

数学精品系列（英文版）

Volume II

# Physics and Partial Differential Equations

物理学与偏微分方程（下册）

Tatsien Li  
Tiehu Qin

*Translated by Yachun Li*

高等教育出版社



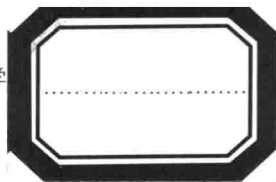
# Physics and Partial Differential Equations

Second Edition

Tianyou Li  
Tiancheng Qiu

Department of Mathematics  
Tsinghua University

数学



Volume II

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WULIXUE YU PIANWEIFEN FANGCHENG

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**Tatsien Li**  
School of Mathematical Sciences  
Fudan University  
Shanghai 200433, China  
Email: dqli@fudan.edu.cn

**Tiehu Qin**  
School of Mathematical Sciences  
Fudan University  
Shanghai 200433, China  
Email: thqin@fudan.edu.cn

### 图书在版编目(CIP)数据

物理学与偏微分方程 = Physics and partial differential equations. 下册: 英文 / 李大潜, 秦铁虎著; 李亚纯译.

— 北京: 高等教育出版社, 2014.9

ISBN 978-7-04-040850-8

I. ①物… II. ①李… ②秦… ③李… III. ①物理学—英文 ②偏微分方程—英文 IV. ①O4 ②O175.2

中国版本图书馆 CIP 数据核字 (2014) 第 175312 号

策划编辑 赵天夫  
责任印制 韩刚

责任编辑 赵天夫

封面设计 张楠

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出版发行	高等教育出版社	咨询电话	400-810-0598
社址	北京市西城区德外大街4号	网址	<a href="http://www.hep.edu.cn">http://www.hep.edu.cn</a>
邮政编码	100120		<a href="http://www.hep.com.cn">http://www.hep.com.cn</a>
印刷	涿州市星河印刷有限公司	网上订购	<a href="http://www.landraco.com">http://www.landraco.com</a>
开本	787×960 1/16		<a href="http://www.landraco.com.cn">http://www.landraco.com.cn</a>
印张	17.75	版次	2014年9月第1版
字数	370千字	印次	2014年9月第1次印刷
购书热线	010-58581118	定价	79.00元

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物料号 40850-00

# Preface to the English Edition

The first volume of the Chinese edition of this book was published in July 1997, and the second volume was published in June 2000. In July 2000, upon the readers' request, we corrected several typographical errors and republished the first volume.

In this edition, only minor typographical errors are corrected.

We would like to take this opportunity to express our sincere thanks to our teachers, friends, and readers for their encouragement and support.

*Tatsien Li*  
*Tiehu Qin*  
Shanghai  
September 2003

# Preface to the Chinese Edition

Partial differential equations are fundamental in many important physical and mechanical disciplines. Although the names of these equations are well known, and they have been the focus of a considerable amount of research, gaining a comprehensive and deep understanding of the related physical and mechanical background is not an easy task. The purpose of this book is to offer some help not only to teachers, graduate students, and senior undergraduate students engaged in studying, researching, and teaching applied partial differential equations, but also to scholars and researchers in other disciplines and application areas. Readers will find the tools to become proficient in the use of important fundamental equations in modern physics, to gain familiarity with the whys, wherefores, and derivation of these equations, and to understand some commonly used mathematical models more easily. They will be able to study and use partial differential equations more consciously, and also learn to grasp some significant problems in order to properly carry out their research. Therefore, our purpose in writing this book is to build a bridge between physics and partial differential equations.

In this book, starting with the most basic concepts of physics, we focus on the whole process of establishing the fundamental equations for physical and mechanical disciplines such as electrodynamics, fluid dynamics, magnetohydrodynamics, reacting fluid dynamics, elastic mechanics, thermoelastic mechanics, viscoelastic mechanics, kinetic theory of gases, special relativity, and quantum mechanics. At the same time, we give a brief description of the mathematical structures and features of these equations, including their types and basic characteristics, the behavior of solutions, and some approaches commonly used to solve these equations. We selectively introduce some worldwide research results from recent years, including those of the authors and their research group. We hope that readers who are unfamiliar with the related physical and mechanical disciplines can gain access to the core of these disciplines in an easy-to-digest way so as to complete as soon as possible their transition from physics to mathematics and from related physical and mechanical fields to their mathematical models described by partial differential equations. On the other hand, for readers who are more familiar with the related physical and mechanical disciplines, we hope that an in-depth understanding of the mathematical structures and features of the fundamental equations will ultimately reveal the advantages of effective mathematical tools and expressions to more clearly present the basic contents of physics; consequently, readers who acquire such an understanding will be able to use modern mathematical concepts, methods, and tools more purposefully to solve related physical and mechanical problems.

This book is divided into two volumes, each consisting of five chapters. The contents of each chapter are relatively independent; however, all of the chapters echo and relate with

each other to a certain extent. Exercises and a bibliography are included in each chapter. Most of the chapters are not meant to be difficult for those readers who have taken basic undergraduate courses in mathematics and physics. This book can be used as a textbook for graduate courses or elective senior undergraduate courses, as well as a reference book or extracurricular reading material.

Since the second half of 1987, the contents of this book have been continuously and successfully taught in Fudan University as both an elective senior undergraduate course and a required graduate degree course. The lecture notes have been constantly supplemented and revised, and it is on these that the final version of this book is based.

The authors would like to thank Higher Education Press for its enthusiastic support of the publication of this book, and Professor Sixu Guo for careful and meticulous typesetting. Thanks also go to Dr. Zhijie Cai for his responsible and proficient typing of the entire manuscript, and to Dr. Yingqiu Gu for his assistance in ensuring all the physical units in this book conform to the international system of units (SI units). In particular, we are grateful to Minyou Qi, Professor of Mathematics in the Department of Mathematics at Wuhan University, and Guangjiong Ni, Professor of Physics in the Department of Physics at Fudan University. They have carefully reviewed the manuscript and supplied many helpful comments and suggestions. Their hard work enriched this book.

As mathematicians, the authors may have a superficial understanding of physics to a certain extent, so errors and omissions are inevitable. We hope that readers will not spare their comments and corrections.

*Tatsien Li*  
*Tiehu Qin*

November 10, 1996

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## Chapter 6

# Thermoelasticity

### 6.1 Introduction

In Chapter 5, we discussed the deformation and stress of an elastic body under mechanical loads without considering the impact of temperature change. In fact, a temperature change in the elastic body will generate additional strain and stress. In this chapter, we will focus our attention on the impact of temperature change on the deformation of an elastic body; i.e., we will discuss the laws obeyed by the deformation of a thermoelastic body and its interior temperature distribution.

What is a thermoelastic body? A *thermoelastic body* is an object with the following characteristics: for an object that was originally in the natural state (there was neither deformation nor a temperature gradient inside), the deformation and the change of temperature field affected by mechanical loads and thermal environment will disappear immediately on being withdrawn from the mechanical loads and thermal environment, so that the object is restored to its original natural state with neither permanent deformation nor permanent temperature gradient.

To obtain the law of deformation and temperature distribution of a thermoelastic body, i.e., to establish the mathematical model of thermoelasticity—the system of thermoelasticity—the corresponding conservation laws and constitutive relations must be given, as was done for the elastic body in Chapter 5. Since the conservation laws of mass and momentum have nothing to do with the temperature change of an elastic body, the two conservation laws obtained in Chapter 5 are still valid for thermoelastic bodies. But since temperature changes must be taken into account in the discussion of thermoelastic bodies, and there is heat transfer among different parts of the thermoelastic body, in addition to the above two conservation laws, we should establish the corresponding *conservation law of energy*. Moreover, from the thermodynamical point of view, heat conduction is an irreversible process, and the conservation law of energy itself is not enough to determine whether a process can be carried on or not, so we have to establish the corresponding *entropy inequality*. Meanwhile, the quantities involved in the constitutive relations in elasticity are expressed as functions of only the deformation gradient tensor  $\mathbf{F}$ , while in thermoelasticity they are functions of the deformation gradient tensor, the absolute temperature, and the temperature gradient, since the deformation and heat

conduction caused by the temperature and the temperature gradient have to be taken into consideration.

In this chapter, we will adopt relevant notation as in Chapter 5.

## 6.2 The Conservation Law of Energy and the Entropy Inequality

### 6.2.1 The Conservation Law of Energy

Let  $\Omega \subset \mathbb{R}^3$  be the reference configuration of the thermoelastic body, i.e., the domain occupied by the thermoelastic body before deformation (assumed to be time  $t = 0$ ). Denote by  $\boldsymbol{x} = (x_1, x_2, x_3)$  a point in this domain. At time  $t (> 0)$ , assume that the domain occupied by the thermoelastic body turns into  $\Omega_t$  from  $\Omega$ . This deformation is described by

$$\boldsymbol{y} = \boldsymbol{y}(t, \boldsymbol{x}),$$

where  $\boldsymbol{y} = (y_1, y_2, y_3) \in \Omega_t$ .

For any given domain  $G_t \subset \Omega_t$ , let us examine the changes in the total energy in  $G_t$ . For the elastic body, the total energy is the sum of the kinetic energy and the strain energy. However, for the thermoelastic body, the strain energy should be replaced by the internal energy, due to the heat transfer caused by the temperature change among different parts. Thus, the total energy in  $G_t$  should be the sum of the total kinetic energy and the total internal energy. Here the internal energy, of course, includes both the thermodynamic internal energy and the strain energy.

The total kinetic energy in  $G_t$  is

$$\int_{G_t} \frac{1}{2} \rho |\boldsymbol{v}|^2 dy,$$

where  $\rho$  is the mass density and  $\boldsymbol{v}$  is the velocity vector. Suppose that the internal energy per unit mass of the thermoelastic body is  $e$ . Then the total internal energy in  $G_t$  is

$$\int_{G_t} \rho e dy.$$

From the conservation law of energy, the rate of change of the total energy in  $G_t$  should be equal to the sum of the following four parts: the work done by the volume force in  $G_t$ , the work done by the stress on the boundary  $S_t$  of  $G_t$ , the heat produced by the heat source in  $G_t$ , and the heat flowing into  $G_t$  across  $S_t$  in unit time.

Suppose that the volume force density, i.e., the volume force per unit mass, is  $\boldsymbol{b} = (b_1, b_2, b_3)$ . Then the work done by the volume force in unit time is

$$\int_{G_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} dy.$$

Moreover, the stress on the boundary  $S_t$  of  $G_t$  received from the exterior of  $G_t$  in  $\Omega_t$  is  $\boldsymbol{T}\boldsymbol{\nu}$ , where  $\boldsymbol{T}$  is the Cauchy stress tensor (see section 5.3.5 in Chapter 5) and  $\boldsymbol{\nu}$  is the unit outward normal vector on  $S_t$ . Thus, the work done by the stress on  $S_t$  in unit time is

$$\int_{S_t} (\boldsymbol{T}\boldsymbol{\nu}) \cdot \boldsymbol{v} dS_t.$$

Suppose that the heat source density of the thermoelastic body, i.e., the heat generated per unit mass in unit time, is  $\gamma$ ; then the heat produced by the heat source in  $G_t$  in unit time is

$$\int_{G_t} \rho \gamma dy.$$

Suppose furthermore that  $\mathbf{q} = (q_1, q_2, q_3)^T$  is the heat flux density vector: its direction is that of the heat flux, i.e., that of the heat conduction, while its length stands for the heat across a unit area perpendicular to the direction of heat flux in unit time. Thus, for any given area element  $dS_t$  on  $S_t$ , the heat flowing across  $dS_t$  along the normal vector  $\boldsymbol{\nu}$  in unit time is given by

$$\mathbf{q} \cdot \boldsymbol{\nu} dS_t.$$

Therefore, the heat flowing into  $G_t$  across  $S_t$  in unit time is

$$-\int_{S_t} \mathbf{q} \cdot \boldsymbol{\nu} dS_t.$$

In summary, from the conservation law of energy we obtain

$$\begin{aligned} & \frac{d}{dt} \left( \int_{G_t} \frac{1}{2} \rho |\mathbf{v}|^2 dy + \int_{G_t} \rho e dy \right) \\ &= \int_{S_t} (\mathbf{T}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} dS_t + \int_{G_t} \rho \mathbf{b} \cdot \mathbf{v} dy + \int_{G_t} \rho \gamma dy - \int_{S_t} \mathbf{q} \cdot \boldsymbol{\nu} dS_t. \end{aligned} \quad (6.1)$$

From Lemma 5.2 in Chapter 5, we have

$$\frac{d}{dt} \int_{G_t} \left( \frac{1}{2} \rho |\mathbf{v}|^2 + \rho e \right) dy = \int_{G_t} \rho \frac{d}{dt} \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) dy,$$

where  $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_y)$  and  $\nabla_y = \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3} \right)$ . Then (6.1) can be rewritten as

$$\begin{aligned} & \int_{G_t} \rho \frac{d}{dt} \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) dy \\ &= \int_{S_t} (\mathbf{T}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} dS_t + \int_{G_t} \rho \mathbf{b} \cdot \mathbf{v} dy + \int_{G_t} \rho \gamma dy - \int_{S_t} \mathbf{q} \cdot \boldsymbol{\nu} dS_t. \end{aligned} \quad (6.2)$$

This is the integral form of the conservation law of energy under the spatial description (see section 5.1 in Chapter 5).

To obtain the differential form of (6.2), similarly to Chapter 5, we need to reduce the surface integral on its right-hand side to the volume integral on the domain enclosed by this

surface. Assume that the integrands under discussion are smooth. Then by Green's formula we have

$$\int_{S_t} (\mathbf{T}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} dS_t = \int_{G_t} \operatorname{div}_y(\mathbf{T}\boldsymbol{v}) dy, \quad (6.3)$$

$$\int_{S_t} \mathbf{q} \cdot \boldsymbol{\nu} dS_t = \int_{G_t} \operatorname{div}_y \mathbf{q} dy, \quad (6.4)$$

where  $\operatorname{div}_y$  stands for the divergence with respect to  $\mathbf{y} = (y_1, y_2, y_3)$ . We also used the symmetry of the Cauchy stress tensor  $\mathbf{T}$  when deriving (6.3) (see Theorem 5.2 in Chapter 5).

Using (6.3) and (6.4), it follows from (6.2) that

$$\begin{aligned} & \int_{G_t} \rho \frac{d}{dt} \left( \frac{1}{2} |\boldsymbol{v}|^2 + e \right) dy \\ &= \int_{G_t} (\operatorname{div}_y(\mathbf{T}\boldsymbol{v}) - \operatorname{div}_y \mathbf{q} + \rho \mathbf{b} \cdot \boldsymbol{v} + \rho \gamma) dy. \end{aligned} \quad (6.5)$$

Since (6.5) holds for any given domain  $G_t \subset \Omega_t$ , we have

$$\rho \frac{d}{dt} \left( \frac{1}{2} |\boldsymbol{v}|^2 + e \right) = \operatorname{div}_y(\mathbf{T}\boldsymbol{v}) - \operatorname{div}_y \mathbf{q} + \rho \mathbf{b} \cdot \boldsymbol{v} + \rho \gamma. \quad (6.6)$$

This is exactly the differential form of the conservation law of energy under the spatial description.

Obviously, we have

$$\frac{d}{dt} \left( \frac{1}{2} |\boldsymbol{v}|^2 \right) = \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}$$

and

$$\operatorname{div}_y(\mathbf{T}\boldsymbol{v}) = \operatorname{div}_y \mathbf{T} \cdot \boldsymbol{v} + \sum_{i,j=1}^3 t_{ij} \frac{\partial v_i}{\partial y_j}, \quad (6.7)$$

where  $t_{ij}$  are components of  $\mathbf{T}$  and  $\operatorname{div}_y \mathbf{T} = \left( \sum_{j=1}^3 \frac{\partial t_{ij}}{\partial y_j} \right)$  is a vector (see Appendix A in Volume I). We used the symmetry of  $\mathbf{T}$  again when obtaining (6.7). Thus, using the differential form of the conservation laws of momentum obtained in Chapter 5 (see (5.57) in Chapter 5),

$$\rho \frac{d\boldsymbol{v}}{dt} - \operatorname{div}_y \mathbf{T} - \rho \mathbf{b} = 0, \quad (6.8)$$

equation (6.6) can also be rewritten in the following simpler form:

$$\rho \frac{de}{dt} = \sum_{i,j=1}^3 t_{ij} \frac{\partial v_i}{\partial y_j} - \operatorname{div}_y \mathbf{q} + \rho \gamma. \quad (6.9)$$

This is another expression of the differential form of the conservation law of energy under the spatial description.

Just as in elasticity, the form of the above equations under the material description (see section 5.1 in Chapter 5) is the most common one used in thermoelasticity. Let the domain  $G_0 \subset \Omega$  in the reference configuration correspond to  $G_t \subset \Omega_t$ . In order to obtain the conservation equation of energy under the material description, we need to reduce the volume integral with respect to variables  $(y_1, y_2, y_3)$  over  $G_t$  in (6.2) to the volume integral with respect to variables  $(x_1, x_2, x_3)$  over  $G_0$ , and the surface integral over  $S_t$  into the surface integral over the boundary  $S_0$  of  $G_0$ . The former can easily be done by the usual substitution of integral variables and by noticing that

$$\rho J = \rho_0 \quad (6.10)$$

(see (5.34) in Chapter 5), where  $J = \det \mathbf{F}$ , and  $\mathbf{F} = (f_{ij})$ ,  $f_{ij} = \frac{\partial y_i}{\partial x_j}$ , while  $\rho_0$  represents the mass density of the thermoelastic body before the deformation (depending only on  $x$ , and independent of  $t$ ). For example,

$$\begin{aligned} & \int_{G_t} \rho \frac{d}{dt} \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) dy \\ &= \int_{G_0} \rho \frac{d}{dt} \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) J dx \\ &= \int_{G_0} \rho_0 \frac{d}{dt} \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) dx \\ &= \int_{G_0} \frac{\partial}{\partial t} \left( \rho_0 e + \frac{1}{2} \rho_0 |\mathbf{v}|^2 \right) dx. \end{aligned} \quad (6.11)$$

Similarly, we have

$$\int_{G_t} \rho \mathbf{b} \cdot \mathbf{v} dy = \int_{G_0} \rho_0 \mathbf{b} \cdot \mathbf{v} dx, \quad (6.12)$$

$$\int_{G_t} \rho \gamma dy = \int_{G_0} \rho_0 \gamma dx. \quad (6.13)$$

Now we inspect the corresponding change of the surface integral over  $S_t$  in (6.2). Noting that

$$\boldsymbol{\nu} dS_t = J \mathbf{F}^{-T} \mathbf{n} dS_0 \quad (6.14)$$

(see (5.63) of Lemma 5.3 in Chapter 5), where  $\mathbf{n} = (n_1, n_2, n_3)$  is the unit outward normal vector of  $S_0$ , and using the symmetry of  $\mathbf{T}$ , we have

$$\begin{aligned} \int_{S_t} (\mathbf{T}\boldsymbol{\nu}) \cdot \mathbf{v} dS_t &= \int_{S_t} (\mathbf{T}\mathbf{v}) \cdot \boldsymbol{\nu} dS_t \\ &= \int_{S_0} J (\mathbf{T}\mathbf{v}) \cdot (\mathbf{F}^{-T} \mathbf{n}) dS_0 \\ &= \int_{S_0} \sum_{i,j=1}^3 p_{ij} v_i n_j dS_0, \end{aligned} \quad (6.15)$$

where  $p_{ij}$  are the components of the Piola stress tensor

$$\mathbf{P} = J\mathbf{T}\mathbf{F}^{-T} \quad (6.16)$$

(see (5.69) in Chapter 5). Similarly, it is easy to see that

$$\int_{S_t} \mathbf{q} \cdot \boldsymbol{\nu} dS_t = \int_{S_0} \mathbf{h} \cdot \mathbf{n} dS_0, \quad (6.17)$$

where

$$\mathbf{h} = J\mathbf{F}^{-1}\mathbf{q}. \quad (6.18)$$

It is not hard to see that  $\mathbf{q} \cdot \boldsymbol{\nu}$  stands for the heat across unit area on  $S_t$  along  $\boldsymbol{\nu}$  in unit time, while  $\mathbf{h} \cdot \mathbf{n}$  is the heat flux in unit time measured by unit area before deformation (i.e., on  $S_0$ ). Plugging (6.11)–(6.13), (6.15), and (6.17) into (6.2), we then get

$$\begin{aligned} & \int_{G_0} \frac{\partial}{\partial t} \left( \rho_0 e + \frac{1}{2} \rho_0 |\mathbf{v}|^2 \right) dx \\ &= \int_{S_0} \sum_{i,j=1}^3 p_{ij} v_i n_j dS_0 + \int_{G_0} \rho_0 \mathbf{b} \cdot \mathbf{v} dx \\ & \quad + \int_{G_0} \rho_0 \gamma dx - \int_{S_0} \mathbf{h} \cdot \mathbf{n} dS_0. \end{aligned} \quad (6.19)$$

This is just the integral form of the conservation law of energy under the material description.

To obtain the differential form of (6.19), under the assumption that the integrands are smooth, we reduce the surface integrals on its right-hand side to the volume integrals over  $G_0$  by using Green's formula:

$$\int_{S_0} \sum_{i,j=1}^3 p_{ij} v_i n_j dS_0 = \int_{G_0} \sum_{i,j=1}^3 \frac{\partial}{\partial x_j} (p_{ij} v_i) dx, \quad (6.20)$$

$$\int_{S_0} \mathbf{h} \cdot \mathbf{n} dS_0 = \int_{G_0} \operatorname{div} \mathbf{h} dx, \quad (6.21)$$

where  $\operatorname{div}$  denotes the divergence with respect to  $\mathbf{x} = (x_1, x_2, x_3)$ . Plugging (6.20) and (6.21) into (6.19), we obtain

$$\begin{aligned} & \int_{G_0} \frac{\partial}{\partial t} \left( \rho_0 e + \frac{1}{2} \rho_0 |\mathbf{v}|^2 \right) dx \\ &= \int_{G_0} \left( \sum_{i,j=1}^3 \frac{\partial}{\partial x_j} (p_{ij} v_i) - \operatorname{div} \mathbf{h} + \rho_0 \mathbf{b} \cdot \mathbf{v} + \rho_0 \gamma \right) dx. \end{aligned} \quad (6.22)$$

Since (6.22) holds for any given domain  $G_0 \subset \Omega$ , we have

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \rho_0 e + \frac{1}{2} \rho_0 |\mathbf{v}|^2 \right) \\ &= \sum_{i,j=1}^3 \frac{\partial}{\partial x_j} (p_{ij} v_i) - \operatorname{div} \mathbf{h} + \rho_0 \mathbf{b} \cdot \mathbf{v} + \rho_0 \gamma. \end{aligned} \quad (6.23)$$

This is exactly the differential form of the conservation law of energy under the material description. It is a conservation equation of divergence form.

Using the conservation equations of momentum,

$$\rho_0 \frac{\partial v_i}{\partial t} = \sum_{j=1}^3 \frac{\partial p_{ij}}{\partial x_j} + \rho_0 b_i \quad (i = 1, 2, 3), \quad (6.24)$$

obtained in Chapter 5 (see (5.74) in Chapter 5), it is easy to see that (6.23) can also be written in the following simpler equivalent form:

$$\rho_0 \frac{\partial e}{\partial t} = \sum_{i,j=1}^3 p_{ij} \frac{\partial v_i}{\partial x_j} - \operatorname{div} \mathbf{h} + \rho_0 \gamma. \quad (6.25)$$

This is another expression of the conservation law of energy under the material description.

### 6.2.2 Entropy Inequality

Let  $\eta$  be the entropy density, i.e., the entropy per unit mass. Now we consider the change of entropy in any given domain  $G_t \subset \Omega_t$ . From the second law of thermodynamics (see Appendix B in Volume I) we know that the increment of entropy in  $G_t$  in unit time is not less than the sum of the entropy supplied by the entropy source in  $G_t$  and the entropy flowing into  $G_t$  across the boundary  $S_t$  of  $G_t$  during this time period.

The total entropy in  $G_t$  is

$$\int_{G_t} \rho \eta \, dy.$$

The entropy source in  $G_t$  is exactly the heat source. It is known that the heat produced by the volume element  $dy = dy_1 dy_2 dy_3$  in  $G_t$  in unit time is  $\rho \gamma \, dy$ , the entropy supplied by which is  $\rho \gamma \, dy / \theta$ , where  $\theta$  is the absolute temperature of the element  $dy$  (see (B.4) in Appendix B in Volume I). Then the entropy supplied by the entropy source in  $G_t$  in unit time is

$$\int_{G_t} \frac{\rho \gamma}{\theta} \, dy.$$

Moreover, it is known that the heat flowing into  $G_t$  across the boundary area element  $dS_t$  in unit time is  $-\mathbf{q} \cdot \boldsymbol{\nu} \, dS_t$ , and thus the entropy is  $-\mathbf{q} \cdot \boldsymbol{\nu} \, dS_t / \theta$ . Hence, the entropy flowing into  $G_t$  across  $S_t$  in unit time is

$$-\int_{S_t} \frac{\mathbf{q} \cdot \boldsymbol{\nu}}{\theta} \, dS_t.$$



From the above analysis and the second law of thermodynamics, we obtain

$$\frac{d}{dt} \int_{G_t} \rho \eta dy \geq \int_{G_t} \frac{\rho \gamma}{\theta} dy - \int_{S_t} \frac{\mathbf{q} \cdot \boldsymbol{\nu}}{\theta} dS_t. \quad (6.26)$$

Applying Lemma 5.2 in Chapter 5 to the left-hand side of the above formula, (6.26) can be rewritten as

$$\int_{G_t} \rho \frac{d\eta}{dt} dy \geq \int_{G_t} \frac{\rho \gamma}{\theta} dy - \int_{S_t} \frac{\mathbf{q} \cdot \boldsymbol{\nu}}{\theta} dS_t. \quad (6.27)$$

This is exactly the integral form of the entropy inequality under the spatial description.

Under the assumption that the integrand is smooth, reducing the surface integral on the right-hand side of (6.27) into the volume integral over  $G_t$  by Green's formula, (6.27) can then be written in the following form:

$$\int_{G_t} \rho \frac{d\eta}{dt} dy \geq \int_{G_t} \left( \frac{\rho \gamma}{\theta} - \operatorname{div}_y \left( \frac{\mathbf{q}}{\theta} \right) \right) dy. \quad (6.28)$$

Since (6.28) holds for any given domain  $G_t \subset \Omega_t$ , we then have

$$\rho \frac{d\eta}{dt} \geq \frac{\rho \gamma}{\theta} - \operatorname{div}_y \left( \frac{\mathbf{q}}{\theta} \right). \quad (6.29)$$

This is exactly the differential form of the entropy inequality under the spatial description.

To obtain the entropy inequality under the material description, we need to reduce the volume integral and the surface integral in (6.27) to the volume integral over  $G_0$  and the surface integral on  $S_0$ , respectively. Similarly to what we have done to the conservation equation of energy, (6.27) can be reduced to

$$\int_{G_0} \rho_0 \frac{\partial \eta}{\partial t} dx \geq \int_{G_0} \frac{\rho_0 \gamma}{\theta} dx - \int_{S_0} \frac{\mathbf{h} \cdot \mathbf{n}}{\theta} dS_0, \quad (6.30)$$

where  $\mathbf{h}$  is defined by (6.18). This is just the integral form of the entropy inequality under the material description.

Reducing the surface integral on  $S_0$  on the right-hand side of (6.30) to the volume integral over  $G_0$  by Green's formula, (6.30) can also be rewritten as

$$\int_{G_0} \rho_0 \frac{\partial \eta}{\partial t} dx \geq \int_{G_0} \left( \frac{\rho_0 \gamma}{\theta} - \operatorname{div} \left( \frac{\mathbf{h}}{\theta} \right) \right) dx. \quad (6.31)$$

Since (6.31) holds for any given domain  $G_0 \subset \Omega$ , we then have

$$\rho_0 \frac{\partial \eta}{\partial t} \geq \frac{\rho_0 \gamma}{\theta} - \operatorname{div} \left( \frac{\mathbf{h}}{\theta} \right). \quad (6.32)$$

This is just the differential form of the second law of thermodynamics under the material description—the *entropy inequality*, also called the *Clausius–Duhem inequality*.