数理逻辑论文选

第一集

数 理 邏 輯 論 文 选

第 一 集

（选 自 外 文 杂 志）

前言

本书是为了数理邏輯課程的討論班作为参考資料以及数理邏輯工作者对能行性問題以及不可判定問題作深入研究的讀物而編輯的。

本书中所收集的論文一般符合以下的原則：

1. 这些論文主要是关于能行性問題以及不可判定性問題的重要文献。至于数理邏輯的其他方面：比如集合論、公理学、演繹論、归納邏輯、模态邏輯、数理邏輯中的哲学問題，以及在数学中已作出的一些判定問題，数理邏輯的应用，特别是計算机方面的应用等方面的論文，都不在本专集之內，我們盼望將来再出有关这些方面的专集。

2. 就是关于能行性問題以及不可判定性問題的重要文献，本书也不能說都搜罗已尽，这主要是由于：

(1)限于我們的見聞，可能遺漏不少；

(2)有些我們很希望收进来的論文，在我們力量所能及的几个图书館內，目前还沒能找到登載这些論文的书刊杂志，这只有待以后找到了再設法补印；

(3)也还有一些論文是可以收到本集中的，但因登載那些論文的书刊在我国已影印过了或卽將影印，讀者容易购到，因此，我們覚得沒有在本集中重印的必要。比如，收在 Automata Studies 一书中的关于 Turing Machine 的論文，Church 的 λ-Conversion, Марков 的 Теория Алгорифмов. Péter 的 Rekursive Funktionen, Kleene 的 Two Papers on the predicate Calculus 以及收在荷兰出版的 Studies in Logic and the Foundation of Mathematics 从书里的一些論文等等。

3. 本书中所收集的論文也未必都是直接关于能行性問題以及不可判定性問題的，有少数几篇論文或者是因为与这些問題間接有关，或者是因为它本身非常重要。本书服务的对象急需用它，因此我們也收在本集里，比如 Church 的 A Set of Postulates for the Foundation of Logic, Gödel 的 Die Vollständigkeit der Axiome de logischen Funktionenkalküls 等等。

最后，还要說明一点，本书因需急于付印，編輯是很忽促的，可供編选的資料也很不完备，再以我們人力薄弱，缺点一定很多，希各方面学者多提意見，以便將来再編續集或再版时能夠改进。

編　者　1958 年 7 月

附注：1. Kleene, S.C., General Recursive Functions of Natural Numbrs 一文有更正和簡化数处，載于 *The Journal of symbolic Logic* vol. 3 p. 152, vol. 2 p. 38, vol. 4 top p. iv at end, 本书未列入，讀者可自已査对。

2. Kleene, S.C., Recursive Predicates and Quantifiers 一文 §15 据作者原注應略去不讀。

CONTENTS

数理邏輯論文选

第一集

（选自外文杂志）

A SET OF POSTULATES FOR THE FOUNDATION OF LOGIC.[1]

BY ALONZO CHURCH.[2]

1. **Introduction.** In this paper we present a set of postulates for the foundation of formal logic, in which we avoid use of the free, or real, variable, and in which we introduce a certain restriction on the law of excluded middle as a means of avoiding the paradoxes connected with the mathematics of the transfinite.

Our reason for avoiding use of the free variable is that we require that every combination of symbols belonging to our system, if it represents a proposition at all, shall represent a particular proposition, unambiguously, and without the addition of verbal explanations. That the use of the free variable involves violation of this requirement, we believe is readily seen. For example, the identity

$$a(b+c) = ab+ac \tag{1}$$

in which a, b, and c are used as free variables, does note state a definite proposition unless it is known what values may be taken on by these variables, and this information, if not implied in the context, must be given by a verbal addition. The range allowed to the variables a, b, and c might consist of all real numbers, or of all complex numbers, or of some other set, or the ranges allowed to the variables might differ, and for each possibility equation (1) has a different meaning. Clearly, when this equation is written alone, the proposition intended has not been completely translated into symbolic language, and, in order to make the translation complete, the necessary verbal addition must be expressed by means of the symbols of formal logic and included, with the equation, in the formula used to represent the proposition. When this is done we obtain, say,

$$R(a)\,R(b)\,R(c) \supset_{abc} \cdot\, a(b+c) = ab+ac \tag{2}$$

where $R(x)$ has the meaning, "x is a real number," and the symbol $\supset_{abc}$ has the meaning described in §§ 5 and 6 below. And in this expression there are no free variables.

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[1] Received October 5, 1931.

[2] This paper contains, in revised form, the work of the author while a National Research Fellow in 1928–29.

A further objection to the use of the free variable is contained in the duplication of symbolism which arises when the free, or real, variable and the bound, or apparent, variable are used side by side.[3] Corresponding to the proposition, represented by equation (1) when a, b, and c stand for any three real numbers, there is also a proposition expressed without the use of free variables, namely (2), and between these two propositions we know of no convincing distinction. An attempt to identify the two propositions is, however, unsatisfactory, because substitution of (1) for (2), when the latter occurs as a part of a more complicated expression, cannot always be allowed without producing confusion. In fact, the only feasible solution seems to be the complete abandonment of the free variable as a part of the symbolism of formal logic.[4]

Rather than adopt the method of Russell for avoiding the familiar paradoxes of mathematical logic,[5] or that of Zermelo,[6] both of which appear somewhat artificial, we introduce for this purpose, as we have said, a certain restriction on the law of excluded middle. This restriction consists in leaving open the possibility that a propositional function F may, for some values X of the independent variable, represent neither a true proposition nor a false proposition. For such a value X of the independent variable we suppose that $\mathsf{F}(\mathsf{X})$ is undefined and represents nothing, and we use a system of logical symbols capable of dealing with propositional functions whose ranges of definition are limited.

In the case of the Russell paradox the relevance of this proposed restriction on the law of excluded middle is evident. The formula P which leads to this paradox may be written, in the notation explained below, $\{\lambda\varphi . \sim\varphi(\varphi)\}(\lambda\varphi . \sim\varphi(\varphi))$. It has the property that if we assume $\sim\mathsf{P}$ then we can infer P and if we assume P then we can infer $\sim\mathsf{P}$. On ordinary assumptions both the truth and the falsehood of P can be deduced in consequence of this property, but the system of this paper, while it provides for the existence of a propositional function $\lambda\varphi . \sim\varphi(\varphi)$ does not provide either that this propositional function shall be true or that it shall be false, for the value $\lambda\varphi . \sim\varphi(\varphi)$ of the independent variable.

Other paradoxes either disappear in the same way, or else, as in the case of the Epimenides or the paradox of the least undefinable ordinal,

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[3] Cf. the introduction to the second edition of Whitehead and Russell's *Principia Mathematica*.

[4] Unless it is retained as a mere abbreviation of notation.

[5] B. Russell, Mathematical Logic as based on the Theory of Types, Amer. Jour. Math., vol. 30 (1908), pp. 222–262. A list of some of these paradoxes, with a reference to the source of each, will be found in this article, or in Whitehead and Russell, *Principia Mathematica*, vol. 1, pp. 63–64.

[6] E. Zermelo, Untersuchungen über die Grundlagen der Mengenlehre, Math. Annalen, vol. 65 (1908), pp. 261–281.

they contain words which are not definable in terms of the undefined symbols of our system, and hence need not concern us.

The paradox of Burali-Forti is not, however, so readily disposed of. The question whether this paradox is a consequence of our postulates, or what modification of them will enable us to avoid it, probably must be left open until the theory of ordinal numbers which results from the postulates has been developed.

Whether the system of logic which results from our postulates is adequate for the development of mathematics, and whether it is wholly free from contradiction, are questions which we cannot now answer except by conjecture. Our proposal is to seek at least an empirical answer to these questions by carrying out in some detail a derivation of the consequences of our postulates, and it is hoped either that the system will turn out to satisfy the conditions of adequacy and freedom from contradiction or that it can be made to do so by modifications or additions.

2. **Relation to intuitionism.** Since, in the postulate set which is given below, the law of the excluded middle is replaced by weaker assumptions, the question arises what the relation is between the system of logic which results from this set, and the intuitionism of L. E. J. Brouwer.[7]

The two systems are not the same, because, although both reject a certain part of the principle of the excluded middle, the parts rejected are different. The law of double negation, denied by Brouwer, is preserved in the system of this paper, and the principle, which Brouwer accepts, that a statement[8] from which a contradiction can be inferred is false, we find it necessary to abandon in certain cases.

Our system appears, however, to have the property, which relates it to intuitionism, that a statement of the form $\Sigma x . F(x)$ (read, "there exists x such that $F(x)$") is never provable unless there exists a formula M such that $F(M)$ is provable.

3. **The abstract character of formal logic.** We do not attach any character of uniqueness or absolute truth to any particular system of logic. The entities of formal logic are abstractions, invented because of their use in describing and systematizing facts of experience or observation, and their properties, determined in rough outline by this intended use, depend for their exact character on the arbitrary choice of the inventor.

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[7] See L. E. J. Brouwer, Intuitionistische Mengenlehre, Jahresbericht der D. Math. Ver., vol. 28 (1919), pp. 203–208, and Mathematik, Wissenschaft und Sprache, Monatshefte für Math. u. Phys., vol. 36 (1929), pp. 153–164, and many other papers.

[8] We purposely use the word, "statement", because we wish to reserve the word, "proposition", for something either true or false. A statement, in the form of a proposition, which fails to be either true or false, we regard as a mere group of symbols, without significance.

We may draw the analogy of a three dimensional geometry used in describing physical space, a case for which, we believe, the presence of such a situation is more commonly recognized. The entities of the geometry are clearly of abstract character, numbering as they do planes without thickness and points which cover no area in the plane, point sets containing an infinitude of points, lines of infinite length, and other things which cannot be reproduced in any physical experiment. Nevertheless the geometry can be applied to physical space in such a way that an extremely useful correspondence is set up between the theorems of the geometry and observable facts about material bodies in space. In building the geometry, the proposed application to physical space serves as a rough guide in determining what properties the abstract entities shall have, but does not assign these properties completely. Consequently there may be, and actually are, more than one geometry whose use is feasible in describing physical space. Similarly, there exist, undoubtedly, more than one formal system whose use as a logic is feasible, and of these systems one may be more pleasing or more convenient than another, but it cannot be said that one is right and the other wrong.

In consequence of this abstract character of the system which we are about to formulate, it is not admissible, in proving theorems of the system, to make use of the meaning of any of the symbols, although in the application which is intended the symbols do acquire meanings. The initial set of postulates must of themselves define the system as a formal structure, and in developing this formal structure reference to the proposed application must be held irrelevant. There may, indeed, be other applications of the system than its use as a logic.

4. Intuitive logic. It is clear, however, that formulas composed of symbols to which no meaning is attached cannot define a procedure of proof or justify an inference from one formula to another. If our postulates were expressed wholly by means of the symbols of formal logic without use of any words or symbols having a meaning, there would be no theorems except the postulates themselves. We are therefore obliged to use in some at least of our postulates other symbols than the undefined terms of the formal system, and to presuppose a knowledge of the meaning of these symbols, as well as to assume an understanding of a certain body of principles which these symbols are used to express, and which belong to what we shall call intuitive logic.[9] It seems desirable

[9] The principles of intuitive logic which we assume initially form, of course, a part of the body of facts to which the formal system, when completed, is to be applied. We should not, however, allow this to confuse us as to the clear cut distinction between intuitive logic and formal logic.

to make these presuppositions as few and as simple as we can, but there is no possibility of doing without them.

Before proceeding to the statement of our postulates, we shall attempt to make a list of these principles of intuitive logic which we find it necessary to assume and of the symbols a knowledge of whose meanings we presuppose. The latter belong to what we shall call the language of intuitive logic, as distinguished from the language of formal logic which is made up of the undefined terms of our abstract system.

We assume that we know the meaning of the words *symbol* and *formula* (by the word formula we mean a set of symbols arranged in an order of succession, one after the other). We assume the ability to write symbols and to arrange them in a certain order on a page, and the ability to recognize different occurrences of *the same* symbol and to distinguish between such a double occurrence of a symbol and the occurrence of *distinct* symbols. And we assume the possibility of dealing with a formula as a unit, of copying it at any desired point, and of recognizing other formulas as being *the same* or *distinct*.

We assume that we know what it means to say that a certain symbol or formula *occurs* in a given formula, and also that we are able to pick out and discuss a particular *occurrence* of one formula in another.

We assume an understanding of the operation of *substituting* a given symbol or formula *for a particular occurrence* of a given symbol or formula.

And we assume also an understanding of the operation of substitution throughout a given formula, and this operation we indicate by an S, $S^{\mathbf{X}}_{\mathbf{Y}}\mathbf{U}|$ representing the formula which results when we operate on the formula $\mathbf{U}$ by replacing $\mathbf{X}$ by $\mathbf{Y}$ throughout, where $\mathbf{Y}$ may be any symbol or formula but $\mathbf{X}$ must be a single symbol, not a combination of several symbols.

We assume that we know how to recognize a given formula as *being obtainable* from the formulas of a certain set by *repeated combinations* of the latter according to a given law. This assumption is used below in defining the term "well-formed". It may be described as an assumption of the ability to make a definition by induction, when dealing with groups of symbols.

We assume the ability to make the assertion that a given formula is one of those belonging to the abstract system which we are constructing, and this assertion we indicate by the words *is true*. As an abbreviation, however, we shall usually omit the words *is true*, the mere placing of the formula in an isolated position being taken as a sufficient indication of them.

We assume further the meaning and use of the word *every* as part of the language of intuitive logic, and the use in connection with it of variable letters, which we write in bold face type to distinguish them from variable letters used in the language of formal logic. These variable letters, written in bold face type, stand always for a variable (or undetermined) symbol or formula.

We assume the meaning and use of the following words from the language of intuitive logic: *there is, and, or, if ··· then, not,* and *is* in the sense of identity. That is, we assume that we know what combinations of these words with themselves and our other symbols constitute propositions, and, in a simple sense, what such propositions mean.

We assume that we know how to distinguish between the words and symbols we have been enumerating, which we shall describe as *symbols of intuitive logic,* and other symbols, which are mere symbols, without meaning, and which we shall describe as *formal symbols.*

We assume that we know what it is to be a proposition of intuitive logic, and that we are able to assert such propositions, not merely one proposition, but various propositions in succession. And, finally, we assume the permanency of a proposition once asserted, so that it may at any later stage be reverted to and used as if just asserted.

In making the preceding statements it becomes clear that a certain circle is unavoidable in that we are unable to make our explanations of the ideas in question intelligible to any but those who already understand at least a part of these ideas. For this reason we are compelled to assume them as known in the beginning independently of our statement of them. Our purpose has been, not to explain or convey these ideas, but to point out to those who already understand them what the ideas are to which we are referring and to explain our symbolism for them.

5. **Undefined terms.** We are now ready to set down a list of the undefined terms of our formal logic. They are as follows:

$$\{\ \}(\ ),\ \lambda[\ ],\ \Pi,\ \Sigma,\ \&,\ \sim,\ \iota,\ A.$$

The expressions $\{\ \}(\ )$ and $\lambda[\ ]$ are not, of course, single symbols, but sets of several symbols, which, however, in every formula which will be provable as a consequence of our postulates, always occur in groups in the order here given, with other symbols or formulas between as indicated by the blank spaces.

In addition to the undefined terms just set down, we allow the use, in the formulas belonging to the system which we are constructing, of any other formal symbol, and these additional symbols used in our formulas we call *variables.*

An occurrence of a variable **x** in a given formula is called an occurrence of **x** as a *bound variable* in the given formula if it is an occurrence of **x** in a part of the formula of the form λ**x**[**M**]; that is, if there is a formula **M** such that λ**x**[**M**] occurs in the given formula and the occurrence of **x** in question is an occurrence in λ**x**[**M**]. All other occurrences of a variable in a formula are called occurrences as a *free variable*.

A formula is said to be *well-formed* if it is a variable, or if it is one of the symbols Π, Σ, &, ~, ι, A, or if it is obtainable from these symbols by repeated combinations of them of one of the forms {**M**}(**N**) and λ**x**[**M**], where **x** is any variable and **M** and **N** are symbols or formulas which are being combined. This is a definition by induction. It implies the following rules: (1) a variable is well-formed (2) Π, Σ, &, ~, ι, and A are well-formed (3) if **M** and **N** are well-formed then {**M**}(**N**) is well-formed (4) if **x** is a variable and **M** is well-formed then λ**x**[**M**] is well-formed.

All the formulas which will be provable as consequences of our postulates will be well-formed and will contain no free variables.

The undefined terms of a formal system have, as we have explained, no meaning except in connection with a particular application of the system. But for the formal system which we are engaged in constructing we have in mind a particular application, which constitutes, in fact, the motive for constructing it, and we give here the meanings which our undefined terms are to have in this intended application.

If **F** is a function and **A** is a value of the independent variable for which the function is defined, then {**F**}(**A**) represents the value taken on by the function **F** when the independent variable takes on the value **A**. The usual notation is **F**(**A**). We introduce the braces on account of the possibility that **F** might be a combination of several symbols, but, in the case that **F** is a single symbol, we shall often use the notation **F**(**A**) as an abbreviation for the fuller expression.[10]

Adopting a device due to Schönfinkel,[11] we treat a function of two variables as a function of one variable whose values are functions of one variable, and a function of three or more variables similarly. Thus, what is usually written **F**(**A**,**B**) we write {{**F**}(**A**)}(**B**), and what is usually written **F**(**A**, **B**, **C**) we write {{{**F**}(**A**)}(**B**)}(**C**), and so on. But again we frequently find it convenient to employ the more usual notations as abbreviations.

If **M** is any formula containing the variable **x**, then λ**x**[**M**] is a symbol for the function whose values are those given by the formula. That is,

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[10] The braces { } are, as a matter of fact, superfluous and might have been omitted from our list of undefined terms, but their inclusion makes for readability of formulas.

[11] M. Schönfinkel, Über die Bausteine der mathematischen Logik, Math. Annalen, vol. 92 (1924) pp. 305–316.

$\lambda x[\mathbf{M}]$ represents a function, whose value for a value $\mathbf{L}$ of the independent variable is equal to the result $S_{\mathbf{L}}^{x}\mathbf{M}|$ of substituting $\mathbf{L}$ for x throughout $\mathbf{M}$, whenever $S_{\mathbf{L}}^{x}\mathbf{M}|$ turns out to have a meaning, and whose value is in any other case undefined.

The symbol Π stands for a certain propositional function of two independent variables, such that $\Pi(\mathbf{F}, \mathbf{G})$ denotes, "$\mathbf{G}(x)$ is a true proposition for all values of x for which $\mathbf{F}(x)$ is a true proposition." It is necessary to distinguish between the proposition $\Pi(\mathbf{F}, \mathbf{G})$ and the proposition $x \cdot \mathbf{F}(x) \supset \mathbf{G}(x)$ (read, "For every x, $\mathbf{F}(x)$ implies $\mathbf{G}(x)$"). The latter proposition justifies, for any value $\mathbf{M}$ of x, the inference $\mathbf{F}(\mathbf{M}) \supset \mathbf{G}(\mathbf{M})$, and hence can be used only in the case that the functions $\mathbf{F}$ and $\mathbf{G}$ are defined for all values of their respective independent variables. The proposition $\Pi(\mathbf{F}, \mathbf{G})$ does not, on the other hand, justify this inference, although, when $\{\mathbf{F}\}(\mathbf{M})$ is known to be true, it does justify the inference $\{\mathbf{G}\}(\mathbf{M})$. And the proposition $\Pi(\mathbf{F}, \mathbf{G})$ is, therefore, suitable for use in the case that the ranges of definition of the functions $\mathbf{F}$ and $\mathbf{G}$ are limited.

The symbol Σ stands for a certain propositional function of one independent variable, such that $\Sigma(\mathbf{F})$ denotes, "There exists at least one value of x for which $\mathbf{F}(x)$ is true."

The symbol & stands for a certain propositional function of two independent variables, such that, if $\mathbf{P}$ and $\mathbf{Q}$ are propositions, $\&(\mathbf{P}, \mathbf{Q})$ is the logical product $\mathbf{P}$-and-$\mathbf{Q}$.

The symbol $\sim$ stands for a certain propositional function of one independent variable, such that, if $\mathbf{P}$ is a proposition, then $\sim(\mathbf{P})$ is the negation of $\mathbf{P}$ and may be read, "Not-$\mathbf{P}$".

The symbol ι stands for a certain function of one independent variable, such that, if $\mathbf{F}$ is a propositional function of one independent variable, then $\iota(\mathbf{F})$ denotes, "The object x such that $\{\mathbf{F}\}(x)$ is true."

The symbol A stands for a certain function of two independent variables, the formula $A(\mathbf{F}, \mathbf{M})$ being read, "The abstraction from $\mathbf{M}$ with respect to $\mathbf{F}$."

6. **Abbreviations and definitions.** In practice we do not use actually the notation just described, but introduce various abbreviations and substituted notations, partly for the purpose of shortening our formulas and partly in order to render them more readable. We do not, however, regard these abbreviations as an essential part of our theory but rather as extraneous. When we use them we do not literally carry out the development of our system, but we do indicate in full detail how this development can be carried out, and this is for our purpose sufficient.

As has been said above, we use $\{\mathbf{F}\}(\mathbf{A}, \mathbf{B})$ as an abbreviation for $\{\{\mathbf{F}\}(\mathbf{A})\}(\mathbf{B})$ and similarly in the case of functions of larger numbers of variables. Moreover, alike in the case of functions of one variable and in the case of functions of two or more variables, we omit the braces { } whenever the function is represented by a single symbol rather than by an expression consisting of several symbols. Thus, if F is a single symbol, we write $\mathsf{F}(\mathsf{A})$ instead of $\{\mathsf{F}\}(\mathsf{A})$ and $\mathsf{F}(\mathsf{A}, \mathsf{B})$ instead of $\{\mathsf{F}\}(\mathsf{A}, \mathsf{B})$ or $\{\{\mathsf{F}\}(\mathsf{A})\}(\mathsf{B})$.

We shall usually write $[\mathsf{M}][\mathsf{N}]$ instead of $\&(\mathsf{M}, \mathsf{N})$ or $\{\{\&\}(\mathsf{M})\}(\mathsf{N})$, and $\sim[\mathsf{M}]$ instead of $\sim(\mathsf{M})$ or $\{\sim\}(\mathsf{M})$.

When F is a single symbol, we shall often write $\mathsf{F}\mathsf{x}[\mathsf{M}]$ instead of $\mathsf{F}(\lambda\mathsf{x}[\mathsf{M}])$.

Instead of $\Pi(\lambda\mathsf{x}[\mathsf{M}], \lambda\mathsf{x}[\mathsf{N}])$, we shall write $[\mathsf{M}] \supset_{\mathsf{x}} [\mathsf{N}]$. And $[\Sigma\mathsf{y}[\mathsf{M}]] \supset_{\mathsf{x}} [[\mathsf{M}] \supset_{\mathsf{y}} [\mathsf{N}]]$ we abbreviate further to $[\mathsf{M}] \supset_{\mathsf{xy}} [\mathsf{N}]$, and $[\Sigma\mathsf{y}[\Sigma\mathsf{z}[\mathsf{M}]]] \supset_{\mathsf{x}} [[\Sigma\mathsf{z}[\mathsf{M}]] \supset_{\mathsf{y}} [[\mathsf{M}] \supset_{\mathsf{z}} [\mathsf{N}]]]$ we abbreviate to $[\mathsf{M}] \supset_{\mathsf{xyz}} [\mathsf{N}]$ and so on.

Moreover, whenever possible without ambiguity, we omit square brackets [ ], whether the brackets belong to the undefined term $\lambda[\ ]$ or whether they appear as a part of one of our abbreviations. In order to allow the omission of square brackets as often as possible, we adopt the convention that, whenever there are more than one possibility, the extent of the omitted square brackets shall be taken as the shortest. And when the omission of the square brackets is not possible without ambiguity, we can sometimes substitute for them a dot, or period. This dot, when it occurs within a parenthesis, enclosed by either square brackets [ ], round parentheses ( ), or braces { }, stands for square brackets extending from the place where the dot occurs and up to the end of the parenthesis, or, if the parenthesis is divided into sections by commas as in the case of functions of two or more variables, extending from the place where the dot occurs and up to the first of these commas or to the end of the parenthesis, whichever is first reached. And when the dot is not within any parenthesis, it stands for square brackets extending from the place where it occurs and up to the end of the entire formula. In other words, a dot represents square brackets extending the greatest possible distance forward from the point where it occurs.

In addition to these abbreviations, we allow freely the introduction of abbreviations of a simpler sort, which we call definitions,[12] and which consist in the substitution of a particular single symbol for a particular well-formed formula containing no free variables.

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[12] There seems to be, as a matter of fact, no serious objection to treating definitions as an essential part of the system rather than as extraneous, but we believe it more consistent to class them with our other and more complicated abbreviations.

We introduce at once the following definitions, using an arrow $\longrightarrow$ to mean "Stands for", or, "Is an abbreviation for":

$V \longrightarrow \lambda\mu\lambda\nu . \sim . \sim\mu . \sim\nu$

$U \longrightarrow \lambda\mu\lambda\nu . \sim . \mu . \sim\nu$

$Q \longrightarrow \lambda\mu\lambda\nu . \Pi(\mu, \nu) . \Pi(\nu, \mu)$

$E \longrightarrow \lambda\pi\Sigma\varphi . \varphi(\pi)$

$V(\mathbf{P}, \mathbf{Q})$ is to be read, "**P** or **Q**", $U(\mathbf{P}, \mathbf{Q})$ is to be read, "**P** implies **Q**", $Q(\mathbf{F}, \mathbf{G})$ is to be read "**F** and **G** are equivalent" and $E(\mathbf{M})$ is to be read "**M** exists"

And in connection with these symbols just defined we introduce some further abbreviations. For $V(\mathbf{P}, \mathbf{Q})$ we write $[\mathbf{P}] \vee [\mathbf{Q}]$, and for $U(P, Q)$ we write $[\mathbf{P}] \supset [\mathbf{Q}]$. We abbreviate $Q(\lambda\mathbf{x} . \mathbf{M}, \lambda\mathbf{x} . \mathbf{N})$ to $[\mathbf{M}] \equiv_{\mathbf{x}} [\mathbf{N}]$. And we abbreviate $E(\mathbf{x}) \supset_{\mathbf{x}} [\mathbf{M}]$ to $`\mathbf{x}[\mathbf{M}]$, which may be read, "For every **x**, **M**".

The notion of a class may be introduced by means of the definition:

$$K \longrightarrow A(Q).$$

The formula $K(\mathbf{F})$ is then to be read, "the class of x's such that $\{\mathbf{F}\}(x)$ true."

7. Postulates. We divide our postulates into two groups, of which the first consists of what we shall call rules of procedure and the second of what we shall call formal postulates. The latter assert that a given formula is true, and contain nothing from the language of intuitive logic other than the words *is true* (and even these words, as already explained, we leave unexpressed when we write the postulates). And the former, the rules of procedure, contain other words from the language of intuitive logic.

The theorems which are proved as consequences of these postulates are of the same form as the postulates of the first group, namely, that a certain formula is true. And the proof of a theorem consists of a series of steps which, from a set of one or more postulates of the first group as a starting point, leads us to the theorem, each step being justified by an appeal to one of the rules of procedure.

The postulates of our first group, the rules of procedure, are five in number:

I. *If* **J** *is true, if* **L** *is well-formed, if all the occurrences of the variable* **x** *in* **L** *are occurrences as a bound variable, and if the variable* **y** *does not occur in* **L**, *then* **K**, *the result of substituting* $S^{\mathbf{x}}_{\mathbf{y}}\mathbf{L}|$ *for a particular occurrence of* **L** *in* **J**, *is also true.*

II. *If* **J** *is true, if* **M** *and* **N** *are well-formed, if the variable* **x** *occurs in* **M**, *and if the bound variables in* **M** *are distinct both from the variable* **x** *and from the free variables in* **N**, *then* **K**, *the result of substituting* $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ *for a particular occurrence of* $\{\lambda\mathbf{x} . \mathbf{M}\}(\mathbf{N})$ *in* **J**, *is also true.*

III. *If* J *is true, if* M *and* N *are well-formed, if the variable* x *occurs in* M, *and if the bound variables in* M *are distinct both from* x *and from the free variables in* N, *then* K, *the result of substituting* $\{\lambda \mathsf{x}\,.\,\mathsf{M}\}(\mathsf{N})$ *for a particular occurrence of* $S^{\mathsf{x}}_{\mathsf{N}}\mathsf{M}|$ *in* J, *is also true.*

IV. *If* $\{\mathsf{F}\}(\mathsf{A})$ *is true and* F *and* A *are well-formed, then* $\Sigma(\mathsf{F})$ *is true.*

V. *If* $\Pi(\mathsf{F}, \mathsf{G})$ *and* $\{\mathsf{F}\}(\mathsf{A})$ *are true, and* F, G, *and* A *are well-formed, then* $\{\mathsf{G}\}(\mathsf{A})$ *is true.*

And our formal postulates are the thirty-seven following:

1. $\Sigma(\varphi) \supset_\varphi \Pi(\varphi, \varphi)$.
2. $`x \,.\, \varphi(x) \supset_\varphi . \Pi(\varphi, \psi) \supset_\psi \psi(x)$.
3. $\Sigma(\sigma) \supset_\sigma . [\sigma(x) \supset_x \varphi(x)] \supset_\varphi . \Pi(\varphi, \psi) \supset_\psi . \sigma(x) \supset_x \psi(x)$.
4. $\Sigma(\varrho) \supset_\varrho . \Sigma y[\varrho(x) \supset_x \varphi(x, y)] \supset_\varphi . [\varrho(x) \supset_x \Pi(\varphi(x), \psi(x))] \supset_\psi .$ $[\varrho(x) \supset_x \varphi(x, y)] \supset_y . \varrho(x) \supset_x \psi(x, y)$.
5. $\Sigma(\varphi) \supset_\varphi . \Pi(\varphi, \psi) \supset_\psi . \varphi(f(x)) \supset_{fx} \psi(f(x))$.
6. $`x \,.\, \varphi(x) \supset_\varphi . \Pi(\varphi, \psi(x)) \supset_\psi \psi(x, x)$.
7. $\varphi(x, f(x)) \supset_{\varphi f x} . \Pi(\varphi(x), \psi(x)) \supset_\psi \psi(x, f(x))$.
8. $\Sigma(\varrho) \supset_\varrho . \Sigma y[\varrho(x) \supset_x \varphi(x, y)] \supset_\varphi . [\varrho(x) \supset_x \Pi(\varphi(x), \psi)] \supset_\psi .$ $[\varrho(x) \supset_x \varphi(x, y)] \supset_y \psi(y)$.
9. $`x \,.\, \varphi(x) \supset_\varphi \Sigma(\varphi)$.
10. $\Sigma x \varphi(f(x)) \supset_{f\varphi} \Sigma(\varphi)$.
11. $\varphi(x, x) \supset_{\varphi x} \Sigma(\varphi(x))$.
12. $\Sigma(\varphi) \supset_\varphi \Sigma x \varphi(x)$.
13. $\Sigma(\varphi) \supset_\varphi . [\varphi(x) \supset_x \psi(x)] \supset_\psi \Pi(\varphi, \psi)$.
14. $p \supset_p . q \supset_q pq$.
15. $pq \supset_{qp} p$.
16. $pq \supset_{pq} q$.
17. $\Sigma x \Sigma \theta\, [\varphi(x) \,.\, \sim \theta(x) \,.\, \Pi(\psi, \theta)] \supset_{\varphi\psi} \sim \Pi(\varphi, \psi)$.
18. $\sim \Pi(\varphi, \psi) \supset_{\varphi\psi} \Sigma x \Sigma \theta \,.\, \varphi(x) \,.\, \sim \theta(x) \,.\, \Pi(\psi, \theta)$.
19. $\Sigma x \Sigma \theta\, [\sim \varphi(u, x) \,.\, \sim \theta(u) \,.\, \Sigma(\varphi(y)) \supset_y \theta(y)] \supset_{\varphi u} \sim \Sigma(\varphi(u))$.
20. $\sim \Sigma(\varphi) \supset_\varphi \Sigma x \,.\, \sim \varphi(x)$.
21. $p \supset_p . \sim q \supset_q \sim . pq$.
22. $\sim p \supset_p . q \supset_q \sim . pq$.
23. $\sim p \supset_p . \sim q \supset_q \sim . pq$.
24. $p \supset_p . [\sim . pq] \supset_q \sim q$.

25. $[\sim . \varphi(u) . \psi(u)] \supset_{\varphi\psi u} . [[\varphi(x) . \sim \psi(x)] \supset_x \varrho(x)] .$
$[[\sim \varphi(x) . \psi(x)] \supset_x \varrho(x)] . [\sim \varphi(x) . \sim \psi(x)] \supset_x \varrho(x)] \supset_\varrho \varrho(u).$

26. $p \supset_p \sim \sim p.$

27. $\sim \sim p \supset_p p.$

28. $\sim \Sigma(\varphi) \supset_\varphi . \Sigma(\psi) \supset_\psi \Pi(\varphi, \psi).$

29. $\sim \Sigma(\varphi) \supset_\varphi . \sim \Sigma(\psi) \supset_\psi \Pi(\varphi, \psi).$

30. $'x . \varphi(x) \supset_\varphi . [\theta(x) . \psi(x) \supset_\psi \Pi(\theta, \psi)] \supset_\theta \varphi(\iota(\theta)).$

31. $\varphi(\iota(\theta)) \supset_{\theta\varphi} \Pi(\theta, \varphi).$

32. $E(\iota(\theta)) \supset_\theta \Sigma(\theta).$

33. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z)] \; [\varphi(x, y) \supset_{xy} \varphi(y, x)] \supset_\varphi . \varphi(u, v) \supset_{uv} .$
$\psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v)).$

34. $[\varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z)] \; [\varphi(x, y) \supset_{xy} \varphi(y, x)] \supset_\varphi .$
$[\varphi(x, y) \supset_{xy} \theta(x, y)] \supset_\theta . \sim \theta(u, v) \supset_{uv} \sim . \psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v)).$

35. $[\psi(A(\varphi, u)) \supset_\psi \psi(A(\varphi, v))] \supset_{\varphi uv} \varphi(u, v).$

36. $E(A(\varphi)) \supset_\varphi . \varphi(x, y) \supset_{xy} . \varphi(y, z) \supset_z \varphi(x, z).$

37. $E(A(\varphi)) \supset_\varphi . \varphi(x, y) \supset_{xy} \varphi(y, x).$

8. **The relation between free and bound variables.** By a *step* in a proof we mean an application of one of the rules of procedure IV or V, occurring in the course of the proof. And in counting the *number of steps* in a proof, each step is to be counted with its proper multiplicity. That is, if a formula **M** is proved and then used r times as a premise for subsequent steps of the proof, then each step in the proof of **M** is to be counted r times.

If **M** and **N** are well-formed and if **N** can be derived from **M** by successive applications of the rules of procedure I, II, and III, then **M** is said to be *convertible* into **N**, and the process is spoken of as a *conversion* of **M** into **N**.

The formula **N** is said to be *provable as a consequence of* the formula **M**, if **M** is well-formed, and **N** could be made a provable formula by adding **M** to our list of formal postulates as a thirty-eighth postulate. Either of the formulas, **M** or **N**, or both, may contain free variables, since although none of our formal postulates contains free variables, there is, formally, nothing to prevent our adding a thirty-eighth postulate which does contain free variables.

We conclude by proving about our system of postulates the three following theorems: