

ORTHOGONAL POLYNOMIALS OF SEVERAL VARIABLES

多变量的正交多项式

CHARLES F.DUNKL and YUAN XU

CAMBRIDGE

世界图书出版公司

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Orthogonal Polynomials of Several Variables

CHARLES F. DUNKL

University of Virginia

YUAN XU

University of Oregon



CAMBRIDGE
UNIVERSITY PRESS

世界图书出版公司

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

<http://www.cambridge.org>

© Cambridge University Press 2001

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2001

Typeface Computer Modern 10/12pt *System* L^AT_EX [UPH]

A catalogue record of this book is available from the British Library

ISBN 0 521 80043 9 hardback

This edition of *Orthogonal Polynomials of Several Variables* by C.
F. Dunkl and Y. Xu is published by arrangement with the Syndicate
of the Press of University of Cambridge, Cambridge, England.

Licensed edition for sale in the People's Republic of China only.
Not for export elsewhere.

书 名: Orthogonal Polynomials of Several Variables
作 者: C. F. Dunkl, Y. Xu
中 译 名: 多变量的正交多项式
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京世图印刷厂
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
联系电话: 010-64015659, 64038347
电子信箱: kjsk@vip.sina.com
开 本: 24 印 张: 17
出版年代: 2003 年 9 月
书 号: 7-5062-5942-7/O • 361
版权登记: 图字: 01-2003-5540
定 价: 98.00 元

世界图书出版公司北京公司已获得 Cambridge University Press 授权在中国大陆
独家重印发行。

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

FOUNDED BY G.-C. ROTA

Editorial Board

R. S. Doran, P. Flajolet, M. Ismail, T.-Y. Lam, E. Lutwak, R. Spigler

Volume 81

Orthogonal Polynomials of Several Variables

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

- 4 W. Miller, Jr. *Symmetry and separation of variables*
- 6 H. Minc *Permanents*
- 11 W. B. Jones and W. J. Thron *Continued fractions*
- 12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
- 18 H. O. Fattorini *The Cauchy problem*
- 19 G. G. Lorentz, K. Jetter and S. D. Riemenschneider *Birkhoff interpolation*
- 21 W. T. Tutte *Graph theory*
- 22 J. R. Bastida *Field extensions and Galois theory*
- 23 J. R. Cannon *The one-dimensional heat equation*
- 25 A. Salomaa *Computation and automata*
- 26 N. White (ed.) *Theory of matroids*
- 27 N. H. Bingham, C. M. Goldie and J. L. Teugels *Regular variation*
- 28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
- 29 N. White (ed.) *Combinatorial geometries*
- 30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
- 31 J. Aczel and J. Dhombres *Functional equations containing several variables*
- 32 M. Kuczma, B. Chozewski and R. Ger *Iterative functional equations*
- 33 R. V. Ambartzumian *Factorization calculus and geometric probability*
- 34 G. Gripenberg, S.-O. Londen and O. Staffans *Volterra integral and functional equations*
- 35 G. Gasper and M. Rahman *Basic hypergeometric series*
- 36 E. Torgersen *Comparison of statistical experiments*
- 37 A. Neumaier *Intervals methods for systems of equations*
- 38 N. Korneichuk *Exact constants in approximation theory*
- 39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
- 40 N. White (ed.) *Matroid applications*
- 41 S. Sakai *Operator algebras in dynamical systems*
- 42 W. Hodges *Model theory*
- 43 H. Stahl and V. Totik *General orthogonal polynomials*
- 44 R. Schneider *Convex bodies*
- 45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
- 46 A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented matroids*
- 47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
- 48 C. Sims *Computation with finitely presented groups*
- 49 T. Palmer *Banach algebras and the general theory of *-algebras*
- 50 F. Borceux *Handbook of categorical algebra I*
- 51 F. Borceux *Handbook of categorical algebra II*
- 52 F. Borceux *Handbook of categorical algebra III*
- 54 A. Katok and B. Hasselblatt *Introduction to the modern theory of dynamical systems*
- 55 V. N. Sachkov *Combinatorial methods in discrete mathematics*
- 56 V. N. Sachkov *Probabilistic methods in discrete mathematics*
- 57 P. M. Cohn *Skew Fields*
- 58 Richard J. Gardner *Geometric tomography*
- 59 George A. Baker, Jr. and Peter Graves-Morris *Padé approximants*
- 60 Jan Krajčiček *Bounded arithmetic, propositional logic, and complex theory*
- 61 H. Gromer *Geometric applications of Fourier series and spherical harmonics*
- 62 H. O. Fattorini *Infinite dimensional optimization and control theory*
- 63 A. C. Thompson *Minkowski geometry*
- 64 R. B. Bapat and T. E. S. Raghavan *Nonnegative matrices and applications*
- 65 K. Engel *Sperner theory*
- 66 D. Cvetković, P. Rowlinson and S. Simić *Eigenspaces of graphs*
- 67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial species and tree-like structures*
- 68 R. Goodman and N. Wallach *Representations of the classical groups*
- 69 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume I 2 ed.*
- 70 A. Pietsch and J. Wenzel *Orthonormal systems and Banach space geometry*
- 71 George E. Andrews, Richard Askey and Ranjan Roy *Special Functions*
- 72 R. Ticciati *Quantum field theory for mathematicians*
- 76 A. A. Ivanov *Geometry of sporadic groups I*
- 78 T. Beth, D. Jungnickel and H. Lenz *Design Theory volume II 2 ed.*
- 80 O. Stormark *Lie's Structural Approach to PDE Systems*

To our wives
Phil and Litian
with deep appreciation

Preface

The study of orthogonal polynomials of several variables goes back at least as far as Hermite. There have been only a few books on the subject since: Appell and de Fériet [1926] and Erdélyi et al. [1953]. Twenty-five years have gone by since Koornwinder's survey article [1975]. A number of individuals who need techniques from this topic have approached us and suggested (even asked) that we write a book accessible to a general mathematical audience.

It is our goal to present the developments of very recent research to a readership trained in classical analysis. We include applied mathematicians and physicists, and even chemists and mathematical biologists, in this category.

While there is some material about the general theory, the emphasis is on classical types, by which we mean families of polynomials whose weight functions are supported on standard domains such as the simplex and the ball, or of Gaussian type, which satisfy differential-difference equations, and for which fairly explicit formulae exist. The phrase 'difference' refers to operators associated to reflections in hyperplanes. The most desirable situation is when there is a set of commuting self-adjoint operators whose simultaneous eigenfunctions form an orthogonal basis of polynomials. As will be seen, this is still an open area of research for some families.

With the intention of making this book useful to a wide audience, for both reference and instruction, we use familiar and standard notation for analysis on Euclidean space, and assume basic knowledge of Fourier and functional analysis, matrix theory, and elementary group theory. We have been influenced by the important books of Bailey [1935], Szegő [1975] and Lebedev [1972] in style and taste.

Here is an overview of the contents: Chapter 1 is a summary of the

key one variable methods and definitions: gamma and beta functions, the classical and related orthogonal polynomials and their structure constants, hypergeometric and Lauricella series. The multi-variable analysis begins in Chapter 2 with some examples of orthogonal polynomials and spherical harmonics, and specific two variable examples such as Jacobi polynomials on various domains and disc polynomials. There is a discussion of the moment problem, general properties of orthogonal polynomials of several variables and matrix three term recurrences in Chapter 3. Coxeter groups are treated systematically in a self-contained way, in a style suitable for the analyst, in Chapter 4 (knowledge of representation theory is not necessary). The chapter goes on to introduce differential-difference operators, the intertwining operator, and the analogue of the exponential function, and concludes with the construction of invariant differential operators. Chapter 5 is a presentation of h -harmonics, the analogue of harmonic homogeneous polynomials associated with reflection groups; there are some examples for specific reflection groups as well as the application to proving the isometric properties of the generalized Fourier transform. This transform uses the analogue of the exponential function. It contains the classical Hankel transform as a special case. Chapter 6 is a detailed treatment of orthogonal polynomials on the simplex, the ball, and of Hermite type. Then summability theorems for expansions in terms of these polynomials are presented in Chapter 7; the main method is Cesàro (C, δ) summation, and there are precise results on which values of δ give positive or bounded linear operators. The nonsymmetric Jack polynomials appear in Chapter 8; this chapter contains all necessary details for their derivation, formulae for norms, hook length products, and computation of the structure constants. There is a proof of the Macdonald-Mehta-Selberg integral formula. Finally Chapter 9 shows how to use the nonsymmetric Jack polynomials to produce bases associated with the octahedral groups. This chapter has a short discussion of how these polynomials and related operators are used to solve the Schrödinger equations of the Calogero-Sutherland systems; these are exactly solvable models of quantum mechanics involving identical particles in a one dimensional space. Both Chapters 8 and 9 discuss orthogonal polynomials on the torus, and of Hermite type.

The bibliography is intended to be reasonably comprehensive into the near past; the reader is referred to Erdélyi et al. [1953] for older papers, and Internet data bases for the newest articles. There are occasions in the book where we suggest some algorithms for possible symbolic algebra use; the reader is encouraged to implement them in his/her

favorite computer algebra system; but again the reader is referred to the Internet for specific published software.

There are several areas of related current research that we have deliberately avoided: the role of special functions in the representation theory of Lie groups (see Dieudonné [1980], Hua [1963], Vilenkin [1968], Vilenkin and Klimyk [1991a,b,c, 1995]), basic hypergeometric series and orthogonal polynomials of q type (see Gasper and Rahman [1990], Andrews, Askey and Roy [1999]), quantum groups (Koornwinder [1992], Noumi [1996], Koelink [1996] and Stokman [1997]), Macdonald symmetric polynomials (a generalization of the q type) (see Macdonald [1995, 1998]). These topics touch on algebra, combinatorics and analysis; and some classical results can be obtained as limiting cases for $q \rightarrow 1$. Nevertheless, the material in this book can stand alone and ' q ' is not needed in the proofs.

We gratefully acknowledge support from the National Science Foundation over the years for our original research, some of which is described in this book. Also we are grateful to the mathematics departments of the University of Oregon for granting sabbatical leave and the University of Virginia for inviting Y. X. to visit for a year, which provided the opportunity for this collaboration.

Charles F. Dunkl
Yuan Xu

Contents

<i>Preface</i>	<i>page</i> xiii
1 Background	1
1.1 The Gamma and Beta Functions	1
1.2 Hypergeometric Series	3
1.3 Orthogonal Polynomials of One Variable	7
1.3.1 General properties	7
1.3.2 Three term recurrence	9
1.4 Classical Orthogonal Polynomials	13
1.4.1 Hermite polynomials	14
1.4.2 Laguerre polynomials	15
1.4.3 Gegenbauer polynomials	17
1.4.4 Jacobi polynomials	21
1.5 Modified Classical Polynomials	23
1.5.1 Generalized Hermite polynomials	25
1.5.2 Generalized Gegenbauer polynomials	26
1.5.3 A limiting relation	28
1.6 Notes	28
2 Examples of Orthogonal Polynomials in Several Variables	30
2.1 Notation and Preliminary	30
2.2 Spherical Harmonics	33
2.3 Classical Orthogonal Polynomials	37
2.3.1 Multiple Jacobi polynomials on the cube	37
2.3.2 Classical orthogonal polynomials on the unit ball	38
2.3.3 Classical orthogonal polynomials on the simplex	46
2.3.4 Multiple Hermite polynomials on \mathbb{R}^d	49
2.3.5 Multiple Laguerre polynomials on \mathbb{R}_+^d	51

2.4	Other Examples of Orthogonal Polynomials	51
2.4.1	Two general families of orthogonal polynomials	51
2.4.2	A method for generating orthogonal polynomials of two variables	55
2.4.3	Disc polynomials	57
2.5	Van der Corput-Schaaake Inequality	58
2.6	Notes	60
3	General Properties of Orthogonal Polynomials in Several Variables	63
3.1	Moment Functionals and Orthogonal Polynomials in Several Variables	64
3.1.1	Definition of orthogonal polynomials	64
3.1.2	Orthogonal polynomials and moment matrices	69
3.1.3	The moment problem	72
3.2	The Three Term Relation	75
3.2.1	Definition and basic properties	75
3.2.2	Favard's theorem	79
3.2.3	Centrally symmetric integrals	82
3.2.4	Examples	85
3.3	Jacobi Matrices and Commuting Operators	88
3.4	Further Properties of the Three Term Relation	95
3.4.1	Recurrence formula	95
3.4.2	General solutions of the three-term relation	102
3.5	Reproducing Kernels and Fourier Orthogonal Series	105
3.5.1	Reproducing kernels	106
3.5.2	Fourier orthogonal series	110
3.6	Common Zeros of Orthogonal Polynomials in Several Variables	113
3.7	Gaussian Cubature Formulae	117
3.7.1	Characterization of Gaussian cubature formulae	118
3.7.2	Examples of Gaussian cubature formulae	123
3.8	Orthogonal Polynomials on the Unit Sphere	126
3.8.1	Orthogonal structures on S^d and on B^d	126
3.8.2	Orthogonal structure on B^d and on S^{d+m}	132
3.9	Notes	134
4	Root Systems and Coxeter groups	137
4.1	Introduction and Overview	137
4.2	Root Systems	139
4.2.1	Type A_{d-1}	142

4.2.2	Type B_d	143
4.2.3	Type $I_2(m)$	144
4.2.4	Type D_d	144
4.2.5	Type H_3	145
4.2.6	Type F_4	146
4.2.7	Other types	146
4.2.8	Miscellaneous results	146
4.3	Invariant Polynomials	147
4.3.1	Type A_{d-1} invariants	149
4.3.2	Type B_d invariants	150
4.3.3	Type D_d invariants	150
4.3.4	Type $I_2(m)$ invariants	151
4.3.5	Type H_3 invariants	151
4.3.6	Type F_4 invariants	151
4.4	Differential-Difference Operators	151
4.5	The Intertwining Operator	157
4.6	The κ -Analog of the Exponential	167
4.7	Invariant Differential Operators	169
4.8	Notes	173
5	Spherical Harmonics Associated with Reflection Groups	175
5.1	h -Harmonic Polynomials	175
5.2	Inner Products on Polynomials	185
5.3	Reproducing Kernels and the Poisson Kernel	189
5.4	Integration of the Intertwining Operator	192
5.5	Example: Abelian Group \mathbb{Z}_2^d	197
5.6	Example: Dihedral Groups	205
5.6.1	An orthonormal basis of $\mathcal{H}_n(h_{\alpha,\beta}^2)$	206
5.6.2	Cauchy and Poisson kernels	214
5.7	The Fourier Transform	216
5.8	Notes	224
6	Classical and Generalized Classical Orthogonal Polynomials	225
6.1	Generalized Classical Orthogonal Polynomials on the Ball	225
6.1.1	Definition and differential-difference equations	225
6.1.2	Bases, reproducing kernels, and the Funk-Hecke formula	231
6.2	Orthogonal Polynomials on the Simplex	236
6.2.1	General weight functions on T^d	236
6.2.2	Generalized classical orthogonal polynomials	239

6.3	Generalized Hermite Polynomials	243
6.4	Generalized Laguerre Polynomials	249
6.5	Notes	253
7	Summability of Orthogonal Expansions	255
7.1	General Results on Orthogonal Expansions	255
7.1.1	Uniform convergence of partial sums	255
7.1.2	Cesàro means of the orthogonal expansion	259
7.2	Orthogonal Expansion on the Sphere	262
7.3	Orthogonal Expansion on the Ball	266
7.4	Orthogonal Expansions on the Simplex	271
7.5	Orthogonal Expansion of Laguerre and Hermite Polynomials	274
7.6	Multiple Jacobi Expansion	279
7.7	Notes	284
8	Orthogonal Polynomials Associated with Symmetric Groups	287
8.1	Introduction	287
8.2	Partitions, Compositions and Orderings	288
8.3	Commuting Self-Adjoint Operators	289
8.4	The Dual Polynomial Basis	292
8.5	S_d Invariant Subspaces	299
8.6	Degree Changing Recurrences	304
8.7	Norm Formulae	308
8.7.1	Hook length products and the pairing norm	308
8.7.2	The biorthogonal type norm	311
8.7.3	The torus inner product	313
8.7.4	Normalizing constants	316
8.8	Symmetric Functions and Jack Polynomials	321
8.9	Miscellaneous Topics	330
8.10	Notes	335
9	Orthogonal Polynomials Associated with Octahedral Groups and Applications	337
9.1	Introduction	337
9.2	Operators of Type B	338
9.3	Polynomial Eigenfunctions of Type B	341
9.4	Generalized Binomial Coefficients	350
9.5	Hermite Polynomials of Type B	359
9.6	Calogero-Sutherland Systems	360

9.6.1	The simple harmonic oscillator	361
9.6.2	Root systems and the Laplacian	362
9.6.3	Type <i>A</i> models on the line	363
9.6.4	Type <i>A</i> models on the circle	365
9.6.5	Type <i>B</i> models on the line	368
9.7	Notes	370
	<i>Bibliography</i>	372
	<i>Author index</i>	384
	<i>Symbol index</i>	387
	<i>Subject index</i>	389

1

Background

The theory of orthogonal polynomials of several variables, especially those of classical type, uses a significant amount of analysis in one variable. In this chapter we give concise descriptions of the needed tools.

1.1 The Gamma and Beta Functions

It is our feeling, or perhaps, our taste, that the most interesting objects of consideration have expressions which are rational functions of the underlying parameters. This immediately leads us to consideration of the gamma function and its relatives.

Definition 1.1.1 *The gamma function is defined for $\operatorname{Re} x > 0$ by the integral*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

It is directly related to the beta function:

Definition 1.1.2 *The beta function is defined for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$ by*

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

By changing variables $s = uv$ and $t = (1-u)v$ in the integral $\Gamma(x)\Gamma(y) = \int_0^{\infty} \int_0^{\infty} s^{x-1} t^{y-1} e^{-(s+t)} ds dt$, one obtains

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x, y).$$

This leads to several useful definite integrals (all valid for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$):

- (i) $\int_0^{\pi/2} \sin^{x-1} \theta \cos^{y-1} \theta d\theta = \frac{1}{2} B\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{y}{2}\right)}{\Gamma\left(\frac{x+y}{2}\right)}$;
- (ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (set $x = y = 1$ in the previous integral);
- (iii) $\int_0^\infty t^{x-1} \exp(-at^2) dt = \frac{1}{2} a^{-x/2} \Gamma\left(\frac{x}{2}\right)$, for $a > 0$;
- (iv) $\int_0^1 t^{x-1} (1-t^2)^{y-1} dt = \frac{1}{2} B\left(\frac{x}{2}, y\right) = \frac{1}{2} \Gamma\left(\frac{x}{2}\right) \Gamma(y) / \Gamma\left(\frac{x}{2} + y\right)$;
- (v) $\Gamma(x) \Gamma(1-x) = B(x, 1-x) = \frac{\pi}{\sin \pi x}$.

The last equation can be proven by restricting x by $0 < x < 1$, in the beta integral $\int_0^1 (t/(1-t))^{x-1} (1-t)^{-1} dt$ making the substitution $s = t/(1-t)$ and computing the resulting integral by residues. Of course one of the fundamental properties of the gamma function is the recurrence formula (integration by parts)

$$\Gamma(x+1) = x\Gamma(x),$$

which leads to the fact that Γ can be analytically continued to a meromorphic function on the complex plane; also $1/\Gamma$ is entire with (simple) zeros exactly at $\{0, -1, -2, \dots\}$. Note that Γ interpolates the factorial, indeed $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \dots$

Definition 1.1.3 *The Pochhammer symbol, also called the shifted factorial, is defined for all x by*

$$(x)_0 = 1, (x)_n = \prod_{i=1}^n (x+i-1) \quad \text{for } n = 1, 2, 3, \dots$$

Alternatively one recursively defines $(x)_n$ by $(x)_0 = 1$ and $(x)_{n+1} = (x)_n(x+n)$ for $n = 0, 1, 2, 3, \dots$. Here are some important consequences of the definition:

- (i) $(x)_{m+n} = (x)_m(x+m)_n$, for $m, n \in \mathbb{N}_0$;
- (ii) $(x)_n = (-1)^n (1-n-x)_n$ (writing the product in reverse order);
- (iii) $(x)_{n-i} = (x)_n (-1)^i / (1-n-x)_i$.

The Pochhammer symbol incorporates binomial coefficient and factorial notation:

$$(i) \quad (1)_n = n!, \quad 2^n \left(\frac{1}{2}\right)_n = 1 \times 3 \times 5 \times \dots \times (2n-1);$$