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Mark R. Sepanski

Compact Lie Groups

紧李群

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《国外数学名著系列》(影印版) 序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

To Laura, Sarah, Ben, and Shannon

Preface

As an undergraduate, I was offered a reading course on the representation theory of finite groups. When I learned this basically meant studying homomorphisms from groups into matrices, I was not impressed. In its place I opted for a reading course on the much more glamorous sounding topic of multilinear algebra. Ironically, when I finally took a course on representation theory from B. Kostant in graduate school, I was immediately captivated.

In broad terms, representation theory is simply the study of symmetry. In practice, the theory often begins by classifying all the ways in which a group acts on vector spaces and then moves into questions of decomposition, unitarity, geometric realizations, and special structures. In general, each of these problems is extremely difficult. However in the case of compact Lie groups, answers to most of these questions are well understood. As a result, the theory of compact Lie groups is used extensively as a stepping stone in the study of noncompact Lie groups.

Regarding prerequisites for this text, the reader must first be familiar with the definition of a group and basic topology. Secondly, elementary knowledge of differential geometry is assumed. Students lacking a formal course in manifold theory will be able to follow most of this book if they are willing to take a few facts on faith. This mostly consists of accepting the existence of an invariant integral in §1.4.1. In a bit more detail, the notion of a submanifold is used in §1.1.3, the theory of covering spaces is used in §1.2, §1.3, §4.2.3, and §7.3.6, integral curves are used in §4.1.2, and Frobenius' theorem on integral submanifolds is used in the proof of Theorem 4.14. A third prerequisite is elementary functional analysis. Again, students lacking formal course work in this area can follow most of the text if they are willing to assume a few facts. In particular, the Spectral Theorem for normal bounded operators is used in the proof of Theorem 3.12, vector-valued integration is introduced in §3.2.2, and the Spectral Theorem for compact self-adjoint operators is used in the proof of Lemma 3.13.

The text assumes no prior knowledge of Lie groups or Lie algebras and so all the necessary theory is developed here. Students already familiar with Lie groups can quickly skim most of Chapters 1 and 4. Similarly, students familiar with Lie algebras can quickly skim most of Chapter 6.

The book is organized as follows. Chapter 1 lays out the basic definitions, examples, and theory of compact Lie groups. Though the construction of the spin groups in §1.3 is very important to later representation theory and mathematical physics, this material can be easily omitted on a first reading. Doing so allows for a more rapid transition to the harmonic analysis in Chapter 3. A similar remark holds for the construction of the spin representations in §2.1.2.4. Chapter 2 introduces the concept of a finite-dimensional representation. Examples, Schur's Lemma, unitarity, and the canonical decomposition are developed here. Chapter 3 begins with matrix coefficients and character theory. It culminates in the celebrated Peter–Weyl Theorem and its corresponding Fourier theory.

Up through Chapter 3, the notion of a Lie algebra is unnecessary. In order to progress further, Chapter 4 takes up their study. Since this book works with compact Lie groups, it suffices to consider linear Lie groups which allows for a fair amount of differential geometry to be bypassed. Chapter 5 examines maximal tori and Cartan subalgebras. The Maximal Torus Theorem, Dynkin's Formula, the Commutator Theorem, and basic structural results are given. Chapter 6 introduces weights, roots, the Cartan involution, the Killing form, the standard $\mathfrak{sl}(2, \mathbb{C})$, various lattices, and the Weyl group. Chapter 7 uses all this technology to prove the Weyl Integration Formula, the Weyl Character Formula, the Highest Weight Theorem, and the Borel–Weil Theorem.

Since this work is intended as a textbook, most references are given only in the bibliography. The interested reader may consult [61] or [34] for brief historical outlines of the theory. With that said, there are a number of resources that had a powerful impact on this work and to which I am greatly indebted. First, the excellent lectures of B. Kostant and D. Vogan shaped my view of the subject. Notes from those lectures were used extensively in certain sections of this text. Second, any book written by A. Knapp on Lie theory is a tremendous asset to all students in the field. In particular, [61] was an extremely valuable resource. Third, although many other works deserve recommendation, there are four outstanding texts that were especially influential: [34] by Duistermaat and Kolk, [72] by Rossmann, [70] by Onishchik and Vinberg, and [52] by Hoffmann and Morris. Many thanks also go to C. Conley who took up the onerous burden of reading certain parts of the text and making helpful suggestions. Finally, the author is grateful to the Baylor Sabbatical Committee for its support during parts of the preparation of this text.

Mark Sepanski
March 2006

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1

Compact Lie Groups

1.1 Basic Notions

1.1.1 Manifolds

Lie theory is the study of symmetry springing from the intersection of algebra, analysis, and geometry. Less poetically, Lie groups are simultaneously groups and manifolds. In this section, we recall the definition of a manifold (see [8] or [88] for more detail). Let $n \in \mathbb{N}$.

Definition 1.1. An n-dimensional *topological manifold* is a second countable (i.e., possessing a countable basis for the topology) Hausdorff topological space M that is locally homeomorphic to an open subset of $\mathbb{R}^n$.

This means that for all $m \in M$ there exists a homeomorphism $\varphi : U \to V$ for some open neighborhood U of m and an open neighborhood V of $\mathbb{R}^n$. Such a homeomorphism φ is called a *chart*.

Definition 1.2. An n-dimensional smooth *manifold* is a topological manifold M along with a collection of charts, $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$, called an *atlas*, so that
(1) $M = \cup_\alpha U_\alpha$ and
(2) For all α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the *transition map* $\varphi_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map on $\mathbb{R}^n$.

It is an elementary fact that each atlas can be completed to a unique maximal atlas containing the original. By common convention, a manifold's atlas will always be extended to this completion.

Besides $\mathbb{R}^n$, common examples of manifolds include the *n-sphere*,

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\},$$

where $\|\cdot\|$ denotes the standard Euclidean norm, and the *n-torus*,

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ copies}}.$$

Another important manifold is real *projective space*, $\mathbb{P}(\mathbb{R}^n)$, which is the n-dimensional compact manifold of all lines in $\mathbb{R}^{n+1}$. It may be alternately realized as $\mathbb{R}^{n+1}\backslash\{0\}$ modulo the equivalence relation $x \sim \lambda x$ for $x \in \mathbb{R}^{n+1}\backslash\{0\}$ and $\lambda \in \mathbb{R}\backslash\{0\}$, or as S^n modulo the equivalence relation $x \sim \pm x$ for $x \in S^n$. More generally, the *Grassmannian*, $\mathrm{Gr}_k(\mathbb{R}^n)$, consists of all k-planes in $\mathbb{R}^n$. It is a compact manifold of dimension $k(n-k)$ and reduces to $\mathbb{P}(\mathbb{R}^{n-1})$ when $k = 1$.

Write $M_{n,m}(\mathbb{F})$ for the set of $n \times m$ matrices over $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. By looking at each coordinate, $M_{n,m}(\mathbb{R})$ may be identified with $\mathbb{R}^{nm}$ and $M_{n,m}(\mathbb{C})$ with $\mathbb{R}^{2nm}$. Since the determinant is continuous on $M_{n,n}(\mathbb{F})$, we see $\det^{-1}\{0\}$ is a closed subset. Thus the *general linear group*

$$GL(n,\mathbb{F}) = \{g \in M_{n,n}(\mathbb{F}) \mid g \text{ is invertible}\} \tag{1.3}$$

is an open subset of $M_{n,n}(\mathbb{F})$ and therefore a manifold. In a similar spirit, for any finite-dimensional vector space V over $\mathbb{F}$, we write $GL(V)$ for the set of invertible linear transformations on V.

1.1.2 Lie Groups

Definition 1.4. A *Lie group* G is a group and a manifold so that
(1) the *multiplication* map $\mu : G \times G \to G$ given by $\mu(g, g') = gg'$ is smooth and
(2) the *inverse* map $\iota : G \to G$ by $\iota(g) = g^{-1}$ is smooth.

A trivial example of a Lie group is furnished by $\mathbb{R}^n$ with its additive group structure. A slightly fancier example of a Lie group is given by S^1. In this case, the group structure is inherited from multiplication in $\mathbb{C}\backslash\{0\}$ via the identification

$$S^1 \cong \{z \in \mathbb{C} \mid |z| = 1\}.$$

However, the most interesting example of a Lie group so far is $GL(n,\mathbb{F})$. To verify $GL(n,\mathbb{F})$ is a Lie group, first observe that multiplication is smooth since it is a polynomial map in the coordinates. Checking that the inverse map is smooth requires the standard linear algebra formula $g^{-1} = \mathrm{adj}(g)/\det g$, where the $\mathrm{adj}(g)$ is the transpose of the matrix of cofactors. In particular, the coordinates of $\mathrm{adj}(g)$ are polynomial functions in the coordinates of g and $\det g$ is a nonvanishing polynomial on $GL(n,\mathbb{F})$ so the inverse is a smooth map.

Writing down further examples of Lie groups requires a bit more machinery. In fact, most of our future examples of Lie groups arise naturally as subgroups of $GL(n,\mathbb{F})$. To this end, we next develop the notion of a Lie subgroup.

1.1.3 Lie Subgroups and Homomorphisms

Recall that an (immersed) *submanifold* N of M is the image of a manifold N' under an injective immersion $\varphi : N' \to M$ (i.e., a one-to-one smooth map whose differential has full rank at each point of N') together with the manifold structure on N

making $\varphi : N' \to N$ a diffeomorphism. It is a familiar fact from differential geometry that the resulting topology on N may not coincide with the relative topology on N as a subset of M. A submanifold N whose topology agrees with the relative topology is called a *regular* (or *imbedded*) submanifold.

Defining the notion of a Lie subgroup is very similar. Essentially the word homomorphism needs to be thrown in.

Definition 1.5. A *Lie subgroup* H of a Lie group G is the image in G of a Lie group H' under an injective immersive homomorphism $\varphi : H' \to G$ together with the Lie group structure on H making $\varphi : H' \to H$ a diffeomorphism.

The map φ in the above definition is required to be smooth. However, we will see in Exercise 4.13 that it actually suffices to verify that φ is continuous.

As with manifolds, a Lie subgroup is *not* required to be a regular submanifold. A typical example of this phenomenon is constructed by wrapping a line around the torus at an irrational angle (Exercise 1.5). However, regular Lie subgroups play a special role and there happens to be a remarkably simple criterion for determining when Lie subgroups are regular.

Theorem 1.6. *Let G be a Lie group and $H \subseteq G$ a subgroup (with no manifold assumption). Then H is a regular Lie subgroup if and only if H is closed.*

The proof of this theorem requires a fair amount of effort. Although some of the necessary machinery is developed in §4.1.2, the proof lies almost entirely within the purview of a course on differential geometry. For the sake of clarity of exposition and since the result is only used to efficiently construct examples of Lie groups in §1.1.4 and §1.3.2, the proof of this theorem is relegated to Exercise 4.28. While we are busy putting off work, we record another useful theorem whose proof, for similar reasons, can also be left to a course on differential geometry (e.g., [8] or [88]). We note, however, that a proof of this result follows almost immediately once Theorem 4.6 is established.

Theorem 1.7. *Let H be a closed subgroup of a Lie group G. Then there is a unique manifold structure on the quotient space G/H so the projection map $\pi : G \to G/H$ is smooth, and so there exist local smooth sections of G/H into G.*

Pressing on, an immediate corollary of Theorem 1.6 provides an extremely useful method of constructing new Lie groups. The corollary requires the well-known fact that when $f : H \to M$ is a smooth map of manifolds with $f(H) \subseteq N$, N a *regular* submanifold of M, then $f : H \to N$ is also a smooth map (see [8] or [88]).

Corollary 1.8. *A closed subgroup of a Lie group is a Lie group in its own right with respect to the relative topology.*

Another common method of constructing Lie groups depends on the Rank Theorem from differential geometry.

Definition 1.9. A *homomorphism of Lie groups* is a smooth homomorphism between two Lie groups.

Theorem 1.10. *If G and G' are Lie groups and $\varphi : G \to G'$ is a homomorphism of Lie groups, then φ has constant rank and* $\ker \varphi$ *is a (closed) regular Lie subgroup of G of dimension* $\dim G - \operatorname{rk}\varphi$ *where* $\operatorname{rk}\varphi$ *is the rank of the differential of φ.*

Proof. It is well known (see [8]) that if a smooth map φ has constant rank, then $\varphi^{-1}\{e\}$ is a closed regular submanifold of G of dimension $\dim G - \operatorname{rk}\varphi$. Since $\ker\varphi$ is a subgroup, it suffices to show that φ has constant rank. Write l_g for left translation by g. Because φ is a homomorphism, $\varphi \circ l_g = l_{\varphi(g)} \circ \varphi$, and since l_g is a diffeomorphism, the rank result follows by taking differentials. □

1.1.4 Compact Classical Lie Groups

With the help of Corollary 1.8, it is easy to write down new Lie groups. The first is the *special linear group*

$$SL(n, \mathbb{F}) = \{g \in GL(n, \mathbb{F}) \mid \det g = 1\}.$$

As $SL(n, \mathbb{F})$ is a closed subgroup of $GL(n, \mathbb{F})$, it follows that it is a Lie group.

Using similar techniques, we next write down four infinite families of compact Lie groups collectively known as the *classical compact Lie groups:* $SO(2n+1)$, $SO(2n)$, $SU(n)$, and $Sp(n)$.

1.1.4.1 $SO(n)$ The *orthogonal group* is defined as

$$O(n) = \{g \in GL(n, \mathbb{R}) \mid g^t g = I\},$$

where g^t denotes the transpose of g. The orthogonal group is a closed subgroup of $GL(n, \mathbb{R})$, so Corollary 1.8 implies that $O(n)$ is a Lie group. Since each column of an orthogonal matrix is a unit vector, we see that topologically $O(n)$ may be thought of as a closed subset of $S^{n-1} \times S^{n-1} \times \cdots \times S^{n-1} \subseteq \mathbb{R}^{n^2}$ (n copies). In particular, $O(n)$ is a compact Lie group.

The *special orthogonal group* (or *rotation group*) is defined as

$$SO(n) = \{g \in O(n) \mid \det g = 1\}.$$

This is a closed subgroup of $O(n)$, and so $SO(n)$ is also a compact Lie group.

Although not obvious at the moment, the behavior of $SO(n)$ depends heavily on the parity of n. This will become pronounced starting in §6.1.4. For this reason, the special orthogonal groups are considered to embody two separate infinite families: $SO(2n+1)$ and $SO(2n)$.

1.1.4.2 $SU(n)$ The *unitary group* is defined as

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid g^*g = I\},$$

where g^* denotes the complex conjugate transpose of g. The unitary group is a closed subgroup of $GL(n, \mathbb{C})$, and so $U(n)$ is a Lie group. As each column of a unitary matrix is a unit vector, we see that $U(n)$ may be thought of, topologically, as a closed subset of $S^{2n-1} \times S^{2n-1} \times \cdots \times^{2n-1} \subseteq \mathbb{R}^{2n^2}$ (n copies). In particular, $U(n)$ is a compact Lie group.

Likewise, the *special unitary group* is defined as

$$SU(n) = \{g \in U(n) \mid \det g = 1\}.$$

As usual, this is a closed subgroup of $U(n)$, and so $SU(n)$ is also a compact Lie group. The special case of $n = 2$ will play an especially important future role. It is straightforward to check (Exercise 1.8) that

$$SU(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\} \tag{1.11}$$

so that topologically $SU(2) \cong S^3$.

1.1.4.3 $Sp(n)$ The final compact classical Lie group, the *symplectic group*, ought to be defined as

$$Sp(n) = \{g \in GL(n, \mathbb{H}) \mid g^*g = I\}, \tag{1.12}$$

where $\mathbb{H} = \{a + ib + jc + kd \mid a, b, c, d \in \mathbb{R}\}$ denotes the *quaternions* and g^* denotes the quaternionic conjugate transpose of g. However, $\mathbb{H}$ is a noncommutative division algebra, so understanding the meaning of $GL(n, \mathbb{H})$ takes a bit more work. Once this is done, Equation 1.12 will become the honest definition of $Sp(n)$.

To begin, view $\mathbb{H}^n$ as a *right* vector space with respect to scalar multiplication and let $M_{n,n}(\mathbb{H})$ denote the set of $n \times n$ matrices over $\mathbb{H}$. By using matrix multiplication on the *left*, $M_{n,n}(\mathbb{H})$ may therefore be identified with the set of $\mathbb{H}$-linear transformations of $\mathbb{H}^n$. Thus the old definition of $GL(n, \mathbb{F})$ in Equation 1.3 can be carried over to define $GL(n, \mathbb{H}) = \{g \in M_{n,n}(\mathbb{H}) \mid g \text{ is an invertible transformation of } \mathbb{H}^n\}$.

Verifying that $GL(n, \mathbb{H})$ is a Lie group, unfortunately, requires more work. In the case of $GL(n, \mathbb{F})$ in §1.1.2, that work was done by the determinant function which is no longer readily available for $GL(n, \mathbb{H})$. Instead, we embed $GL(n, \mathbb{H})$ into $GL(2n, \mathbb{C})$ as follows.

Observe that any $v \in \mathbb{H}$ can be uniquely written as $v = a + jb$ for $a, b \in \mathbb{C}$. Thus there is a well-defined $\mathbb{C}$-linear isomorphism $\vartheta : \mathbb{H}^n \to \mathbb{C}^{2n}$ given by $\vartheta(v_1, \ldots, v_n) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ where $v_p = a_p + jb_p$, $a_p, b_p \in \mathbb{C}$. Use this to define a $\mathbb{C}$-linear injection of algebras $\widetilde{\vartheta} : M_{n,n}(\mathbb{H}) \to M_{n,n}(\mathbb{C})$ by $\widetilde{\vartheta} X = \vartheta \circ X \circ \vartheta^{-1}$ for $X \in M_{n,n}(\mathbb{H})$ with respect to the usual identification of matrices as linear maps. It is straightforward to verify (Exercise 1.12) that when X is uniquely written as $X = A + jB$ for $A, B \in M_{n,n}(\mathbb{C})$, then

$$\widetilde{\vartheta}(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \tag{1.13}$$

where $\overline{A}$ denotes complex conjugation of A. Thus $\widetilde{\vartheta}$ is a $\mathbb{C}$-linear algebra isomorphism from $M_{n,n}(\mathbb{H})$ to

$$M_{2n,2n}(\mathbb{C})_{\mathbb{H}} \equiv \{\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \mid A, B \in M_{n,n}(\mathbb{C})\}.$$

An alternate way of checking this is to first let r_j denote scalar multiplication by j on $\mathbb{H}^n$, i.e., *right* multiplication by j. It is easy to verify (Exercise 1.12) that $\vartheta r_j \vartheta^{-1} z = J\overline{z}$ for $z \in \mathbb{C}^{2n}$ where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Since ϑ is a $\mathbb{C}$-linear isomorphism, the image of $\widetilde{\vartheta}$ consists of all $Y \in M_{2n,2n}(\mathbb{C})$ commuting with $\vartheta r_j \vartheta^{-1}$ so that $M_{2n,2n}(\mathbb{C})_{\mathbb{H}} = \{Y \in M_{2n}(\mathbb{C}) \mid YJ = J\overline{Y}\}$.

Finally, observe that X is invertible if and only if $\widetilde{\vartheta} X$ is invertible. In particular, $M_{n,n}(\mathbb{H})$ may be thought of as $\mathbb{R}^{4n^2}$ and, since $\det \circ \widetilde{\vartheta}$ is continuous, $GL(n, \mathbb{H})$ is the open set in $M_{n,n}(\mathbb{H})$ defined by the complement of $(\det \circ \widetilde{\vartheta})^{-1}\{0\}$. Since $GL(n, \mathbb{H})$ is now clearly a Lie group, Equation 1.12 shows that $Sp(n)$ is a Lie group by Corollary 1.8. As with the previous examples, $Sp(n)$ is compact since each column vector is a unit vector in $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

As an aside, Dieudonné developed the notion of determinant suitable for $M_{n,n}(\mathbb{H})$ (see [2], 151–158). This quaternionic determinant has most of the nice properties of the usual determinant and it turns out that elements of $Sp(n)$ always have determinant 1.

There is another useful realization for $Sp(n)$ besides the one given in Equation 1.12. The isomorphism is given by $\widetilde{\vartheta}$ and it remains only to describe the image of $Sp(n)$ under $\widetilde{\vartheta}$. First, it is easy to verify (Exercise 1.12) that $\widetilde{\vartheta}(X^*) = (\widetilde{\vartheta} X)^*$ for $X \in M_{n,n}(\mathbb{H})$, and thus $\widetilde{\vartheta} Sp(n) = U(2n) \cap M_{2n,2n}(\mathbb{C})_{\mathbb{H}}$. This answer can be reshaped further. Define

$$Sp(n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) \mid g^t J g = J\}$$

so that $U(2n) \cap M_{2n,2n}(\mathbb{C})_{\mathbb{H}} = U(2n) \cap Sp(n, \mathbb{C})$. Hence $\widetilde{\vartheta}$ realizes the isomorphism:

$$\begin{aligned} Sp(n) &\cong U(2n) \cap M_{2n,2n}(\mathbb{C})_{\mathbb{H}} \\ &= U(2n) \cap Sp(n, \mathbb{C}). \end{aligned} \tag{1.14}$$

1.1.5 Exercises

Exercise 1.1 Show that S^n is a manifold that can be equipped with an atlas consisting of only two charts.