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Preface

One of the mathematical miracles of the 20th century was the discovery of a group of nonlinear wave equations being integrable. These integrable systems are the infinite dimensional counterpart of the finite dimensional integrable Hamiltonian systems of classical mechanics. Icons of integrable systems are the KdV equation, sine-Gordon equation, nonlinear Schrödinger equation etc. The beauty of the integrable theory is reflected by the explicit formulas of nontrivial solutions to the integrable systems. These explicit solutions bear the iconic names of soliton, multi-soliton, breather, quasi-periodic orbit, homoclinic orbit (the focus of this book) etc. There are several ways now available for obtaining these explicit solutions: Bäcklund transformation, Darboux transformation, and inverse scattering transform. The clear connection among these transforms is still an open question although they are certainly closely related. These transformations can be regarded as the counterpart of the canonical transformation of the finite dimensional integrable Hamiltonian system. Bäcklund transformation originated from a quest for Lie's second type invariant transformation rather than his tangent transformation. That brings the title of this book: Lie-Bäcklund-Darboux Transformations which refer to both Bäcklund transformations and Darboux transformations.

The most famous mathematical miracle of the 20th century was probably the discovery of chaos. When the finite dimensional integrable Hamiltonian systems are under perturbations, their regular solutions can turn into chaotic solutions. For such near integrable systems, existence of chaos can sometimes be proved mathematically rigorously. Following the same spirit, one may attempt to prove the existence of chaos for near integrable nonlinear wave equations viewed as near integrable Hamiltonian partial differential equations. This has been accomplished as summarized in the book [69]. The key ingredients in this theory of chaos in partial differential equations are the explicit formulas for the homoclinic orbit and Melnikov integral. The first author's taste is to use Darboux transformation to obtain the homoclinic orbit and Melnikov integral. This will be the focus of the first part of this book.

The second author's taste is to use Darboux transformation in a diversity of applications especially in higher spatial dimensions. The range of applications crosses many different fields of physics. This will be the focus of the second part of this book. This book is a result of the second author's several visits at University of Missouri as a Miller scholar.

The first author would like to thank his wife Sherry and his son Brandon, and the second author would like to thank his wife Alla and his son Valerian, for their loving support during this work.

Contents

Chapter 1

Introduction

The so-called Bäcklund transformation originated from the studies by S. Lie [80–82] and A. V. Bäcklund [11–15] on the Lie's second question on the existence of invariant multi-valued surface transformations [5]. Lie's first question was on the well-known Lie's tangent transformations. The first example of a Bäcklund transformation was studied on the Bianchi's geometrical construction of surfaces of constant negative curvatures – pseudospheres [18]. The Gauss equation of a pseudosphere can be rewritten as the sine-Gordon equation. The Bäcklund transformation for the sine-Gordon equation is an invariant transformation with a so-called Bäcklund parameter first introduced by Bäcklund. The Bäcklund parameter is particularly important in Bianchi's diagram of iterating the Bäcklund transformation to generate a so-called nonlinear superposition law [18, 19]. Immediate further studies on Bäcklund transformations were conducted by J. Clairin [26] and E. Goursat [40].

Darboux transformation was first introduced by Gaston Darboux [29] for the nowadays well-known one-dimensional linear Schrödinger equation – a special form of the Sturm-Liouville equations [84]. Darboux found a covariant transformation for the eigenfunction and the potential. The covariant transformation was built upon a particular eigenfunction at a particular value of the spectral parameter.

At the beginning, it seemed that Bäcklund transformation and Darboux transformation are irrelevant. The first link of the two came about in 1967 when Gardner, Greene, Kruskal, and Miura related KdV equation to its Lax pair of which the spatial part is the one-dimensional linear Schrödinger equation [38]. Soon afterwards, the Bäcklund transformation for the KdV equation was found. This was the beginning of a renaissance of Bäcklund transformations and Darboux transformations. It turned out that the existence of a Lax pair for a nonlinear wave equation, the solvability of the Cauchy problem for the nonlinear wave equation by the inverse scattering transform [38], the existence of a Bäcklund transformation for the nonlinear wave equation, and the existence of a Darboux transformation for the nonlinear wave equation and its Lax pair are closely related (although clear relation is still not known). Up to now, Bäcklund transformations and Darboux transformations for most of the nonlinear wave equations solvable by the inverse scattering transform, have been found [84, 96]. The potential lies at utilizing these transformations. All the earlier books [3, 5, 27, 41, 84, 91, 95, 96] focus on using Bäcklund or Darboux or inverse scattering transformation to construct multi-soliton solutions. Such multi-soliton solutions are defined on the whole spatial space with decaying boundary conditions. When the integrable system is posed with periodic boundary conditions, the solutions are temporally quasi-periodic or periodic or homoclinic. The first part of this book will focus on homoclinic orbit. Chapters 3–9 contain many valuable formulas for homoclinic orbits and Melnikov integrals. Here the Darboux transformations are not only used to generate explicit formulas for the homoclinic orbits but also interlaced with the isospectral theory of the corresponding Lax pairs to generate Melnikov vectors crucial for building the Melnikov integrals. The integrable systems studied in Chapters 3–9 are the so-called canonical systems each of which models a variety of different phenomena. The formulas can be directly used by the readers to study their own near integrable systems. In Chapter 2, we briefly summarize various methods for deriving Bäcklund transformations. These are all "experimental" or "trial-correction" methods. For a more detailed account on these methods, we refer the readers to the book [96]. There are not many methods for deriving Darboux transformations. The commonly used one is the dressing method [84]. Sometimes Chen's method in Chapter 2 can be effective too. Again these are all "trial-correction" methods. Chapters 10–16 contain applications of Darboux transformations in more specific physics problems, and various connections among different systems. Here no specific boundary condition is posed.

The future of Lie-Bäcklund-Darboux transformations is very bright. Besides the potential of their important applications and new transformations, it is possible to broaden their notion and still end up with useful transformations. This broadening process had begun long ago, e.g. the group notion in [5], the jet bundle in [96], and the Moutard transformation in this book. The broadened transformations even reached the Euler equations of fluids [63, 79].

Chapter 2

A Brief Account on Bäcklund Transformations

Bäcklund transformation is an invariant transformation that transforms one solution into another of a differential equation. It usually involves partial derivatives, and is easily solvable once the initial solution is given. Since J. Clairin [26], researchers have been trying to find an effective systematic way of obtaining a Bäcklund transformation. It was not successful. The commonly employed approaches are Clairin's method [5, 26, 96], Chen's method [25, 91], Hirota's bilinear operator approach [96], and Wahlquist-Estabrook procedure [96, 103, 104]. These are all "trail and correction" approaches. The most common use of Bäcklund transformations is to obtain multi-soliton solutions to integrable systems. For a detailed account, we refer the readers to [96]. Here we shall briefly go over the above mentioned derivation methods.

2.1 A Warm-Up Approach

Take the sine-Gordon equation for example

$$u_{xy} = \sin u, \tag{2.1}$$

where u is a real-valued function of two variables x and y. Let us assume that a Bäcklund transformation transforms u into v where v also satisfies the same sine-Gordon equation

$$v_{xy} = \sin v, \tag{2.2}$$

and our goal is to find the Bäcklund transformation. Let $w^+ = \frac{1}{2}(u+v)$ and $w^- = \frac{1}{2}(u-v)$ [59], then w^+ and w^- satisfy

$$w^+_{xy} = \sin w^+ \cos w^-, \quad w^-_{xy} = \cos w^+ \sin w^-. \tag{2.3}$$

We assume that the Bäcklund transformation has the trial form

$$w_x^- = f(w^+), \tag{2.4}$$

then by the second equation of (2.3), one has

$$f' w_y^+ = \cos w^+ \sin w^-. \tag{2.5}$$

By substituting (2.5) into the first equation of (2.3), one has

$$-\frac{w_x^+ \sin w^-}{f'}\left(\frac{f''}{f'}\cos w^+ + \sin w^+\right) + \cos w^-\left(\frac{f}{f'}\cos w^+ - \sin w^+\right) = 0,$$

which can be satisfied if one demands

$$\frac{f''}{f'}\cos w^+ + \sin w^+ = 0, \quad \text{and} \quad \frac{f}{f'}\cos w^+ - \sin w^+ = 0.$$

A solution of this over-determined system can be found

$$f = a\sin w^+,$$

where a is an arbitrary constant called a Bäcklund parameter. (2.4) and (2.5) now take the form

$$w_x^- = a\sin w^+, \quad w_y^+ = \frac{1}{a}\sin w^-. \tag{2.6}$$

(2.6) implies (2.3) which is equivalent to (2.1) and (2.2). (2.6) is the Bäcklund transformation for sine-Gordon equation (2.1). The above approach of derivation was developed in [59]. After experimenting with different forms of assumptions like (2.4), one hopes to find a Bäcklund transformation for a given equation.

2.2 Chen's Method

Chen's method [25] is a quite efficient method. Take the KdV equation for example,

$$u_t + 6uu_x + u_{xxx} = 0,$$

its Lax Pair is given by

$$L\psi = \lambda\psi, \quad \partial_t\psi = A\psi,$$

where

$$L = \partial_x^2 + u, \quad A = -4\partial_x^3 - 3(u\partial_x + \partial_x u).$$

Let $v = \psi_x/\psi$, one gets

$$v_x + v^2 + u = \lambda, \tag{2.7}$$

$$\begin{aligned} -v_t = {} & 4v_{xxx} + 12vv_{xx} + 12v^2v_x + 12v_x^2 \\ & + 6u_xv + 6uv_x + 3u_{xx}. \end{aligned} \tag{2.8}$$

Eliminating u, one finds that v satisfies the modified KdV equation

$$v_t - 6v^2 v_x + 6\lambda v_x + v_{xxx} = 0.$$

Relation (2.7) is the Miura transformation between KdV and modified KdV. If v solves the modified KdV, so does $-v$, then one can find $\hat{u}$ by solving KdV such that

$$-v_x + v^2 + \hat{u} = \lambda, \tag{2.9}$$

$$\begin{aligned} v_t = &- 4v_{xxx} + 12vv_{xx} - 12v^2 v_x + 12v_x^2 \\ &- 6\hat{u}_x v - 6\hat{u}v_x + 3\hat{u}_{xx}. \end{aligned} \tag{2.10}$$

Subtracting (2.9) from (2.7), one gets $v_x = \frac{1}{2}(\hat{u} - u)$. Let $w_x = \frac{1}{2}u$ and $\hat{w}_x = \frac{1}{2}\hat{u}$, then from (2.7) and (2.8), one gets the Bäcklund transformation for KdV equation

$$\begin{aligned} (\hat{w} + w)_x &= \lambda - (\hat{w} - w)^2, \\ (w - \hat{w})_t &= 4(\hat{w} - w)_{xxx} + 12(\hat{w} - w)(\hat{w} - w)_{xx} \\ &\quad + 12(\hat{w} - w)^2(\hat{w} - w)_x + 12(\hat{w} - w)_x^2 \\ &\quad + 12(\hat{w} - w)w_{xx} + 12w_x(\hat{w} - w)_x + 6w_{xxx}, \end{aligned}$$

where λ is the Bäcklund parameter.

2.3 Clairin's Method

Clairin's method is a direct trial method which involves tedious calculations. In general, let $u = u(t, x)$ satisfy some equations, and let $v = v(t, x)$ be another variable linked to u through a transformation of the form

$$u_x = f(u, v, v_x, v_t), \quad u_t = g(u, v, v_x, v_t).$$

The compatibility condition

$$f_t = g_x,$$

hopefully can lead to an equation for v, of course, by virtue of the equation satisfied by u. Consider the focusing cubic nonlinear Schrödinger equation

$$iq_t + q_{xx} + |q|^2 q = 0,$$

Lamb started with a transformation of the form [55]

$$\begin{aligned} q_x &= f(q, \bar{q}, p, \bar{p}, p_x, \bar{p}_x), \\ q_t &= g(q, \bar{q}, p, \bar{p}, p_x, \bar{p}_x, p_t, \bar{p}_t). \end{aligned}$$

After some lengthy calculation, Lamb obtained the Bäcklund transformation [55],

$$q_x = p_x - \frac{1}{2}iw\xi + ikv,$$
$$q_t = p_t + \frac{1}{2}\xi w_x - k\zeta + \frac{1}{4}iv(|w|^2 + |v|^2),$$

where

$$\xi = \pm i(b - 2|v|^2)^{1/2}, \quad w = q + p,$$
$$v = q - p, \quad \zeta = -\frac{1}{2}iw\xi + ikv,$$

b and k are arbitrary real constants called Bäcklund parameters, and p and q satisfy the same equation.

2.4 Hirota's Bilinear Operator Method

Hirota [45, 46] introduced certain bilinear operators to convert the nonlinear wave equations into bilinear equations for which Bäcklund transformations can be constructed. The Hirota bilinear operators D_t and D_x act according to

$$D_t^m D_x^n f \circ g = (\partial_t - \partial_{\hat{t}})^m (\partial_x - \partial_{\hat{x}})^n f(t,x) g(\hat{t},\hat{x})|_{\hat{t}=t,\hat{x}=x}.$$

Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

by setting $u = 2(\ln f)_{xx}$, one obtains

$$u_t + 6uu_x + u_{xxx} = \partial_x \left[\frac{1}{f^2} D_x(D_t + D_x^3) f \circ f \right].$$

Thus, if f solves the associated bilinear equation

$$D_x(D_t + D_x^3) f \circ f = 0,$$

then u solves the KdV equation. A Bäcklund transformation for the bilinear equation can be obtained as [46]

$$(D_x^2 - \beta) f \circ \tilde{f} = 0, \quad (D_t + 3\beta D_x + D^3) f \circ \tilde{f} = 0,$$

where β is the Bäcklund parameter.

2.5 Wahlquist-Estabrook Procedure

Wahlquist-Estabrook Procedure [103, 104] offers a relatively more systematic approach than that of Clairin [96]. Consider the equation (which is essentially the stationary 2D Euler equation) [58]

$$\Delta u = f(u), \tag{2.11}$$

where $\Delta = \partial_x^2 + \partial_y^2$, and f is an arbitrary function. Let $M = \mathbb{R}^2$ (with coordinates x, y), $N = \mathbb{R}^1$ (with coordinate u), $N' = \mathbb{R}^1$ (with coordinate v), w denote the volume form on M, $w = dx \wedge dy$, and θ denote the contact 1-form on the 1-jet $J^1(M, N)$, $\theta = du - u_x dx - u_y dy$. The exterior differential system of 2-forms on $J^1(M, N)$ associated with (2.11) is generated by

$$\begin{aligned}
\sigma &= du_x \wedge dy - du_y \wedge dx - f(u)w, \\
\eta_1 &= du \wedge dy - u_x dx \wedge dy, \\
\eta_2 &= du \wedge dx + u_y dx \wedge dy.
\end{aligned}$$

We seek the following form of Bäcklund transformations for equation (2.11),

$$v_z = \psi_z(u, u_x, u_y, v), \quad (z = x, y).$$

The Wahlquist-Estabrook procedure [96, 103, 104] requires that

$$d\psi_x \wedge dx + d\psi_y \wedge dy = f_1\eta_1 + f_2\eta_2 + g\sigma + \xi \wedge \zeta,$$

where f_1, f_2, and g are arbitrary functions on $J^1(M, N) \times J^0(M, N')$, $\zeta = dv - \psi_x dx - \psi_y dy$, and ξ is a 1-form on $J^1(M, N) \times J^0(M, N')$. Solving this equation, one finds that if f satisfies [58, 98]

$$\frac{d^2 f}{du^2} = \lambda f,$$

for an arbitrary constant λ, then there is a Bäcklund transformation, and u and v satisfy the same equation.

Chapter 3

Nonlinear Schrödinger Equation

In the late 19th century, G. Darboux [29] introduced a type of transformations, now called the Darboux transformations, for Sturm-Liouville systems. The Darboux transformation is a covariant transformation which transforms the solution and the coefficient (potential) simultaneously.

For soliton systems, the corresponding Darboux transformation involves both the evolution equation and its Lax pair. In this context, a Darboux transformation is another form of the Lie-Bäcklund-Darboux transformation. Like the Bäcklund transformations, the derivation method for Darboux transformations is often of "experimental" nature. The popular ones include the "dressing method" [84] and the Chen's method [25]. As shown in the previous chapter, the main application of a Bäcklund transformation is at generating multisoliton solutions. Besides generating multisoliton solutions, a Darboux transformation has another novel application—generating homoclinic (heteroclinic) orbits. This new application was not heavily publicized. Its importance is obvious. Homoclinic (heteroclinic) orbits are the fertile ground of chaos when the system is under perturbations [60–62, 66–72, 75–78]. These homoclinic (heteroclinic) orbits form a figure eight structure also called a separatrix.

We take the focusing nonlinear Schrödinger equation (NLS) as our first example to show how to construct figure eight structures [70, 76]. If one starts from the conservation laws of the NLS, it turns out that it is very difficult to get the separatrices. On the contrary, starting from the Darboux transformation to be presented, one can find the separatrices rather easily.

3.1 Physical Background

The term "nonlinear Schrödinger equation" of course comes from the well-known (linear) Schrödinger equation of quantum mechanics. Folklore says that it was artificially written down at first and then discovered in many physical applications.