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ORTHOGONAL POLYNOMIALS OF SEVERAL VARIABLES

多变量的正交多项式

CHARLES F.DUNKL and YUAN XU

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Orthogonal Polynomials of Several Variables

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To our wives

Phil and Litian

with deep appreciation

Preface

The study of orthogonal polynomials of several variables goes back at least as far as Hermite. There have been only a few books on the subject since: Appell and de Fériet [1926] and Erdélyi et al. [1953]. Twenty-five years have gone by since Koornwinder's survey article [1975]. A number of individuals who need techniques from this topic have approached us and suggested (even asked) that we write a book accessible to a general mathematical audience.

It is our goal to present the developments of very recent research to a readership trained in classical analysis. We include applied mathematicians and physicists, and even chemists and mathematical biologists, in this category.

While there is some material about the general theory, the emphasis is on classical types, by which we mean families of polynomials whose weight functions are supported on standard domains such as the simplex and the ball, or of Gaussian type, which satisfy differential–difference equations, and for which fairly explicit formulae exist. The phrase 'difference' refers to operators associated to reflections in hyperplanes. The most desirable situation is when there is a set of commuting self-adjoint operators whose simultaneous eigenfunctions form an orthogonal basis of polynomials. As will be seen, this is still an open area of research for some families.

With the intention of making this book useful to a wide audience, for both reference and instruction, we use familiar and standard notation for analysis on Euclidean space, and assume basic knowledge of Fourier and functional analysis, matrix theory, and elementary group theory. We have been influenced by the important books of Bailey [1935], Szegő [1975] and Lebedev [1972] in style and taste.

Here is an overview of the contents: Chapter 1 is a summary of the key one variable methods and definitions: gamma and beta functions, the classical and related orthogonal polynomials and their structure constants, hypergeometric and Lauricella series. The multi-variable analysis begins in Chapter 2 with some examples of orthogonal polynomials and spherical harmonics, and specific two variable examples such as Jacobi polynomials on various domains and disc polynomials. There is a discussion of the moment problem, general properties of orthogonal polynomials of several variables and matrix three term recurrences in Chapter 3. Coxeter groups are treated systematically in a self-contained way, in a style suitable for the analyst, in Chapter 4 (knowledge of representation theory is not necessary). The chapter goes on to introduce differential–difference operators, the intertwining operator, and the analogue of the exponential function, and concludes with the construction of invariant differential operators. Chapter 5 is a presentation of h-harmonics, the analogue of harmonic homogeneous polynomials associated with reflection groups; there are some examples for specific reflection groups as well as the application to proving the isometric properties of the generalized Fourier transform. This transform uses the analogue of the exponential function. It contains the classical Hankel transform as a special case. Chapter 6 is a detailed treatment of orthogonal polynomials on the simplex, the ball, and of Hermite type. Then summability theorems for expansions in terms of these polynomials are presented in Chapter 7; the main method is Cesàro (C, δ) summation, and there are precise results on which values of δ give positive or bounded linear operators. The nonsymmetric Jack polynomials appear in Chapter 8; this chapter contains all necessary details for their derivation, formulae for norms, hook length products, and computation of the structure constants. There is a proof of the Macdonald–Mehta–Selberg integral formula. Finally Chapter 9 shows how to use the nonsymmetric Jack polynomials to produce bases associated with the octahedral groups. This chapter has a short discussion of how these polynomials and related operators are used to solve the Schrödinger equations of the Calogero-Sutherland systems; these are exactly solvable models of quantum mechanics involving identical particles in a one dimensional space. Both Chapters 8 and 9 discuss orthogonal polynomials on the torus, and of Hermite type.

The bibliography is intended to be reasonably comprehensive into the near past; the reader is referred to Erdélyi et al. [1953] for older papers, and Internet data bases for the newest articles. There are occasions in the book where we suggest some algorithms for possible symbolic algebra use; the reader is encouraged to implement them in his/her favorite computer algebra system; but again the reader is referred to the Internet for specific published software.

There are several areas of related current research that we have deliberately avoided: the role of special functions in the representation theory of Lie groups (see Dieudonné [1980], Hua [1963], Vilenkin [1968], Vilenkin and Klimyk [1991a,b,c, 1995]), basic hypergeometric series and orthogonal polynomials of q type (see Gasper and Rahman [1990], Andrews, Askey and Roy [1999]), quantum groups (Koornwinder [1992], Noumi [1996], Koelink [1996] and Stokman [1997]), Macdonald symmetric polynomials (a generalization of the q type) (see Macdonald [1995, 1998]). These topics touch on algebra, combinatorics and analysis; and some classical results can be obtained as limiting cases for $q \to 1$. Nevertheless, the material in this book can stand alone and 'q' is not needed in the proofs.

We gratefully acknowledge support from the National Science Foundation over the years for our original research, some of which is described in this book. Also we are grateful to the mathematics departments of the University of Oregon for granting sabbatical leave and the University of Virginia for inviting Y. X. to visit for a year, which provided the opportunity for this collaboration.

Charles F. Dunkl
Yuan Xu

Contents

1
Background

The theory of orthogonal polynomials of several variables, especially those of classical type, uses a significant amount of analysis in one variable. In this chapter we give concise descriptions of the needed tools.

1.1 The Gamma and Beta Functions

It is our feeling, or perhaps, our taste, that the most interesting objects of consideration have expressions which are rational functions of the underlying parameters. This immediately leads us to consideration of the gamma function and its relatives.

Definition 1.1.1 *The gamma function is defined for* $\operatorname{Re} x > 0$ *by the integral*

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t.$$

It is directly related to the beta function:

Definition 1.1.2 *The beta function is defined for* $\operatorname{Re} x > 0$ *and* $\operatorname{Re} y > 0$ *by*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t.$$

By changing variables $s = uv$ and $t = (1-u)v$ in the integral $\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty s^{x-1} t^{y-1} \mathrm{e}^{-(s+t)} \mathrm{d}s\, \mathrm{d}t$, one obtains

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x, y).$$

This leads to several useful definite integrals (all valid for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$):

(i) $\displaystyle\int_0^{\pi/2} \sin^{x-1}\theta\cos^{y-1}\theta\,\mathrm{d}\theta = \frac{1}{2}B\left(\frac{x}{2},\frac{y}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{y}{2}\right)}{\Gamma\left(\frac{x+y}{2}\right)}$;

(ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (set $x = y = 1$ in the previous integral);

(iii) $\displaystyle\int_0^{\infty} t^{x-1}\exp(-at^2)\mathrm{d}t = \frac{1}{2}a^{-x/2}\Gamma\left(\frac{x}{2}\right)$, for $a > 0$;

(iv) $\displaystyle\int_0^{1} t^{x-1}(1-t^2)^{y-1}\mathrm{d}t = \frac{1}{2}B\left(\frac{x}{2},y\right) = \frac{1}{2}\Gamma\left(\frac{x}{2}\right)\Gamma(y)/\Gamma\left(\frac{x}{2}+y\right)$;

(v) $\Gamma(x)\Gamma(1-x) = B(x,1-x) = \dfrac{\pi}{\sin\pi x}$.

The last equation can be proven by restricting x by $0 < x < 1$, in the beta integral $\int_0^1 (t/(1-t))^{x-1}(1-t)^{-1}\mathrm{d}t$ making the substitution $s = t/(1-t)$ and computing the resulting integral by residues. Of course one of the fundamental properties of the gamma function is the recurrence formula (integration by parts)

$$\Gamma(x+1) = x\Gamma(x),$$

which leads to the fact that Γ can be analytically continued to a meromorphic function on the complex plane; also $1/\Gamma$ is entire with (simple) zeros exactly at $\{0,-1,-2,\dots\}$. Note that Γ interpolates the factorial, indeed $\Gamma(n+1) = n!$ for $n = 0,1,2,\dots$.

Definition 1.1.3 *The Pochhammer symbol, also called the shifted factorial, is defined for all x by*

$$(x)_0 = 1, (x)_n = \prod_{i=1}^{n}(x+i-1) \qquad \textit{for} \quad n = 1,2,3,\dots.$$

Alternatively one recursively defines $(x)_n$ by $(x)_0 = 1$ and $(x)_{n+1} = (x)_n(x+n)$ for $n = 0,1,2,3,\dots$. Here are some important consequences of the definition:

(i) $(x)_{m+n} = (x)_m(x+m)_n$, for $m,n \in \mathbb{N}_0$;
(ii) $(x)_n = (-1)^n(1-n-x)_n$ (writing the product in reverse order);
(iii) $(x)_{n-i} = (x)_n(-1)^i/(1-n-x)_i$.

The Pochhammer symbol incorporates binomial coefficient and factorial notation:

(i) $(1)_n = n!$, $2^n\left(\frac{1}{2}\right)_n = 1\times 3\times 5\times\cdots\times(2n-1)$;