Lectures in Mathematics ETH Zürich

Raghavan Narasimhan

Compact Riemann Surfaces

紧黎曼曲面

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Preface

These notes form the contents of a *Nachdiplomvorlesung* given at the Forschungsinstitut für Mathematik of the Eidgenössische Technische Hochschule, Zürich from November, 1984 to February, 1985. Prof. K. Chandrasekharan and Prof. Jürgen Moser have encouraged me to write them up for inclusion in the series, published by Birkhäuser, of notes of these courses at the ETH.

Dr. Albert Stadler produced detailed notes of the first part of this course, and very intelligible class-room notes of the rest. Without this work of Dr. Stadler, these notes would not have been written. While I have changed some things (such as the proof of the Serre duality theorem, here done entirely in the spirit of Serre's original paper), the present notes follow Dr. Stadler's fairly closely.

My original aim in giving the course was twofold. I wanted to present the basic theorems about the Jacobian from Riemann's own point of view. Given the Riemann–Roch theorem, if Riemann's methods are expressed in modern language, they differ very little (if at all) from the work of modern authors.

I had hoped to follow this with some of the extensive work relating theta functions and the geometry of algebraic curves to solutions of certain non-linear partial differential equations (in particular KdV and KP). Time did not permit pursuing this subject, and I have contented myself with a couple of references in §17. These references fail to cover much other important work (especially of M. Mulase) but I have not tried to do better because the literature is so extensive.

It is a great pleasure to express my thanks to the ETH for its hospitality, to Prof. J. Moser for his encouragement, and to Dr. A. Stadler for the enormous amount of work he undertook which made these notes easier to write. But special thanks are due to Prof. K. Chandrasekharan. But for him, I would not have been at the ETH, nor would these notes have been written without his advice and encouragement.

Chicago, August 1991 R. Narasimhan

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1. Algebraic Functions

Let $F \in \mathbb{C}[x, y]$ be an irreducible polynomial in two variables (with complex coefficients). We assume that its degree in y is ≥ 1.

Recall that by the so-called Gauss lemma, if we identify $\mathbb{C}[x, y]$ with $\mathbb{C}[x][y]$, and if F is irreducible, it is also irreducible in $\mathbb{C}(x)[y]$, the polynomial ring over the field of rational functions in x. Moreover, $\mathbb{C}[x, y]$ is a factorial ring (i.e. a unique factorisation domain).

An algebraic function is, intuitively, "defined" by an equation $F(x, y) = 0$ (where F is irreducible in $\mathbb{C}[x, y]$).

To make this statement more precise, we begin with the following.

The implicit function theorem. *Let f be a holomorphic function of two complex variables x, y defined on $\{(x, y) \in \mathbb{C}^2 \mid |x| < r_1, |y| < r_2\}$, $r_1, r_2 > 0$. Assume that*

$$f(0,0) = 0\,, \quad \frac{\partial f}{\partial y}(0,0) \neq 0\,.$$

Then, there exist positive numbers $\varepsilon, \delta > 0$ such that for any $x \in D_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, there is a unique solution $y(x)$ of the equation $f(x, y) = 0$ with $|y(x)| < \delta$. The function $x \mapsto y(x)$ is holomorphic on D_ε.

Proof. Since $\frac{\partial f}{\partial y}(0,0) \neq 0$, we can choose $\delta > 0$ such that $f(0, y) \neq 0$ for $0 < |y| \leq \delta$. Choose now $\varepsilon > 0$ such that $f(x, y) \neq 0$ for $|x| \leq \varepsilon$, $|y| = \delta$ (possible since f is non-zero on the compact set $\{0\} \times \{y \mid |y| = \delta\}$).

By the argument principle, if $|x| < \varepsilon$,

$$\frac{1}{2\pi i} \int\limits_{|y|=\delta} \left\{\frac{\partial f}{\partial y}(x, y) \;/\; f(x, y)\right\} dy$$

is an integer $n(x)$ equal to the number of zeros of the function $y \mapsto f(x, y)$ in $|y| < \delta$; by our choice of δ, $n(0) = 1$. On the other hand, since $f(x, y) \neq 0$ for $|x| \leq \varepsilon$, $|y| = \delta$, the integrand, and thus also the integral, is a continuous function of x for $|x| < \varepsilon$. Thus $n(x) = 1$ for $|x| < \varepsilon$, which means precisely that there is a unique zero $y(x)$ of $f(x, y)$ with $|y(x)| < \delta$.

That $x \mapsto y(x)$ is holomorphic follows from the formula

$$y(x) = \frac{1}{2\pi i} \int\limits_{|y|=\delta} y \frac{\frac{\partial f}{\partial y}(x, y)}{f(x, y)}\, dy$$

(which is an immediate consequence of the residue theorem).

Let $F(x,y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \in \mathbb{C}[x,y]$ be an irreducible polynomial with $n \geq 1$; the polynomials $a_0, \ldots, a_n \in \mathbb{C}[x]$ have no non-constant common factor since F is irreducible.

Lemma 1. *Let $a \in \mathbb{C}$ be such that $a_0(a) \neq 0$ and such that there is no $b \in \mathbb{C}$ with $F(a,b) = 0 = \frac{\partial F}{\partial y}(a,b)$. Then, there is $\varepsilon > 0$ and n holomorphic functions $y_1(x), \ldots, y_n(x)$ in the disc $\{x \in \mathbb{C} \mid |x-a| < \varepsilon\}$ with the following properties:*

(i) $y_i(x) \neq y_j(x')$ if $i \neq j$, $|x-a| < \varepsilon$, $|x'-a| < \varepsilon$; moreover

$$F\big(x, y_i(x)\big) \equiv 0 \quad \text{for} \quad |x-a| < \varepsilon\,, \quad i = 1, \ldots, n\,.$$

(ii) if $\eta \in \mathbb{C}$ and $F(x,\eta) = 0$, $|x-a| < \varepsilon$, then $\eta = y_i(x)$ for a unique i between 1 and n.

Proof. Since $\frac{\partial F}{\partial y}(a,b) \neq 0$ for all solutions b of $F(a,b) = 0$, the polynomial $F(a,y)$ has exactly n roots $b_1, \ldots, b_n$. If $\varepsilon > 0$ is small and $y_i(x)$ the holomorphic function on $|x-a| < \varepsilon$ with $y_i(a) = b_i$ and $F\big(x, y_i(x)\big) \equiv 0$ (which exists by the theorem above), then the y_i have property (i) if ε is small enough, and property (ii) since the equation $F(x,\eta) = 0$ has at most n solutions.

Proposition 1. *Let $F \in \mathbb{C}[x,y]$ be irreducible. There are only finitely many $x \in \mathbb{C}$ such that the equations*

$$F(x,y) = 0 = \frac{\partial F}{\partial y}(x,y)$$

have a simultaneous solution $y \in \mathbb{C}$.

Proof. By the division algorithm, there are polynomials $b_i \in \mathbb{C}[x]$ $(i \geq 0)$ with $b_0 = a_0\big[F = a_0(x)y^n + \cdots + a_n(x)\big]$ and polynomials A_j, $Q_j \in \mathbb{C}[x,y]$ $(j \geq 1)$ such that

$$\begin{aligned}
b_0^n F &= A_1 \frac{\partial F}{\partial y} + Q_1\,, \quad \deg_y Q_1 < \deg_y \frac{\partial F}{\partial y} = n-1 \\
b_1 \frac{\partial F}{\partial y} &= A_2 Q_1 + Q_2\,, \quad \deg_y Q_2 < \deg_y Q_1 \\
&\vdots \\
b_{k-1} Q_{k-2} &= A_k Q_{k-1} + Q_k\,, \quad \deg_y Q_k < \deg_y Q_{k-1}\,.
\end{aligned}$$

We may suppose that $\deg_y Q_k = 0$, i.e. that $Q_k \in \mathbb{C}[x]$ (since we can otherwise continue the division process). We claim now that $Q_k(x) \not\equiv 0$. If, in fact, $Q_k \equiv 0$, then from the last of the above equations, any prime factor P of Q_{k-1} with $\deg_y P > 0$ would divide $b_{k-1}Q_{k-2}$, hence Q_{k-2} (since $b_{k-1} \in \mathbb{C}[x]$ and $\deg_y P > 0$). From the equation

$b_{k-2}Q_{k-3} = A_{k-1}Q_{k-2} + Q_{k-1}$, it would follow that P divides $b_{k-2}Q_{k-3}$ and hence Q_{k-3}. Repeating this argument, P would divide all the Q_j ($j \geq 1$), hence also $\frac{\partial F}{\partial y}$ and F, contradicting the irreducibility of F. Thus $Q_k = Q_k(x) \in \mathbb{C}[x]$ is $\not\equiv 0$.

If now $a, b \in \mathbb{C}$ and $F(a,b) = 0 = \frac{\partial F}{\partial y}(a,b)$, we see from the above equations that $Q_1(a,b) = 0$, then that $Q_2(a,b) = 0, \ldots, Q_k(a,b) = Q_k(a) = 0$. Since $Q_k \not\equiv 0$, the set

$$\{x \in \mathbb{C} \mid \exists y \in \mathbb{C} \quad \text{with} \quad F(x,y) = 0 = \frac{\partial F}{\partial y}(x,y)\} \subset \{x \in \mathbb{C} \mid Q_k(x) = 0\}$$

is finite.

Before proceeding further, we insert some toplogical preliminaries. All topological spaces we consider will be Hausdorff.

Definition. A continuous map $p : X \to Y$, where X, Y are locally compact (Hausdorff) spaces, will be called *proper* if, for any compact set $K \subset Y$, the inverse image $p^{-1}(K)$ is compact in X

Lemma 2. *If X, Y are locally compact, a proper map $p : X \to Y$ is necessarily closed, i.e. takes closed sets in X to closed sets in Y.*

Proof. Let $A \subset X$ be closed, and $y_0 \in Y$. Let K be a compact neighbourhood of y_0 in Y. Then $p(A) \cap K = p\big(A \cap p^{-1}(K)\big)$ is compact (since A is closed and $p^{-1}(K)$ is compact), hence closed in K.

Remark. A continuous map $p : X \to Y$ between locally compact spaces X, Y is proper, if and only if, for any locally compact topological space Z, the product

$$p \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z \ , \quad (x,z) \longmapsto \big(p(x), z\big)$$

is closed. If X, Y have countable bases, this can be seen by using the following remark: if $\{x_1, \ldots, x_n, \ldots\}$ is a sequence of points in X, without limit points and such that $\{p(x_n)\}_{n\geq 1}$ converges in Y, then the image of the closed set $\big\{(x_n, \frac{1}{n}) \mid n \geq 1\big\}$ in $X \times \mathbb{R}$ is not closed in $Y \times \mathbb{R}$.

The property in this remark can be used to define proper mappings between spaces which are not locally compact.

Remark. Let $p : X \to Y$ be a proper map between locally compact spaces. Let $Z \subset Y$ be a locally compact space (with the induced topology). Then $p \mid p^{-1}(Z) : p^{-1}(Z) \to Z$ is again proper.

In fact, a compact subset of Z is a compact subset of Y.

Lemma 3. *Let $c_1, \ldots, c_n \in \mathbb{C}$. Let $w \in \mathbb{C}$ and suppose that $w^n + c_1 w^{n-1} + \cdots + c_n = 0$. Then*

$$|w| < 2 \max_\nu |c_\nu|^{1/\nu}$$

(unless $c_1 = \cdots = c_n = 0$).

Proof. Let $c = \max_\nu |c_\nu|^{1/\nu} > 0$. If $z = \frac{w}{c}$, we have $z^n + \frac{c_1}{c} z^{n-1} + \cdots + \frac{c_n}{c^n} = 0$, so that, since $|c_\nu| \leq c^\nu$,

$$|z|^n \leq |z|^{n-1} + \cdots + 1 \, .$$

If $|z| \geq 2$, we would have $1 \leq \frac{1}{|z|} + \cdots + \frac{1}{|z|^n} \leq \frac{1}{2} + \cdots + \frac{1}{2^n} < 1$, a contradiction. Thus $|z| < 2$, i.e. $|w| < 2c$.

Proposition 2. *Let $F \in \mathbb{C}[x,y], F(x,y) = a_0(x)y^n + \cdots + a_n(x), a_0 \not\equiv 0$. Let $V = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0\}$ and $S_0 = \{x \in \mathbb{C} \mid a_0(x) = 0\}$. Let $\pi : V \to \mathbb{C}$ be the projection $(x,y) \mapsto x$. Then $\pi \mid \pi^{-1}(\mathbb{C} - S_0) \to \mathbb{C} - S_0$ is proper.*

Proof. Let $K \subset \mathbb{C} - S_0$ be compact. Then there is $\delta > 0$ so that $|a_0(x)| \geq \delta$ and $|a_\nu(x)| \leq \frac{1}{\delta}$ for $x \in K$. If $(x,y) \in V$, $x \in \pi^{-1}(K)$, we have

$$y^n + \frac{a_1(x)}{a_0(x)} y^{n-1} + \cdots + \frac{a_n(x)}{a_0(x)} = 0 \, ,$$

so that, by (1.8), $|y| \leq 2 \max_\nu \delta^{-2/\nu}$. Thus $\pi^{-1}(K)$ is bounded. Since clearly $\pi^{-1}(K) = (K \times \mathbb{C}) \cap V$ is closed in $\mathbb{C}^2$, $\pi^{-1}(K)$ is compact.

Definition. Let X, Y be (Hausdorff) topological spaces and $p : X \to Y$, a continuous map. p is called a covering map if the following holds: $\forall y_0 \in Y$, there is an open neighbourhood V of y_0 such that $p^{-1}(V)$ is a disjoint union $\bigcup_{j \in J} U_j$ of open sets U_j with the property that $p \mid U_j$ is a homeomorphism onto V $\forall j \in J$. The triple (X, Y, p) is then called an (unramified) covering. We also say that X is a covering of Y.

An open set $V \subset Y$ with the property in the definition is said to be evenly covered by p.

It follows from the definition that the cardinality of $p^{-1}(y)$ is a locally constant function on Y. (With the notation in the definition, the cardinality of $p^{-1}(y)$ is that of $J \forall y \in V$.) Thus, if Y is connected, "the number of points" in $p^{-1}(y)$ is independent of $y \in Y$. The covering is said to be finte (infinite) if the cardinality of $p^{-1}(y)$ is finite (infinite). p is called an n sheeted covering if $p^{-1}(y)$ contains exactly n points for $y \in Y$.

If $p : X \to Y, p : X \to T$ are two coverings of Y, they are said to be isomorphic if there exists a homeomorphism $\varphi : X' \to X$ such that $p \circ \varphi = p'$.

Examples. 1) Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\Delta^* = \Delta - \{0\}$. Then, if $n \geq 1$, the map $p_n : \Delta^* \to \Delta^*$ given by $p_n(z) = z^n$ is an n-sheeted covering.

It is a standard fact in the theory of covering spaces that any connected n-sheeted covering of Δ^* is isomorphic to p_n.

2) $p : \mathbb{C} \to \mathbb{C}^*$, $p(z) = e^z$ is an infinite covering of $\mathbb{C}^*$.

3) Let X, Y be locally compact, let $p : X \to Y$ be a local homeomorphism (i.e. $\forall a \in X$, $\exists$ an open neighbourhood U of a such that $V = p(U)$ is open in Y and $p \mid U$ is a homeomorphism onto V).

Then, p is a finite covering if and only if it is proper.

Proof. If p is a finite covering, if $y_0 \in Y$ and V is an open neighbourhood of y_0 which is evenly covered by p, then $p \mid p^{-1}(V) \to V$ is clearly proper. It follows easily that p is proper.

Conversely, let p be a proper local homeomorphism, let $y_0 \in Y$ and let $p^{-1}(y_0) = \{x_1, \dots, x_n\}$. Let U'_j be an open set with $x_j \in U'_j$ and such that $p \mid U_j$ is a homeomorphism onto the open set $V_j = p(U'_j)$. Since p is proper and $X - \bigcup_1^n U'_j$ is closed in X, $E = p(X - \bigcup_1^n U'_j)$ is closed in Y. Clearly, $y_0 \notin E$. Let $V = Y - E$. Then $p^{-1}(V) \subset U'_1 \cup \dots \cup U'_n$, and we have $V \subset V_1 \cap \dots \cap V_n$. If we set $U_j = U'_j \cap p^{-1}(V)$, then $p^{-1}(V) = \bigcup_1^n U_j$ and $p \mid U_j$ is a homeomorphism onto V.

Let $F \in \mathbb{C}[x, y]$ be irreducible, $F(x, y) = a_0(x)y^n + \cdots + a_n(x)$. Let $S_0 = \{x \in \mathbb{C} \mid a_0(x) = 0\}$ and $S_1 = \{x \in \mathbb{C} \mid \exists y \in \mathbb{C} \text{ with } F(x, y) = 0 = \frac{\partial F}{\partial y}(x, y)\}$. Then, if $V = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\}$ and $\pi : V \to \mathbb{C}$ the projection $(x, y) \mapsto x$, then

$$\pi \mid \pi^{-1}(\mathbb{C} - (S_0 \cup S_1)) \longrightarrow \mathbb{C} - (S_0 \cup S_1)$$

is a finite covering (of n sheets).

This follows from Proposition 2 the implicit function theorem.

Before proceeding to show how the set V can be modified over the points of $S_0 \cup S_1$ and the point at ∞ in $\mathbb{C}$ to define the algebraic function completely, we shall introduce the notion of a Riemann surface and some related topics.

2. Riemann Surfaces

Let X be a 2-dimensional manifold (i.e. X is a Hausdorff space and any point in X has a neighbourhood homeomorphic to an open set in $\mathbb{R}^2$).

Consider pairs (U, φ) where U is open in X and $\varphi : U \to \varphi(U) \subset \mathbb{C}$ is a homeomorphism onto an open set in $\mathbb{C}$.

Two such pairs (U_1, φ_1), (U_2, φ_2) are said to be (holomorphically) compatible if the map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1, \cap U_2)$ is holomorphic; its inverse is also holomorphic by a standard result in complex analysis.

A complex structure on X is a family $\mathcal{S}$ of pairs $\{(U, \varphi)\}$ which are pairwise compatible and such that $\bigcup U = X$; there is then a unique maximal family of pairs with these two properties and containing $\mathcal{S}$; we shall usually assume that the complex structure is maximal. The elements (U, φ) of this (maximal) complex structure are called charts or coordinate neighbourhoods. In a coordinate neighbourhood, we usually identify U with $\varphi(U)$ and write z for φ as one does with the usual complex variable in $\mathbb{C}$.

A Riemann surface is a connected 2-dimensional manifold X with a complex structure $\mathcal{S}$. We shall also assume that X has a countable base of open sets, although a theorem of Radó asserts that this is automatic (for a proof, see e.g. [4]).

If $\Omega \subset X$ is open (X is a Riemann surface) and $f : \Omega \to \mathbb{C}$ is continuous, we say that f is holomorphic if for any chart (U, φ) of X, the function $f \circ \varphi^{-1} : \varphi(\Omega \cap U) \to \mathbb{C}$ is holomorphic.

If X, Y are Riemann surfaces, $f : X \to Y$ a continuous map, f is called holomorphic if, for any chart (V, ψ) of Y, the function $\psi \circ f : f^{-1}(V) \to \psi(V) \subset \mathbb{C}$ is holomorphic.

Non-constant holomorphic maps between Riemann surfaces are open. Also, a bijective holomorphic map $f : X \to Y$ has a holomorphic inverse $f^{-1} : Y \to X$. Such bijective holomorphic maps are called analytic isomorphisms (or biholomorphic maps).

Examples

1. *The complex projective line = Riemann sphere.* Let $\mathbb{P}^1$ be the one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$. We set $U_1 = \mathbb{P}^1 - \{\infty\} = \mathbb{C}$, $\varphi_1 : U_1 \to \mathbb{C}$ being the identity;

$$U_2 = \mathbb{P}^1 - \{0\}, \ \varphi_2(z) = \begin{cases} 1/z & \text{if } z \in \mathbb{C} - \{0\} = \mathbb{C}^* \\ 0 & \text{if } z = \infty\,. \end{cases}$$

The map $\varphi_2 \circ \varphi_1^{-1}$ is the map $z \mapsto 1/z$ of $\mathbb{C}^*$ into itself, so that these two charts define a complex structure on $\mathbb{P}^1$. This Riemann surface is called the projective line or the Riemann sphere.

2. *Tori.* Let $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$. Let $\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\}$. Λ is an additive subgroup of $\mathbb{C}$. Consider the quotient group $X = \mathbb{C}/\Lambda$ and let $\pi : \mathbb{C} \to X$ be the canonical projection. With the quotient topology, X is a compact Hausdorff space, and $\mathbb{C} \to X$ is a local homeomorphism. [These statements are easy consequences of the following two remarks: if $a \in \mathbb{C}$, and we consider the set $U = \{a + \lambda + \mu\tau \mid \lambda, \mu \in \mathbb{R}, -\frac{1}{2} < \lambda, \mu < +\frac{1}{2}\}$, U is open and maps bijectively onto an open set in X; further X is the image of the compact set $\bar{U}$ (closure of U) for any $a \in \mathbb{C}$. π is actually a covering map.]

As charts, we use pairs (U, φ) obtained as follows: let V be any open set in $\mathbb{C}$ such that $\pi|V$ is a homeomorphism onto an open set U in X; set $\varphi = (\pi|V)^{-1} : U \to V \subset \mathbb{C}$. Two such charts (U_1, φ_1), (U_2, φ_2) are holomorphically compatible: we clearly have $\pi\big(\varphi_2 \circ \varphi_1^{-1}(z)\big) = \pi(z)$ for $z \in \varphi_1(U_1 \cap U_2)$ thus $\varphi_2 \circ \varphi_1^{-1}(z) - z \in \Lambda\ \forall z \in \varphi_1(U_1 \cap U_2)$, so must be constant on connected components (because $\varphi_2 \circ \varphi_1^{-1}$ is continuous and Λ is discrete).

The Riemann surfaces X constructed above are called tori or elliptic curves.

3. *Surfaces of "higher genus".* Let g be an integer > 1, and let $0 < r < 1$. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. There is a unique bijective holomorphic (= biholomorphic) map $T : \Delta \to \Delta$ such that $T(r) = re^{3\pi i/2g}$ and $T(re^{\pi i/2g}) = re^{2\pi i/2g}$. Let $\sigma : \Delta \to \Delta$ be the rotation $z \mapsto ze^{2\pi i/4g}$.

For $k \in \mathbb{Z}$, we set

$$A_k = \sigma^{4k} T \sigma^{-4k} , \quad B_k = \sigma^{4k+1} T \sigma^{-4k-1} ,$$

and denote by Γ the group of biholomorphic maps of Δ generated by $A_k, B_k (\forall k \in \mathbb{Z})$.

A special case of a theorem enunciated by Poincaré (for the theorem and its proof, see the elegant article by G. de Rham: *Sur les polygônes générateurs de groupes Fuchsiens*, L'Enseignement Mathématique, 1971, pp. 47–61) implies that there exists an r, $0 < r < 1$, such that Γ acts freely (without fixed points) and discontinuously on Δ, and the quotient Δ/Γ is compact. One sees that the canonical projection $\pi : \Delta \to \Delta/\Gamma$ is a covering map, and obtains a complex structure on Δ/Γ for which the map π is holomorphic as in the case of tori.

4. Let Y be a Riemann surface, X a connected 2-dimensional manifold and $p : X \to Y$ a local homeomorphism. There is a unique complex structure on X for which the map p is holomorphic, obtained as follows: Let U be an open set in X such that $p|U$ is a homeomorphism onto an open set V in Y such that $V \subset V_j$ for some j, where $\{(V_j, \psi_j)_{j \in J}\}$ is the given complex structure on Y. Let $\varphi_U : U \to \mathbb{C}$ be the map $\varphi_U = \psi_j \circ p$. It is easily checked that two such pairs (U, φ_U), $(U', \varphi_{U'})$ are holomorphically compatible, so that one obtains a complex structure on X for which p is holomorphic.

The uniqueness is a consequence of the following remark: let $U \subset X$ be open and $p|U$, a homeomorphism onto $V \subset Y$. Then, if p is holomorphic, the map $(p|U)^{-1} : V \to U$ is again holomorphic.

Consider now a Riemann surface X and a holomorphic map $p : X \to \mathbb{C}$ which is also a local homeomorphism. We consider $\mathbb{C}$ as the complement of $\infty \in \mathbb{P}^1$, and p as a local homeomorphism $X \to \mathbb{P}^1$.

We shall define boundary points of X. Let $\{x_\nu\}_{\nu \geq 1}$ be a sequence of points in X with the following properties:

1) $\{x_\nu\}$ is discrete (i.e. has no limit points in X);

2) $\{p(x_\nu)\}$ converges to a point $a \in \mathbb{P}^1$;

3) Let $D_\varepsilon = \{z \in \mathbb{C} \mid |z - a| < \varepsilon\}$ if $a \in \mathbb{C}$, and let $D_\varepsilon = \{z \in \mathbb{C} \mid |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$ if $a = \infty$. Then, for all sufficiently small $\varepsilon > 0$, all but finitely many of the $\{x_\nu\}$ lie in the same connected component of $p^{-1}(D_\varepsilon)$.

Two such sequences $\{x_\nu\}, \{y_\nu\}$ are called equivalent if the sequence

$$z_\nu = \begin{cases} x_{(\nu+1)/2} & \text{for } \nu \text{ odd} \\ y_{\nu/2} & \text{for } \nu \text{ even} \end{cases}$$

again has the three properties above [i.e. $\lim p(x_\nu) = \lim p(y_\nu) = a$ say, and the connected components of $p^{-1}(D_\varepsilon)$ containing all but finitely many of the x_ν, y_ν respectively are the same].

A *boundary point* of X (relative to the map p) is then an equivalence class of sequences $\{x_\nu\}_{\nu \geq 1}$ with the three properties given above. Set $\tilde{X} = X \cup \{\text{boundary points of } X\}$.

Let P be a boundary point of X, defined by a sequence $\{x_\nu\}_{\nu \geq 1}$. We define neighbourhoods of P in $\tilde{X}$ as follows. Let $\varepsilon > 0$ be small and $D_\varepsilon = \{z \mid |z - a| < \varepsilon\}$ ($a \in \mathbb{C}$) or $D_\varepsilon = \{z \mid |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$ ($a = \infty$), where $a = \lim p(x_\nu)$. Let Ω_ε be the connected component of $p^{-1}(D_\varepsilon)$ containing all but finitely many of the x_ν, and let $\tilde{\Omega}_\varepsilon$ be the union of Ω_ε with those boundary points Q with the following property: if $\{y_\nu\}_{\nu \geq 1}$ defines Q, then $\{\nu \mid y_\nu \notin \Omega_\varepsilon\}$ is finite (this is independent of the sequence $\{y_\nu\}$ defining Q). The $\tilde{\Omega}_\varepsilon$ ($\varepsilon > 0$ small) form a fundamental system of neighbourhood of $P \in \tilde{X} - X$.

This topology is Hausdorff: if P, Q are boundary points defined by $\{x_\nu\}, \{y_\nu\}$ respectively, and $P \neq Q$, then, by the definition of the equivalence relation, there is $\varepsilon > 0$ such that the components $\Omega_{\varepsilon,1}\Omega_{\varepsilon,2}$ of $p^{-1}(D_\varepsilon)$ containing all but finitely many of the x_ν, y_ν respectively are distinct, and $\tilde{\Omega}_{\varepsilon,1} \cap \tilde{\Omega}_{\varepsilon,2} = \emptyset$. Moreover, p clearly extends to a continuous map $\tilde{p} : \tilde{X} \to \mathbb{P}^1 : \tilde{p}(P) = a = \lim p(x_\nu)$.

A boundary point P of X is said to the *algebraic* if the following holds: let D_ε be a small disc around $a = \tilde{p}(P)$ and let Ω be the connected component of $p^{-1}(D_\varepsilon)$ containing all but finitely many points of a sequence defining P; then $p(\Omega) \subset D_\varepsilon - \{a\}$ and the map $p : \Omega \to D_\varepsilon - \{a\}$ is a finite covering.

If we set $\Delta_R = \{z \in \mathbb{C} \mid |z| < R\}$ and $\Delta_R^* = \Delta_R - \{0\}$, then there is $n \geq 1$ such that the map $p : \Omega \to D_\varepsilon - \{a\}$ is isomorphic to the map $p_n : \Delta^*_{\varepsilon^{1/n}} \to D_\varepsilon - \{a\}$ given by $p_n(z) = a + z^n$ if $a \in \mathbb{C}$, $p_n(z) = z^{-n}$ if $a = \infty$ (see Example 1 after Definition (1.10)).