_{i=1}^n$. Put

$$\sigma_F := \frac{\mathrm{Vol}(B^n(1))}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(y^i e_i) < 1\}},$$

where Vol denotes the Euclidean volume and $\mathrm{Vol}(B^n(1))$ denotes the Euclidean volume of the unit ball in $\mathbb{R}^n$. Put

$$dV_F := \sigma_F \theta^1 \wedge \cdots \wedge \theta^n.$$

It is clear that dV_F is well-defined, namely, independent of the choice of a particular basis. dV_F is called the *volume form* of F on V. Put

$$\tau(y) := \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}. \tag{1.12}$$

It is easy to verify that $\tau(y)$ is well-defined. τ is called the *distortion* of F.

If $F = \sqrt{a_{ij} y^i y^j}$ is a Euclidean norm, then

$$\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n | F(y^i e_i) < 1\Big\} = \frac{\mathrm{Vol}(B^n(1))}{\sqrt{\det(a_{ij})}}.$$

Thus

$$\sigma_F = \sqrt{\det(a_{ij})}.$$

Note that $g_{ij}(y) = a_{ij}$. We have

$$\tau(y) = \ln \frac{\sqrt{\det(a_{ij})}}{\sigma_F} = 0.$$

Consider a Randers norm $F = \alpha + \beta$ on an n-dimensional vector space V with $b := \|\beta\|_\alpha < 1$. Let $dV_F = \sigma_F \theta^1 \wedge \cdots \wedge \theta^n$ and $dV_\alpha = \sigma_\alpha \theta^1 \wedge \cdots \wedge \theta^n$ denote the volume forms of F and α, respectively. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for (V, α). Thus $\sigma_\alpha = \sqrt{\det(a_{ij})} = 1$. We may assume that $\beta = b y^1$. Then

$$\Omega := \{(y^i) \in \mathbb{R}^n | F(y^i e_i) < 1\}$$

is a convex body in $\mathbb{R}^n$ and $\sigma_F = \mathrm{Vol}(B^n(1))/\mathrm{Vol}(\Omega)$. Ω is given by

$$(1-b^2)^2\Big(y^1+\frac{b}{1-b^2}\Big)^2+(1-b^2)\sum_{a=2}^{n}(y^a)^2<1.$$

Consider the following coordinate transformation, $\psi:(y^i)\to(u^i)$:

$$u^1=(1-b^2)\Big(y^1+\frac{b}{1-b^2}\Big),\quad u^a=\sqrt{1-b^2}\,y^a. \tag{1.13}$$

ψ sends Ω onto the unit ball $B^n(1)$ and the Jacobian of $\psi:(y^i)\to(u^i)$ is given by

$$(1-b^2)^{\frac{n+1}{2}}.$$

Then

$$\begin{aligned}\mathrm{Vol}(B^n(1)) &= \int_{B^n(1)} du^1\cdots du^n = \int_\Omega (1-b^2)^{\frac{n+1}{2}}\,dy^1\cdots dy^n\\ &= (1-b^2)^{\frac{n+1}{2}}\mathrm{Vol}(\Omega).\end{aligned}$$

Then

$$\sigma_F=\frac{\mathrm{Vol}(B^n(1))}{\mathrm{Vol}(\Omega)}=(1-b^2)^{\frac{n+1}{2}}.$$

Thus for a general base $\{e_i\}_{i=1}^n$, we have

$$\sigma_F=(1-b^2)^{\frac{n+1}{2}}\sigma_\alpha,\quad \sigma_\alpha=\sqrt{\det(a_{ij})}.$$

Therefore

$$dV_F=(1-b^2)^{\frac{n+1}{2}}\,dV_\alpha. \tag{1.14}$$

Note that

$$dV_F\leqslant dV_\alpha.$$

The equality holds if and only if $b=0$ (F is a Euclidean norm).

By (1.9), the distortion of F is given by

$$\tau=(n+1)\ln\sqrt{\frac{1+\beta/\alpha}{1-b^2}}. \tag{1.15}$$

Since $|\beta/\alpha|\leqslant b$, we get

$$(n+1)\ln\frac{1}{\sqrt{1+b}}\leqslant\tau\leqslant(n+1)\ln\frac{1}{\sqrt{1-b}}.$$

1.3 Cartan Torsion

Let (V, F) be an n-dimensional Minkowski space. For a vector $y \in V\backslash\{0\}$, let

$$\mathbf{C}_y(u, v, w) := \frac{1}{4}\frac{\partial^3}{\partial s \partial t \partial r}\left[F^2(y + su + tv + rw)\right]\Big|_{s=t=r=0},$$

where $u, v, w \in V$. It is easy to see that $\mathbf{C}_y$ is a symmetric trilinear form on V and the homogeneity of F implies that

$$\mathbf{C}_y(y, v, w) = 0.$$

The family $\mathbf{C} = \{\mathbf{C}_y | y \in V\backslash\{0\}\}$ is called the *Cartan torsion.*

Let $\{e_i\}_{i=1}^n$ be a basis for V. Put $C_{ijk}(y) := \mathbf{C}_y(e_i, e_j, e_k)$. Then

$$C_{ijk}(y) = \frac{1}{4}\left[F^2\right]_{y^i y^j y^k}(y) = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}(y).$$

It is easy to see that F is Euclidean if and only if $\mathbf{C} = 0$.

The mean of $\mathbf{C}_y$ is defined by

$$\mathbf{I}_y(u) := \sum_{i,j=1}^n g^{ij}(y)\mathbf{C}_y(e_i, e_j, u) = \sum_{i,j=1}^n g^{ij}(y)C_{ijk}(y)u^k, \quad u = u^k e_k. \tag{1.16}$$

The family $\mathbf{I} = \{\mathbf{I}_y | y \in V\backslash\{0\}\}$ is called the *mean Cartan torsion.*

Let $I_i(y) := \mathbf{I}_y(e_i)$. Then

$$I_i(y) = g^{jk}(y)C_{ijk}(y) = \frac{\partial}{\partial y^i}\left[\ln\sqrt{\det\big(g_{jk}(y)\big)}\right]. \tag{1.17}$$

Note from (1.12) that

$$\tau_{y^i} = \frac{\partial}{\partial y^i}\left[\ln\sqrt{\det\big(g_{jk}(y)\big)}\right].$$

We obtain

$$I_i(y) = \tau_{y^i}. \tag{1.18}$$

Theorem 1.3.1 (Deicke Theorem) *For a Minkowski norm on a vector space, it is Euclidean if and only if* $\mathbf{I} = 0$.

We will prove it only for Randers metrics. Consider a Randers norm $F = \alpha + \beta$. By (1.17) and (1.9), we obtain

$$I_i(y) = \frac{\partial}{\partial y^i}\left[\ln\sqrt{\left(\frac{\alpha+\beta}{\alpha}\right)^{n+1}\det(a_{ij})}\right] = \frac{n+1}{2(\alpha+\beta)}\left(b_i - \frac{y_i}{\alpha}\frac{\beta}{\alpha}\right). \tag{1.19}$$

Differentiating (1.5) with respect to y^k and using (1.19), we obtain

$$C_{ijk}(y) = \frac{1}{n+1}\left[I_i(y)h_{jk}(y) + I_j(y)h_{ik}(y) + I_k(y)h_{ij}(y)\right], \tag{1.20}$$

where h_{ij} are given by (1.11). Thus the mean Cartan torsion $\mathbf{I}$ determines the Cartan torsion in higher dimensions.

By (1.20), one can see that F is Euclidean if and only if $\mathbf{I} = 0$. This is Deicke Theorem for Randers norms.

Spired by the identity (1.20) for Randers norms, we consider the following trace-free quantity for Minkowski norm F on V:

$$M_{ijk}(y) := C_{ijk}(y) - \frac{1}{n+1}\left[I_i(y)h_{jk}(y) + I_j(y)h_{ik}(y) + I_k(y)h_{ij}(y)\right]. \tag{1.21}$$

We obtain a symmetric multi-linear form $\mathbf{M}_y : V \times V \times V \to \mathbb{R}$ defined by $\mathbf{M}_y(e_i, e_j, e_k) = M_{ijk}(y)$. The family $\mathbf{M} = \{\mathbf{M}_y \mid y \in V \setminus \{0\}\}$ is called the *Matsumoto torsion* of F.

Theorem 1.3.2 ([Ma2],[MaHo]) *When dimension $n \geqslant 3, F$ is a Randers norm if and only if $\mathbf{M} = 0$.*

The proof is not trivial, so is omitted.

For a Minkowski space (V, F), define the norm of $\mathbf{I}$ and $\mathbf{C}$ in the following natural way:

$$\|\mathbf{I}\| := \sup_{y,u \in V\setminus\{0\}} \frac{F(y)|\mathbf{I}_y(u)|}{\sqrt{g_y(u,u)}},$$

$$\|\mathbf{C}\| := \sup_{y,u,v,w \in V\setminus\{0\}} \frac{F(y)|\mathbf{C}_y(u,v,w)|}{\sqrt{g_y(u,u)g_y(v,v)g_y(w,w)}}.$$

For Randers norms, we have the following lemma:

Lemma 1.3.1 ([ChSh], [Sh]) *Let $F = \alpha + \beta$ be a Randers norm on an n-dimensional vector space V. Then*

$$\|\mathbf{I}\| = \frac{n+1}{\sqrt{2}}\sqrt{1 - \sqrt{1-b^2}} < \frac{n+1}{\sqrt{2}}, \tag{1.22}$$

$$\|\mathbf{C}\| \leqslant \frac{3}{\sqrt{2}}\sqrt{1 - \sqrt{1-b^2}} < \frac{3}{\sqrt{2}}, \tag{1.23}$$

where $b := \|\beta\|_\alpha$.

Proof We have

$$\|\mathbf{I}\| = \sup_{y \in V\setminus\{0\}} F(y)\|\mathbf{I}_y\|,$$

where

$$\|\mathbf{I}_y\| := \sup_{u \in V\setminus\{0\}} \frac{|\mathbf{I}_y(u)|}{\sqrt{g_y(u,u)}} \tag{1.24}$$

$$= \sqrt{g^{ij} I_i I_j}. \tag{1.25}$$

By (1.10) and (1.19), one obtains

$$F(y)^2\|\mathbf{I}_y\|^2 = \left(\frac{n+1}{2}\right)^2 \frac{\alpha(y)}{F(y)}\left[b^2 - \left(\frac{\beta(y)}{\alpha(y)}\right)^2\right] \tag{1.26}$$

Let $s := \beta(y)/\alpha(y)$. Then the range of s is $[-b, b]$. We get

$$F(y)^2\|\mathbf{I}_y\|^2 = \left(\frac{n+1}{2}\right)^2 \frac{b^2 - s^2}{1+s}.$$

Then

$$\sup_{y\in V\setminus\{0\}} F(y)^2\|\mathbf{I}_y\|^2 = \sup_{|s|\leqslant b}\left(\frac{n+1}{2}\right)^2 \frac{b^2-s^2}{1+s} = \frac{(n+1)^2}{2}\Big(1-\sqrt{1-b^2}\Big).$$

This gives (1.22). Now (1.23) follows from (1.20) and (1.22). Q.E.D.

1.4 Duality

Let (V, F) be a Minkowski space, and V^* denote the vector space dual to V. Define

$$F^*(\xi) := \sup_{y\in V\setminus\{0\}} \frac{\xi(y)}{F(y)}. \tag{1.27}$$

F^* is a Minkowski norm on V^* again. Since $V^{**} = V$, we can define a Minkowski norm on V dual to F^*. By an elementary argument, one can show that the dual norm on $V = V^{**}$ must be F, namely,

$$F(y) = \sup_{\xi\in V^*\setminus\{0\}} \frac{\xi(y)}{F^*(\xi)}.$$

Theorem 1.4.1 ([HrSh]) *Let V and V^* be dual vector spaces, and F and F^* be dual Minkowski norms on V and V^*, respectively. Then F is a Randers norm if and only if F^* is a Randers norm. Further, for a Randers norm $F^* = \alpha^* + \beta^*$ on V^*, the dual norm $F = \alpha + \beta$ is determined by*

$$h\Big(\frac{y}{F(y)} - w\Big) = 1, \tag{1.28}$$

where h is the Euclidean norm dual to α^* and $w \in V$ is determined by $\beta^*(\xi) = \xi(w)$ $(\forall \xi \in V^*)$.

Proof Let $\{\mathbf{b}_i\}$ be a basis for V and $\{\theta^i\}$ be the dual basis for V^*. We denote a vector in V by $y = y^i\mathbf{b}_i$ and a covector in V^* by $\xi = \xi_i\theta^i$. Fix any $y \in V \setminus \{0\}$, there is an $\eta \in V^*$ such that

$$F(y) = \frac{\eta(y)}{F^*(\eta)}. \tag{1.29}$$

Thus η is a critical point of $\psi(\xi) := \xi(y)/F^*(\xi)$. Namely,

$$\psi_{\xi^i}(\eta) = \frac{F^*(\eta)y^i - \eta(y)(a^{*ij}\eta_j/\alpha^*(\eta) + w^i)}{F^*(\eta)^2} = 0.$$

We get

$$F^*(\eta)y^i = \eta(y)\Big(\frac{a^{*ij}\eta_j}{\alpha^*(\eta)} + w^i\Big). \tag{1.30}$$

By (1.29), we get

$$y^i = F(y)\Big(\frac{a^{*ij}\eta_j}{\alpha^*(\eta)} + w^i\Big).$$

Rewrite it as follows

$$\frac{y^i}{F(y)} - w^i = \frac{a^{*ij}\eta_j}{\alpha^*(\eta)}. \tag{1.31}$$

Note that the Euclidean norm $h = \sqrt{h_{ij}y^iy^j}$ dual to α^* is given by $h_{ij} := a^*_{ij}$, where $(a^*_{ij}) := (a^{*ij})^{-1}$. Thus (1.28) follows from (1.31).

Solving (1.28), we obtain a formula for the dual norm $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ are given by

$$a_{ij} = \frac{(1 - \|\beta^*\|^2_{\alpha^*})a^*_{ij} + b^*_ib^*_j}{(1 - \|\beta^*\|^2_{\alpha^*})^2}, \tag{1.32}$$

$$b_i = -\frac{b^*_i}{1 - \|\beta^*\|^2_{\alpha^*}}, \tag{1.33}$$

where $(a^*_{ij}) := (a^{*ij})^{-1}$ and $b^*_i := a^*_{ij}b^{*j}$.

Conversely, let $F = \alpha + \beta$ be a Randers norm on V, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$. Then by the same argument as above, one can show that the dual norm on V^* is also a Randers norm $F^* = \alpha^* + \beta^*$, where $\alpha^* = \sqrt{a^{*ij}\xi_i\xi_j}$ and $\beta^* = b^{*i}\xi_i$ are given by

$$a^{*ij} = \frac{(1 - \|\beta\|^2_{\alpha})a^{ij} + b^ib^j}{(1 - \|\beta\|^2_{\alpha})^2}, \tag{1.34}$$

$$b^{*i} = -\frac{b^i}{1 - \|\beta\|^2_{\alpha}}, \tag{1.35}$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. Q.E.D.

Let $F^* = \alpha^* + \beta^*$ be a Randers norm on V^*, where $\alpha^* = \sqrt{a^{*ij}\xi_i\xi_j}$ and $\beta^* = b^{*i}\xi_i$. The norm of β^* with respect to α^* is given by

$$\|\beta^*\|_{\alpha^*} = \sqrt{a^*_{ij}b^{*i}b^{*j}} = \sqrt{a^{*ij}b^*_ib^*_j},$$