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Zhen-Bang Kuang

匡震邦 著

Theory of Electroelasticity

电弹性理论（英文版）

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Preface

Since Pierre Curie discovered the piezoelectric effect in 1880, piezoelectric materials have been widely used to make many electromechanical devices, such as transducers for conversion of electrical and mechanical energies, sensors, actuators, filters, resonators, ultrasonic generators, and piezoelectric biosensors. The performance and reality of devices are established on the foundation of electroelastic analyses due to the electromechanical coupling. The foundations of the electroelastic analyses are the Newton's law, Maxwell electrodynamics equations, Lorentz's law, and constitutive equations of materials. In electrically nonlinear case, different authors give different governing equations in electroelastic analyses. In this book, we give a simple theory to discuss simpler electrically nonlinear problem in engineering.

Using the continuum thermodynamics, it is found that the first law of thermodynamics contains a physical variational principle, which can be used as a fundamental natural principle to derive the governing equations in physics and continuum mechanics. This theory will be used to derive the governing equations of the discussed piezoelectric and pyroelectric body and its environment in Chap. 2. The Maxwell stress can be obtained automatically by the migratory variation of the electric potential.

In literatures many works on the static and dynamic generalized stress and displacement analyses in piezoelectric and electrostrictive materials with and without defects have been published. Some important results of piezoelectric materials will be collected, modified, and discussed in a unified version in Chaps. 3 and 4. The results of the electrostrictive, pyroelectric, and functional graded piezoelectric materials will be given in Chap. 5.

The surface wave propagation in or not in a biasing state is discussed in Chap. 6. The reflection and transmission of waves in piezoelectric and pyroelectric materials are disposed by the inhomogeneous wave theory. We extend the first and second thermodynamic laws to the case with varied temperature and propose an inertial entropy theory due to the heat inertia. The temperature wave equation with a finite propagation velocity can be derived easily from this theory. In the generalized inertial entropy theory an inertial concentration theory is proposed, which can be extended to more extensive area.

In Chap. 7, the three-dimensional and some practical applied electroelastic problems such as plates and shells in electroelastic theory are discussed.

The failure theories published in literatures are also collected in Chap. 8. In the change of the microstructure and failure process, the energy possesses material structure anisotropic behavior and a modal energy density factor theory is proposed, which can also be used in other area, such as in phase transformation theory.

In order to read easily for readers the fundamental knowledge used in this book is given in Chap. 1. Some basic problems are narrated in detail including the formulation of a problem and the mathematical derivation. But for further problems, the narration is simpler. Because the discussed problems in this book are complicated and the check is difficult, some errors may occur. We wish readers will give comments.

The author hopes that this book is useful for graduate students, scientists, and engineers interesting in this area in the fields of continuum mechanics, material science, solid-state physics, and device engineering.

The literatures are very enormous and cannot be all cited, but readers can get more literatures from our cited papers.

This book extensively uses the materials of a Chinese book "Theory of Electro-elasticity" published by Shanghai Jiaotong University Press. The author thanks the support of the Shanghai Jiaotong University Press, Professor Z. Suo of Harvard University, and Professor T-J. Wang of Xian Jiaotong University.

This edition published in Shanghai has minor changes, mainly for continuum thermodynamics in chapter 1.

Shanghai Jiaotong University,
Shanghai, China
July, 2012

Zhen-Bang Kuang

Contents

Chapter 1
Preliminary Knowledge and Continuum Thermodynamics

Abstract In this chapter, some basic knowledge of elastic theory, electrodynamics, and thermodynamics which will be applied in this book are introduced. Some extensions in continuum thermodynamics are proposed. It is shown that together with the first law of thermodynamics, a physical variational principle (PVP) is also held. The physical variational principle gives a true process for all virtual possible process satisfying the geometrical constrained conditions. The physical variational principle is considered to be one of the fundamental physical principles for quasi-static system, which can be used to derive governing equations in continuum mechanics and other fields. When the temperature varies with time, the inertial entropy or inertial heat theory is proposed. This theory modifies the current classical thermodynamic theory. From this theory, the temperature wave equation with finite phase velocity is derived in a very simple fashion. It is shown that the time arrived to equilibrium of the temperature is about 1 ns $\sim$ 1 ps when an internal heat source with a Heaviside step heat function is applied.

Keywords Basic knowledge • Physical variational principle • Inertial entropy

1.1 Background

Jacques and Pierre Curie brothers discovered the piezoelectric effect in 1880 (Sun and Zhang 1984; Ikeda 1990). They found out that a mechanical stress applied on crystals such as tourmaline, quartz, and Rochelle salt could produce electrical charges, and the voltage was proportional to the stress. Piezoelectric can also work in reverse, generating a strain by the application of an electric field. Centrosymmetric classes of crystals are always not piezoelectric, but a few kinds of crystals are still not piezoelectric though lacking a center of symmetry. The pyroelectric effect was found in eighteenth century (Lang 2005), earlier than piezoelectric effect. Most ferroelectric crystals are strongly piezoelectric and pyroelectric. First applications were piezoelectric ultrasonic submarine detector and quartz clocks

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during the First World War. After the Second World War, many new piezoelectric and pyroelectric materials have been discovered in succession, such as $BaTiO_3$, $Pb(Ti.Zr)O_3$-PZT, KDP, PMN, $LiNbO_3$, and $LiTaO_3$. In present time, it has been successfully used in various areas, such as in aerospace, transportation, nuclear, and medical.

It is different with the piezoelectric materials, all of the crystals, especially the isotropic electrostrictive materials, have the electrostrictive effect.

The fundamental phenomenological theory of the piezoelectricity was established by Kelvin (1856), Voigt (1910), etc. In the current time, due to the intrinsic mechanical-electric coupling effects, piezoelectric materials have been widely used in engineering structures to detect the responses of the structure by measuring the electric charge (sensing) or to reduce excessive responses by applying additional electric forces or thermal forces (actuating). By integrating the sensing and actuating, it is possible to create the so-called intelligent structures and systems that can adapt to or correct for changing operating condition. Due to its intrinsic electromechanical coupling behavior and its reliability in performance, the electroelastic analysis is necessary and has been paid much attention. A lot of literatures have appeared in journals and books. Here we cannot review all of these literatures, but reader can find more literatures from our cited papers.

The foundations of the electroelastic analyses are the Newton's law, Maxwell electrodynamics equations, Lorentz's law, and constitutive equations of materials. In electrically nonlinear case, different authors give different governing equations in electroelastic analyses. In this book, we give a simple theory to discuss simpler electrically nonlinear problem in engineering.

Using the continuum thermodynamics, it is found that the first law of thermodynamics contains the physical variational principle, which can be used as a fundamental natural principle to derive the governing equations in physics and continuum mechanics. We also proposed the inertial entropy theory due to the heat inertia. The temperature wave equation with a finite propagation velocity can be derived easily from this theory. A failure theory based on the energy principle is proposed in this book, which can also be used in other area, such as in phase transformation theory. Many works on the static and dynamic generalized stress analyses in piezoelectric and electrostrictive materials with defects, the surface wave propagation, and the failure theory are also discussed in this book.

1.2 Foundations of Classical Electrodynamics

1.2.1 Constitutive (or State) Equations

There are many books that discussed the electrodynamics (Landau et al. 1984; Stratton 1941; Cai and Zhu 1985; Moon 1984) and the electric engineering (Kruck 1954). Here, a short discussion is given only.

The constitutive (or state) equations can be written in the following form:

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, T), \quad \boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H}, T), \quad \boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E}, T) \tag{1.1a}$$

where $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B},$ and T are the electric field intensity, electric displacement or electric flux density, magnetic field intensity, magnetic induction or the magnetic flux density, and the temperature, respectively; $\boldsymbol{J}$ is the total electric current density. When $\boldsymbol{E}$ and $\boldsymbol{H}$ are small, the linear form is used for isothermal case:

$$\begin{aligned} &\boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}) \\ &\boldsymbol{J} = \boldsymbol{J}_{\mathrm{s}} + \boldsymbol{J}_{\mathrm{e}} + \boldsymbol{J}_{\mathrm{v}}, \quad \boldsymbol{J}_{\mathrm{s}} = \boldsymbol{\gamma} \cdot \boldsymbol{E}_{\mathrm{ext}}, \quad \boldsymbol{J}_{\mathrm{e}} = \boldsymbol{\gamma} \cdot \boldsymbol{E}, \quad \boldsymbol{J}_{\mathrm{v}} = \boldsymbol{\gamma} \cdot (v \times \boldsymbol{B}) \end{aligned} \tag{1.1b}$$

where $\boldsymbol{J}_{\mathrm{s}}$, $\boldsymbol{J}_{\mathrm{e}}$, and $\boldsymbol{J}_{\mathrm{v}}$ are the given external exciting current density, the induction or eddy current density, and motional electric current density, respectively; $\boldsymbol{\epsilon}$ is the permittivity, $\boldsymbol{\mu}$ the permeability, $\boldsymbol{\gamma}$ the electric conductivity of a material, respectively, ϵ_0 and μ_0 are values of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in the vacuum or air. $\boldsymbol{E}_{\mathrm{ext}}$ is an external field; v is the velocity of a moving body. $\boldsymbol{P}$ and $\boldsymbol{M}$ are the polarization density and magnetization density, respectively.

1.2.2 Conservation Law of Charge

The conservation law of charge is

$$\nabla \cdot \boldsymbol{J} = -\dot{\rho}_{\mathrm{e}} \tag{1.2}$$

where ρ_{e} is the free electric charge density. A superimposed dot indicates partial differentiation with respect to time, i.e., $\dot{()} = \partial()/\partial t$, such as $\dot{\rho}_{\mathrm{e}} = \partial \rho_{\mathrm{e}}/\partial t$.

1.2.3 The Lorentz Force

For a continuous charge distribution in motion, the Lorentz force equation is

$$\boldsymbol{f} = \rho_{\mathrm{e}}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \quad \text{or} \quad \boldsymbol{f} = \rho_{\mathrm{e}} \boldsymbol{E} + \boldsymbol{J}_{\mathrm{e}} \times \boldsymbol{B}, \quad \boldsymbol{J}_{\mathrm{e}} = \rho_{\mathrm{e}} \boldsymbol{v} \tag{1.3a}$$

where $\boldsymbol{f}$ is the force density (force per unit volume) and $\boldsymbol{J}_{\mathrm{e}}$ is the current density. Equation (1.3a) can be extended to the electromagnetic media and approximately expressed as (Pao 1978; Kuang 2011a)

$$\begin{aligned} &\boldsymbol{f} = \rho_{\mathrm{t}} \boldsymbol{E} + \boldsymbol{J}_{\mathrm{t}} \times \boldsymbol{B}, \quad \rho_{\mathrm{t}} = \rho_{\mathrm{e}} + \rho_{\mathrm{P}}, \quad \rho_{\mathrm{P}} = -\nabla \cdot \boldsymbol{P} \\ &\boldsymbol{J}_{\mathrm{t}} = \boldsymbol{J} + \boldsymbol{J}_{\mathrm{P}} + \boldsymbol{J}_{\mathrm{m}}, \quad \boldsymbol{J}_{\mathrm{P}} = \partial \boldsymbol{P}/\partial t = \dot{\boldsymbol{P}}, \quad \boldsymbol{J}_{\mathrm{m}} = \nabla \times \boldsymbol{M} \end{aligned} \tag{1.3b}$$

where ρ_t is the total electric charge density constituted of free and polarized charges and $\boldsymbol{J}_t$ is the total electric current density constituted of conduction, polarized, and magnetization current densities.

1.2.4 Maxwell Equations

The differential and integral Maxwell equations are as follows:

$$\begin{aligned}
&\nabla \cdot \boldsymbol{D} = \rho_e , && \oint_a \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a} = \int_V \rho_e \, \mathrm{d}V \\
&\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} , && \oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t}\int_a \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \\
&\nabla \cdot \boldsymbol{B} = 0, && \int_a \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = 0\left(\int_V \rho_m \, \mathrm{d}V\right) \\
&\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} , && \oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \int_a \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a} + \frac{\partial}{\partial t}\int_a \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{a}
\end{aligned} \tag{1.4}$$

where V is the volume occupied by the medium; $\boldsymbol{a}$ is the area vector and a is its absolute value; $\boldsymbol{s}$ is a line element vector of a curve C; ∇ is a differential operator vector.

Taking the divergence of the second and the divergence of the fourth in Eq. (1.4) and using the law of conservation of charge we find respectively,

$$\begin{aligned}
&\nabla \cdot \partial \boldsymbol{B}/\partial t = \partial(\nabla \cdot \boldsymbol{B})/\partial t = 0 \\
&\nabla \cdot \boldsymbol{J} + \nabla \cdot \partial \boldsymbol{D}/\partial t = \nabla \partial(\nabla \cdot \boldsymbol{D} - \rho_e)/\partial t = 0
\end{aligned} \tag{1.5}$$

If $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{D} - \rho_e = 0$ at the initial state, they will be held at any time, which are just the third and the first equations in Eq. (1.4). Therefore, the independent equations are the second and the fourth equations in Eq. (1.4) and the charge conservation equation in Eq. (1.2), or other combination.

1.2.5 Electric Scalar Potential and Magnetic Vector Potential

The second and third equations are satisfied automatically if we introduce an electric scalar potential φ and a magnetic vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{E} = -\nabla\varphi - \partial A/\partial t = -\nabla\varphi - \dot{\boldsymbol{A}}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.6}$$

Using the constitutive equation (1.1b) with constant $\boldsymbol{\epsilon} = \epsilon \boldsymbol{I}$, $\boldsymbol{\mu} = \mu \boldsymbol{I}$, $\gamma = \gamma \boldsymbol{I}$ and the relation $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$, the first and forth equations in Maxwell equations are reduced to

$$\begin{aligned}\nabla^2\varphi + \partial(\nabla\cdot\boldsymbol{A})/\partial t &= -\rho_e/\epsilon \\ \nabla^2\boldsymbol{A} - \mu\epsilon\partial^2\boldsymbol{A}/\partial t^2 - \nabla(\nabla\cdot\boldsymbol{A} + \mu\epsilon\partial\varphi/\partial t) &= -\mu\boldsymbol{J}\end{aligned} \tag{1.7}$$

In order to define $\boldsymbol{A}$ uniquely, a supplement gauge condition must be given. Introducing Lorenz gauge $\nabla\cdot\boldsymbol{A} + \mu\cdot\epsilon\partial\varphi/\partial t = 0$, Maxwell equations can be written compactly in the form:

$$\begin{aligned}\nabla^2\varphi - \mu\epsilon\partial^2\varphi/\partial t^2 &= -\rho_e/\epsilon \\ \nabla^2\boldsymbol{A} - \mu\epsilon\partial^2\boldsymbol{A}/\partial t^2 &= -\mu\boldsymbol{J}\end{aligned} \tag{1.8}$$

1.2.6 Quasi-Stationary Electromagnetic Field

If $\partial\boldsymbol{D}/\partial t$ in Maxwell equations can be neglected, the field is called the quasi-stationary magnetic (MQS) field, and in this case, all radiation effects can be negligible. It is also called the eddy current field problem and is important in the electric machine engineering. If $\partial\boldsymbol{B}/\partial t$ in Maxwell equations can be neglected, the field is called quasi-stationary electric (EQS) field which is less important in engineering. For an eddy current field,

$$\partial\boldsymbol{D}/\partial t = -\epsilon\left(\partial^2\boldsymbol{A}/\partial t^2 + \nabla\partial\varphi/\partial t\right) = \boldsymbol{0}, \quad \boldsymbol{J} = \gamma\boldsymbol{E} + \boldsymbol{J}_s + \boldsymbol{J}_v \tag{1.9}$$

so Eq. (1.7) becomes

$$\begin{aligned}\nabla^2\varphi + \partial(\nabla\cdot\boldsymbol{A})/\partial t &= -\rho_e/\epsilon \\ \nabla^2\boldsymbol{A} - \mu\gamma(\partial\boldsymbol{A}/\partial t + \nabla\varphi) - \nabla(\nabla\cdot\boldsymbol{A}) &= -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v)\end{aligned} \tag{1.10}$$

Introducing conductivity gauge $\nabla\cdot\boldsymbol{A} + \mu\gamma\varphi = 0$, Eq. (1.10) is reduced to

$$\begin{aligned}\nabla^2\varphi - \mu\gamma\partial\varphi/\partial t &= -\rho_e/\epsilon \\ \nabla^2\boldsymbol{A} - \mu\gamma\partial\boldsymbol{A}/\partial t &= -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v)\end{aligned} \tag{1.11}$$

Introducing Coulomb gauge $\nabla\cdot\boldsymbol{A} = 0$, Eq. (1.10) is reduced to

$$\begin{aligned}\nabla^2\varphi &= -\rho_e/\epsilon \\ \nabla^2\boldsymbol{A} - \mu\gamma\partial\boldsymbol{A}/\partial t - \mu\gamma\nabla\varphi &= -\mu(\boldsymbol{J}_s + \boldsymbol{J}_v)\end{aligned} \tag{1.12}$$

In current sources and stator laminations, eddy currents are usually neglected. For a constant magnetic field $\boldsymbol{J} = \boldsymbol{J}_S$, $\boldsymbol{J}_e = \boldsymbol{0}$, $\boldsymbol{J}_v = \boldsymbol{0}$, Eq. (1.12) becomes $\nabla^2\boldsymbol{A} = -\mu\boldsymbol{J}_S$.

The finite element analysis shows that the results of calculation sometimes are not fully satisfactory when a supplement gauge condition is used.

When $L/c\tau \ll 1$, then $\partial \boldsymbol{D}/\partial t$ and $\partial \boldsymbol{B}/\partial t$ can all be neglected and we call this field as quasi-static electromagnetic field, where $c = 1/\sqrt{\mu\epsilon}$ is the optic velocity in media, L is the maximum size of the body, and τ is the concerned time interval. Neglecting $\partial \boldsymbol{D}/\partial t$ and $\partial \boldsymbol{B}/\partial t$, the electric and magnetic fields are independent from each other, so the electric field and magnetic field can be solved independently. When material constants are all constant for static case, the Maxwell equations is reduced to

$$\nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{E}) = \rho_e, \quad \nabla \times (\boldsymbol{\mu}^{-1} \cdot \boldsymbol{B}) = \boldsymbol{J} \tag{1.13}$$

For the static electric field, we can always introduce an electric potential or potential φ. For the case without electric current, i.e., $\boldsymbol{J} = \boldsymbol{0}$, in material, the static magnetic field can also be expressed by a scalar magnetic potential ψ. In this case, we have

$$\begin{aligned} \boldsymbol{E} &= -\nabla\varphi, \quad \nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{E}) = \rho_e, \quad \epsilon_{kl}\varphi_{,lk} = \rho_e \\ \boldsymbol{H} &= -\nabla\psi, \quad \nabla \cdot (\boldsymbol{\mu} \cdot \boldsymbol{H}) = 0, \quad \mu_{kl}\psi_{,lk} = 0 \end{aligned} \tag{1.14}$$

The electromagnetic energy $\mathfrak{A}$ and its Legendre transformation, the electromagnetic Gibbs free energy g, stored in the medium are

$$\mathrm{d}\mathfrak{A} = \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{D} + \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{B}, \quad \mathrm{d}g = \mathrm{d}\mathfrak{A} - \mathrm{d}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) = -\boldsymbol{D} \cdot \mathrm{d}\boldsymbol{E} - \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{H} \tag{1.15}$$

1.2.7 Interface Connective (or Continuity), Boundary, and Initial Conditions

The interface connective conditions of $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}$ of electromagnetic media 1 and 2 are

$$\begin{aligned} &(\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \boldsymbol{n} = \sigma_s, \quad \text{or} \quad D_{2n} - D_{1n} = \sigma_s \\ &(\boldsymbol{E}_2 - \boldsymbol{E}_1) \cdot \boldsymbol{n} = (\sigma_s + \sigma_{sp})/\epsilon_0, \quad \sigma_{sp} = -(\boldsymbol{P}_2 - \boldsymbol{P}_1) \cdot \boldsymbol{n} \end{aligned} \tag{1.16}$$

$$\begin{aligned} &(\boldsymbol{B}_2 - \boldsymbol{B}_1) \cdot \boldsymbol{n} = 0, \quad \text{or} \quad B_{2n} - B_{1n} = 0 \\ &(\boldsymbol{H}_2 - \boldsymbol{H}_1) \cdot \boldsymbol{n} = -(\boldsymbol{M}_2 - \boldsymbol{M}_1) \cdot \boldsymbol{n}/\mu_0 \end{aligned} \tag{1.17}$$

$$\begin{aligned} &\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s, \quad \text{or} \\ &\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \mu_0(\boldsymbol{J}_s + \boldsymbol{J}_{sm}), \quad \boldsymbol{J}_{sm} = \boldsymbol{n} \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) \end{aligned} \tag{1.18}$$

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = -\nabla(\pi_s/\epsilon) \tag{1.19}$$

In Eqs. (1.16), (1.17), (1.18), and (1.19), $\boldsymbol{n}$ is the normal of the material 1, σ_{sp} is the surface polarization charge density, $\boldsymbol{J}_{sm}$ is the surface magnetization electric current density, and $\pi_s = \sigma_s d$ is the electric couple density on the interface. There are only two independent interface conditions in Eqs. (1.16), (1.17), (1.18), and (1.19). If material 2 does not exist, let $\boldsymbol{D}_2 = \boldsymbol{E}_2 = \boldsymbol{B}_2 = \boldsymbol{H}_2 = \boldsymbol{0}$; the boundary conditions can be obtained from Eqs. (1.16), (1.17), (1.18), and (1.19).

On the interface, the conservation condition of the electric current is

$$(\boldsymbol{J}_2 - \boldsymbol{J}_1) \cdot \boldsymbol{n} = -\partial \sigma_s / \partial t = -\dot{\sigma}_s \tag{1.20}$$

The initial conditions are

$$\begin{aligned} &\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(0), \quad \dot{\boldsymbol{E}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{E}}_0(0), \quad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(0), \\ &\dot{\boldsymbol{H}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{H}}_0(0), \quad \boldsymbol{x} \in V \end{aligned} \tag{1.21}$$

In Eq. (1.21), there are only still two independent conditions.

1.2.8 Electromagnetic Force

Multiplying the second equation in Eq. (1.4) by $\boldsymbol{D}$ and the fourth by $\boldsymbol{B}$, then adding the results we obtain

$$\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{D} \times \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) + \frac{\partial \boldsymbol{D}}{\partial t} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{0} \tag{1.22}$$

Using

$$\begin{aligned} &\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) = (\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D} - (\boldsymbol{D} \cdot \nabla)\boldsymbol{E}, \quad \nabla \cdot (\boldsymbol{D} \otimes \boldsymbol{E}) = (\nabla \cdot \boldsymbol{D})\boldsymbol{E} + (\boldsymbol{D} \cdot \nabla)\boldsymbol{E} \\ &(\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D} + (\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} = \nabla \cdot [(\boldsymbol{E} \cdot \boldsymbol{D})\boldsymbol{I}], \quad \boldsymbol{I} = \delta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned}$$

and the similar relations for $\boldsymbol{B}, \boldsymbol{H}$, Eq. (1.22) is reduced to

$$\begin{aligned} &\nabla \cdot [(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\boldsymbol{I} - (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H})] + \frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) \\ &\quad = -\rho_e \boldsymbol{E} - \boldsymbol{J} \times \boldsymbol{B} + (\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} + (\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{H} \end{aligned} \tag{1.23}$$

where the notation $\otimes$ is the tensor product, $\boldsymbol{A} \otimes \boldsymbol{B} = A_i B_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, and $\boldsymbol{e}_i$ is the unit vector on coordinate axis x_i. Using the conservation law of the electric charge, Eq. (1.23) can be written in the form of the electromagnetic momentum equation:

$$\begin{aligned} &\boldsymbol{f}^{\mathrm{M}} = \nabla \cdot \boldsymbol{\sigma}^{\mathrm{M}} - \partial \boldsymbol{g}^{\mathrm{M}} / \partial t, \quad \boldsymbol{g}^{\mathrm{M}} = \boldsymbol{D} \times \boldsymbol{B} \\ &\boldsymbol{\sigma}^{\mathrm{M}} = (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H}) - (1/2)(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})\boldsymbol{I} \\ &\boldsymbol{f}^{\mathrm{M}} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - (1/2)[(\nabla \otimes \boldsymbol{D}) \cdot \boldsymbol{E} - (\nabla \otimes \boldsymbol{E}) \cdot \boldsymbol{D}] \\ &\qquad - (1/2)[(\nabla \otimes \boldsymbol{B}) \cdot \boldsymbol{H} - (\nabla \otimes \boldsymbol{H}) \cdot \boldsymbol{B}] \end{aligned} \tag{1.24}$$

where $\boldsymbol{\sigma}^{\mathrm{M}}, \boldsymbol{f}^{\mathrm{M}}$, and $\boldsymbol{g}^{\mathrm{M}}$ are called the Maxwell stress tensor, electromagnetic body force, and electromagnetic momentum density, respectively.

If $\boldsymbol{D} = \epsilon_0 \cdot \boldsymbol{E} + \boldsymbol{P}$, $\boldsymbol{B} = \mu_0 \cdot (\boldsymbol{H} + \boldsymbol{M})$ are used, Maxwell equations become

$$\epsilon_0 \nabla \cdot \boldsymbol{E} = \rho_e - \nabla \cdot \boldsymbol{P}, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \cdot \boldsymbol{B} = 0$$
$$\mu_0^{-1} \nabla \times \boldsymbol{B} = \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M} \tag{1.25}$$

Analogous to the derivation of Eq. (1.24), we get

$$\begin{aligned}
\boldsymbol{f}'^{\mathrm{M}} &= \nabla \cdot \boldsymbol{\sigma}'^{\mathrm{L}} - \partial \boldsymbol{g}'^{\mathrm{L}} / \partial t, \quad \boldsymbol{g}'^{\mathrm{L}} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} \\
\boldsymbol{\sigma}'^{\mathrm{L}} &= (\epsilon_0 \cdot \boldsymbol{E} \otimes \boldsymbol{E} + \mu_0^{-1} \boldsymbol{B} \otimes \boldsymbol{B}) - (1/2)(\epsilon_0 E^2 + \mu_0^{-1} B^2) \boldsymbol{I} \\
\boldsymbol{f}'^{\mathrm{M}} &= \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - (\nabla \cdot \boldsymbol{P}) + (\partial \boldsymbol{P} / \partial t + \nabla \otimes \boldsymbol{M}) \times \boldsymbol{B} = \rho_t \boldsymbol{E} + \boldsymbol{J}_t \times \boldsymbol{B}
\end{aligned} \tag{1.26}$$

Using above method it is found that the Maxwell stress and electromagnetic body forces are not unique (Pao 1978). The reason may be that boundary conditions are not considered. The electromagnetic momentum equation can also be discussed by the macroscopic Lorentz force method in Sect. 1.2.3. Let a dielectric medium occupies a volume V enclosed by its surface a. Noting that on the surface there are polarized electric surface density $\boldsymbol{n} \cdot \boldsymbol{P}$ and magnetic current surface density $-\boldsymbol{n} \times \boldsymbol{M}$, so the force acted on the body is

$$\begin{aligned}
\boldsymbol{F}'' = & \int_V [(\rho_e - \nabla \cdot \boldsymbol{P}) \boldsymbol{E} + (\boldsymbol{J} + \partial \boldsymbol{P} / \partial t + \nabla \times \boldsymbol{M}) \times \boldsymbol{B}] \mathrm{d}V \\
& + \int_a [(\boldsymbol{n} \cdot \boldsymbol{P}) \boldsymbol{E} - (\boldsymbol{n} \times \boldsymbol{M}) \times \boldsymbol{B}] \mathrm{d}a
\end{aligned} \tag{1.27}$$

After some manipulations, we get

$$\begin{aligned}
\boldsymbol{f}''^{\mathrm{M}} &= \nabla \cdot \boldsymbol{\sigma}''^{\mathrm{M}} - \partial \boldsymbol{g}''^{\mathrm{M}} / \partial t, \quad \boldsymbol{g}''^{\mathrm{M}} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} \\
\boldsymbol{\sigma}''^{\mathrm{M}} &= (\boldsymbol{D} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{H}) - (1/2)(\epsilon_0 E^2 + \mu_0^{-1} B^2 - 2\boldsymbol{M} \times \boldsymbol{B}) \boldsymbol{I} \\
\boldsymbol{f}''^{\mathrm{M}} &= \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{P} \cdot (\nabla \otimes \boldsymbol{E}) + (\nabla \otimes \boldsymbol{B}) \times \boldsymbol{M} + \partial \boldsymbol{P} / \partial t \times \boldsymbol{B}
\end{aligned} \tag{1.28}$$

Because the macroscopic Lorentz force is related to the polarization and magnetization of the material, so many different formulas can be got. Eq. (1.28) did not strictly get from complete governing equations. Maugin (1988) considered that in order to avoid arbitrary choice of the ponderomotive force and couple in the electromagnetic contributions, he intend to arrive at expressions for these contributions on the basis of a microscopic model, the electron theory of Lorentz (Eringen and Maugin 1989). In electroelastic analyses only the static electromagnetic force will be discussed by the physical variational principle, and it will be discussed in the next chapter. In this book, the theory concerned with the photon motion is not considered.

1.3 Some Preliminary Knowledge in Electroelasticity

1.3.1 Universal Governing Equations

Universal governing equations must be obeyed by all moving or deforming continuum (Pao 1978; Kuang 2002). In electroelasticity, the universal governing equations are:

(1) *Mass conservation equation*

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{v}) = \dot{\rho} + \rho\nabla\cdot\boldsymbol{v} = 0, \quad \dot{\rho} + \rho v_{k,k} = 0, \quad \dot{\rho} = \partial\rho/\partial t + v_k\rho_{,k} \tag{1.29}$$

where ρ is the mass density, $\boldsymbol{v}$ is the velocity.

(2) *Linear momentum equation*

$$\nabla\cdot\boldsymbol{\sigma} + (\boldsymbol{f}^{\mathrm{m}} + \boldsymbol{f}^{\mathrm{e}}) = \rho\dot{\boldsymbol{v}}, \quad \sigma_{ij,i} + \left(f_j^{\mathrm{m}} + f_j^{\mathrm{e}}\right) = \rho\dot{v}_j \tag{1.30}$$

where $\boldsymbol{f}^{\mathrm{m}}$ and $\boldsymbol{f}^{\mathrm{e}}$ are the mechanical and electromagnetic forces per volume.

(3) *Angular momentum equation*

$$\boldsymbol{\varpi}:\boldsymbol{\sigma} + \boldsymbol{C}^{\mathrm{e}} = 0, \quad \varpi_{kij}\sigma_{ij} + C_k^{\mathrm{e}} = 0 \tag{1.31}$$

where $\boldsymbol{C}^{\mathrm{e}} = \boldsymbol{P}\times\boldsymbol{E} + \mu_0\boldsymbol{M}\times\boldsymbol{H}$ is the body couple density per volume. $\boldsymbol{\varpi}$ is the permutation notation. The asymmetric part of the stress is induced by the polarization and magnetization in electromagnetic material. From Eq. (1.31), it is also found that the asymmetric part of the stress is the second-order effect of the electromagnetic field. Let the symmetric part of the stress σ be denoted by σ^{s}, the asymmetric part by σ^{a}, then we get

$$\begin{aligned} &\sigma_{kl} = \sigma_{kl}^{\mathrm{s}} + \sigma_{kl}^{\mathrm{a}}, \quad \sigma_{lk}^{\mathrm{s}} = (\sigma_{kl} + \sigma_{lk})/2 \\ &\sigma_{kl}^{\mathrm{a}} = (\sigma_{kl} - \sigma_{lk})/2 = (E_kP_l - E_lP_k + \mu_0H_kM_l - \mu_0H_lM_k)/2 \end{aligned} \tag{1.32}$$

1.3.2 Three-Dimensional Governing Equations in Elasticity with Small Deformation

In this chapter, only the case with symmetric stresses is discussed. Let $\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{f}$ be the displacement, stress, strain, and body force per volume, we have (Ogden 1984; Kuang 2002)

Geometric equation $\quad \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad \boldsymbol{\varepsilon} = (\nabla\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\nabla)/2 \qquad (1.33)$

Momentum equation $\quad \sigma_{ij,j} + f_i = \rho\ddot{u}_i, \quad \nabla\boldsymbol{\sigma} + \boldsymbol{f} = \rho\ddot{\boldsymbol{u}} \qquad (1.34)$

Constitutive equation $\quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad \boldsymbol{\sigma} = \boldsymbol{C}:\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{s}:\boldsymbol{\sigma} \qquad (1.35)$