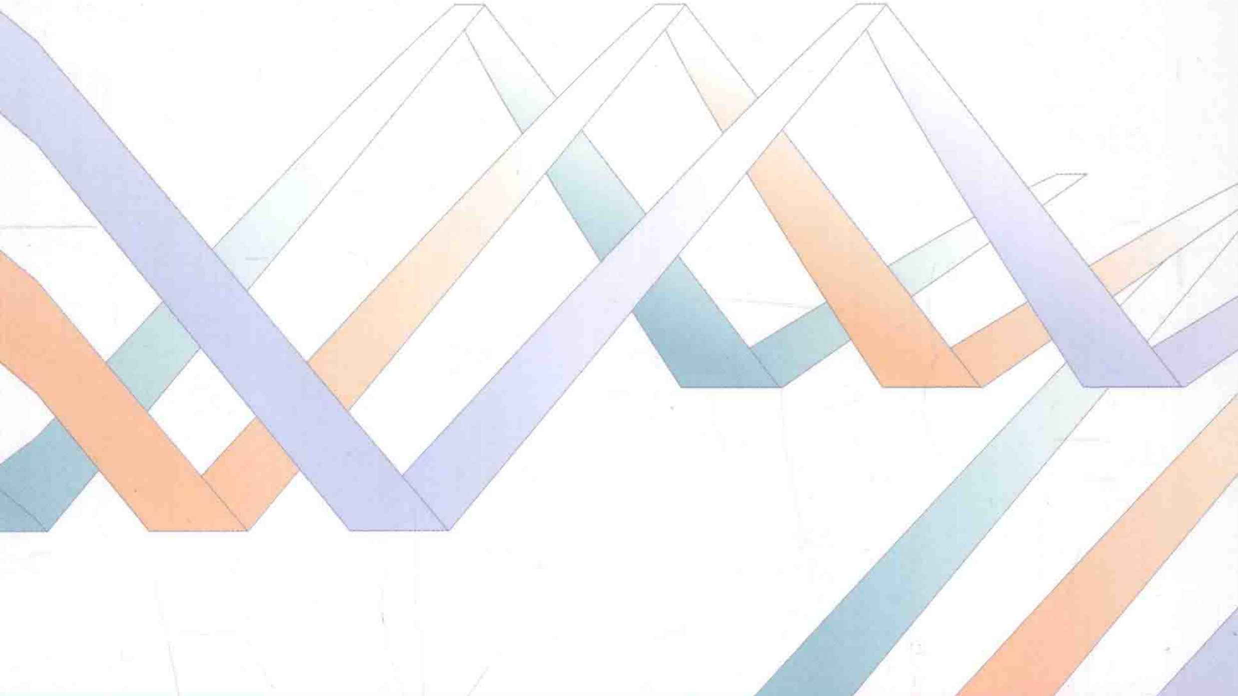
SMM 7
Surveys of Modern Mathematics



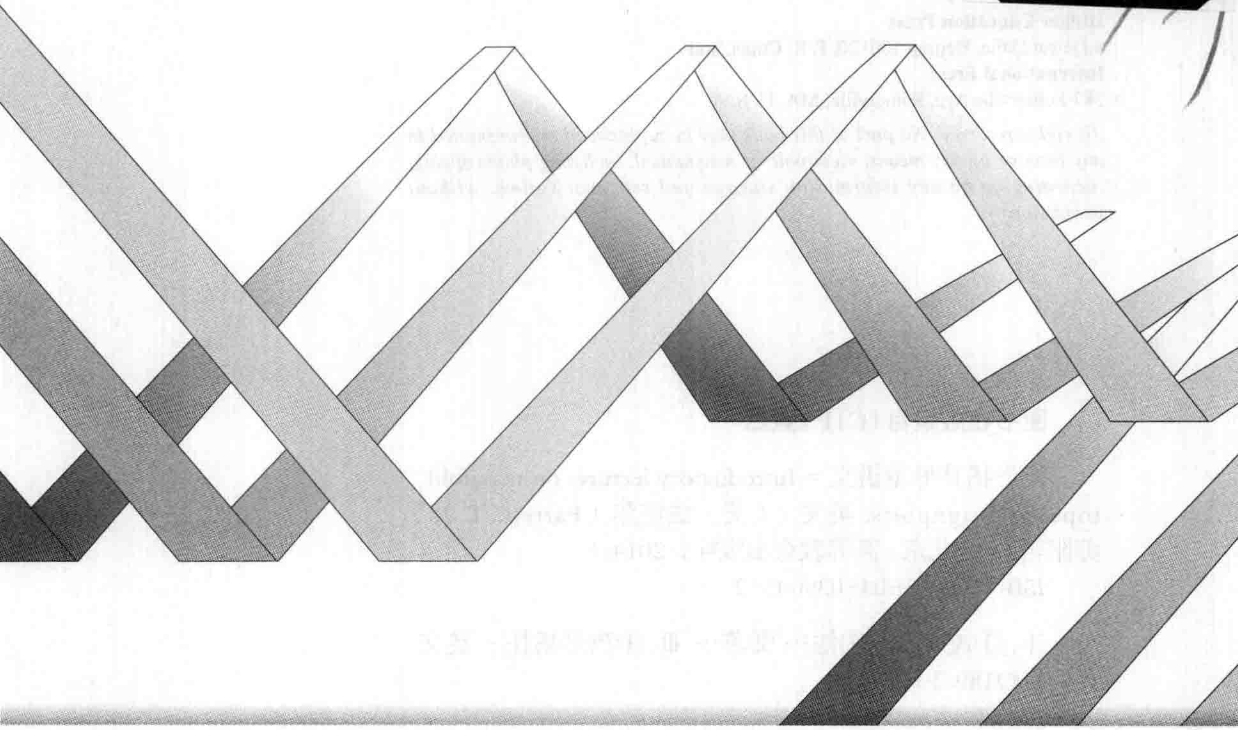
Introductory Lectures on Manifold Topology: Signposts

流形拓扑导论讲义

Thomas Farrell · Yang Su

Higher Education Press

SMM 7
Surveys of Modern M



Introductory Lectures on Manifold Topology: Signposts

流形拓扑导论讲义

LIUXING TUOPU DAOLUN JIANGYI

Thomas Farrell · Yang Su

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

International Press

图书在版编目（CIP）数据

流形拓扑导论讲义 = Introductory lectures on manifold topology: signposts: 英文 /（美）法雷尔（Farrell, T.），苏阳著. — 北京：高等教育出版社，2014.1
ISBN 978-7-04-039003-2

Ⅰ. ①流… Ⅱ. ①法… ②苏… Ⅲ. ①流形拓扑 - 英文 Ⅳ. ① O189.3

中国版本图书馆 CIP 数据核字（2013）第 298425 号

策划编辑 李 鹏　　责任编辑 李 鹏　　封面设计 张申申　　版式设计 童 丹
责任校对 刘娟娟　　责任印制 韩 刚

出版发行	高等教育出版社	咨询电话	400-810-0598
社　　址	北京市西城区德外大街 4 号	网　　址	http://www.hep.edu.cn
邮政编码	100120		http://www.hep.com.cn
印　　刷	涿州市星河印刷有限公司	网上订购	http://www.landraco.com
开　　本	787mm × 1092mm　1/16		http://www.landraco.com.cn
印　　张	8.75	版　　次	2014 年 1 月第 1 版
字　　数	160 千字	印　　次	2014 年 1 月第 1 次印刷
购书热线	010-58581118	定　　价	49.00 元

物 料 号　39003-00

Dedicated to my mother with deepest gratitude

– Y. S.

Surveys of Modern Mathematics

Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

With joint publication by Higher Education Press (HEP) inside China and International Press (IP) in the West with affordable prices, it is expected that books in this series will broadly reach out to the reader, in particular students, around the world, and hence contribute to mathematics and the world mathematics community.

Series Editors

Preface

The purpose of this book is to introduce to advanced graduate students and other interested mathematicians some of the basic technique and results from manifold topology. It is assumed that the reader is familiar with algebraic topology through cup products and Poincaré duality as well as with fiber bundles and characteristic classes; e.g. with the material in the first half of the book "Characteristic Classes" by J. W. Milnor and J. D. Stasheff. A glance at the Contents shows the topics that are covered. The book is based on a course of lectures given by the first author during the fall semester, 2009 at the Morningside Center of the Chinese Academy of Sciences. It was originally planned as a year long course; hence some of the topics alluded to in the Introduction are not covered here. These will be done in a second volume.

The writing of this book was partially supported by a grant from the National Science Foundation of the USA and by a Visiting Professorship at the Chinese Academy of Sciences of the first author, and by a grant from the National Science Foundation of China of the second author.

Surveys of Modern Mathematics

Contents

1

Introduction

The book is devoted to the basic problem of classifying manifolds up to homeomorphism and up to diffeomorphism. Today this classification is pretty well understood for many manifolds of geometric significance, including for example those supporting a Riemannian metric of non-positive sectional curvature (except in dimension 4). The starting point for this classification is due to the work of Kervaire, Milnor and Smale from about 1956—1963. Their work can be thought of as starting to answer the following two questions.

Question 1.1 (*Kervaire-Milnor*). Given a topological manifold M^n (of dimension n), does M^n support a differential structure? And if so, how many non-diffeomorphic structures does M^n support?

Question 1.2 (*Smale*). Given a topological space K, does there exist a compact (topological) manifold (without boundary) homotopy equivalent to K? And if so, how many non-homeomorphic such manifolds are there?

We begin with some explanatory definitions.

An n-dimensional (topological) manifold is a Hausdorff topological space M which is locally homeomorphic to n-dimensional Euclidean space $\mathbb{R}^n$. Additionally one requires the topology on M to have a countable basis; i.e. there exists a countable collection $\{U_n\}$ of open subsets U_n of M such that any open set U in M is a union of sets from this collection. (When M is compact, this condition is superfluous. And we will be mainly interested in compact manifolds which are also called *closed manifolds.*)

But the original reason for thinking about manifolds (probably due to Riemann) was as objects to do calculus on. For this the notion of smooth manifold is needed. And perhaps the most conceptual way to describe a (compact) smooth m-dimensional manifold is as a closed and bounded subset of some Euclidean space $\mathbb{R}^n$ ($n \gg m$) which has an m-dimensional tangent plane T_x at each point $x \in M$; i.e. there exists a pair of perpendicular affine subspaces T_x, N_x passing through x such that

$$\mathbb{R}^n = T_x \times N_x$$

and M is locally the graph of a smooth function

$$\varphi_x : U_x \to N_x$$

where $(D\varphi_x)_{x=0} = 0$. (Here U_x is an open neighborhood of $x \in T_x$.)

A smooth m-manifold is clearly a topological m-manifold; but it has more structure. Namely a cover by open sets $V_x = \text{graph}(\varphi_x)$ and *charts* $\psi_x : V_x \to T_x = \mathbb{R}^m$ where ψ_x is the perpendicular projection into T_x. These charts have the following two properties:

1. $\psi_x : V_x \to U_x \subseteq T_x = \mathbb{R}^m$ is a homeomorphism;
2. $\psi_x \circ \psi_y^{-1} : \psi_y(V_x \cap V_y) \to T_x = \mathbb{R}^m$ is a smooth map.

That is to say, the charts $\{(V_x, \psi_x) | \; x \in M\}$ are a *smooth structure* on M. This is the definition of a smooth manifold given in textbooks; i.e. a smooth manifold is a topological manifold equipped with a smooth structure. Whitney's famous embedding theorem showed that every smooth structure on a topological manifold M arises in the manner described above.

A continuous map $f : M \to N$ between a topological m-manifold M and a topological n-manifold N equipped with smooth structures Φ_M and Φ_N, respectively, is smooth if for each pair of charts $(V, \varphi) \in \Phi_M$ and $(U, \psi) \in \Phi_N$, the composite map

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \supseteq \varphi(V \cap f^{-1}(U)) \to \psi(U) \subseteq \mathbb{R}^n$$

is smooth (in the calculus sense).

In particular smooth structures Φ and Ψ on the *same* topological manifold M are said to be *equivalent* provided both $\text{id} : (M, \Phi) \to (M, \Psi)$ and $\text{id} : (M, \Psi) \to (M, \Phi)$ are smooth maps.

Example 1.3. Consider $M = \mathbb{R}$, with smooth structures

$$\Phi = \{(\mathbb{R}, \varphi(x) = x)\}, \quad \Psi = \{(\mathbb{R}, \psi(x) = x^3)\}.$$

Now $\text{id} : (M, \Phi) \to (M, \Psi)$ is smooth; but $\text{id} : (M, \Psi) \to (M, \Phi)$ is *not*. Therefore Φ and Ψ are inequivalent smooth structures on $\mathbb{R}$.

Hence even the simplest topological manifold supports inequivalent smooth structures. But there is a weaker equivalence relation between smooth structures which is one of the primary interest in differential topology; namely that of diffeomorphic structures.

A homeomorphism $f : (M, \Phi) \to (N, \Psi)$ between topological manifolds M and N equipped with smooth structures Φ and Ψ, respectively, is a *diffeomorphism* provided both

$$f : (M, \Phi) \to (M, \Psi)$$

and

$$f^{-1} : (M, \Psi) \to (M, \Phi)$$

are smooth. When a diffeomorphism exists we say that the two differentiable manifolds (M, Φ) and (N, Ψ) are diffeomorphic. This is the weaker equivalence relation mentioned above.

Note that the two differential structures Φ and Ψ on the manifold $\mathbb{R}$ in the above example are diffeomorphic even though they are inequivalent; in fact $f(x) = x^{1/3}$ is a diffeomorphism $f\colon (\mathbb{R}, \Phi) \to (\mathbb{R}, \Psi)$.

Sixty years ago it was generally assumed that every topological manifold M supports a differential structure and that the structure is *unique* up to diffeomorphism. Furthermore this belief had been recently strengthened by the proof of Hilbert's 5th Problem (Gleason, Montgomery-Zippen) which implied in the case that M is a topological group, then M supports a smooth structure such that both multiplication: $M \times M \to M$ and inverse: $M \to M$ are smooth maps. Furthermore any two smooth structures on M satisfying these conditions are diffeomorphic.

But in 1956 Milnor startled the mathematical community by constructing an "exotic" differential structure Ψ on the 7-dimensional sphere $S^7 = \{x \in \mathbb{R}^8 |\ |x| = 1\}$. S^7 has a canonical differential structure Φ determined by its "birth certificate" embedding in $\mathbb{R}^8$, as explained above. But Milnor's structure comes from some other subspace $\Sigma \subseteq \mathbb{R}^{15}$ having 7-dimensional tangent planes T_x, $x \in \Sigma$, again as explained above. There is a homeomorphism $f\colon S^7 \to \Sigma$; but *no* diffeomorphism. Formally Ψ is the pull to S^7 via f of the structure on Σ induced by its embedding $\Sigma \subseteq \mathbb{R}^{15}$. And four years later Kervaire constructed a 10-dimensional topological manifold which does *not* support any differential structure.

To state the following conjectures we need more explanatory definitions.

A continuous map $f\colon X \to Y$ between topological spaces X and Y is a *homotopy equivalence* provided there exists a second continuous map $g\colon Y \to X$ such that both $g \circ f$ is homotopic to id_X (written $g \circ f \sim \mathrm{id}_X$) and $f \circ g \sim \mathrm{id}_Y$. And such a map g (if it exists) is called a homotopy inverse to f. A homotopy inverse is unique up to homotopy.

We say that two topological spaces X and Y are homotopy equivalent if there exists a homotopy equivalence $f\colon X \to Y$. Since every homeomorphism is clearly a homotopy equivalence, homotopy equivalence puts a weaker equivalence relation on the collection of all topological spaces than homeomorphism; i.e. if X and Y are homeomorphic then they are homotopy equivalent. But algebraic topology tools are quite effective for determining when two spaces are homotopy equivalent. Hence the following conjecture:

Naive Conjecture 1.4. *Homotopy equivalent closed (i.e. compact) manifolds are homeomorphic.*

This conjecture is true for 2-dimensional manifolds. However in the 1930s some simple natural examples were exhibited showing this Naive Conjecture to be false even for 3-dimensional manifolds. But only non-simply connected examples were known. This prompted Hurewicz to make the following conjecture.

Hurewicz Conjecture 1.5. *Simply-connected homotopy equivalent closed manifolds are homeomorphic.*

Remark 1.6. This well-known conjecture is explicitly stated and discussed as Problem 31 on the list compiled by R. Lashof , "Problems in differential and algebraic topology", Seattle conference, 1963, Ann. of Math. 81 (1965), 565–591.

His conjecture was a vast generalization of Poincaré's conjecture that every simply-connected closed 3-dimensional manifold M^3 is homeomorphic to the 3-sphere S^3. To see this one of course needs only show that M is homotopy equivalent to S^3. But this follows from Poincaré duality, Hurewicz Theorem and Whitehead Theorem. We briefly sketch the argument since similar arguments will frequently occur. First of all, M^3 is orientable, because its orientation (2-sheeted) covering space is trivial since $\pi_1(M^3) = 1$. By Poincaré duality

$$H_2(M^3) \cong H^1(M^3) = \mathrm{Hom}(\pi_1 M, \mathbb{Z}) = 0.$$

By Hurewicz Theorem $\pi_3(M^3) = H_3(M^3) = \mathbb{Z}$. Let $f\colon S^3 \to M^3$ generate $\pi_3(M^3)$. Then f is a homotopy equivalence by Whitehead Theorem which asserts that any map between simply-connected "nice spaces" which induces an isomorphism between their homology groups is a homotopy equivalence. And manifolds are particular instances of nice spaces.

2

The h-Cobordism Theorem

2.1 The h-Cobordism Theorem and Generalized Poincaré Conjecture

Smale proved a very important special case of the Hurewicz Conjecture in 1961. Namely he showed that any closed (smooth) manifold M homotopy equivalent to the m-sphere S^m, where $m \geq 6$, is homeomorphic to S^m. This verified what had been known as the Generalized Poincaré Conjecture (GPC).

Remark 2.1. Stallings (1965) improved Smale's result to include $m = 5$, and Newman (1966) showed the assumption that M has a smooth structure is superfluous. They both used a technique much different from Smale's, called engulfing.

The fundamental result from which the Generalized Poincaré Conjecture follows is Smale's h-Cobordism Theorem, which we proceed to formulate. For this we need to generalize the notion of m-dimensional manifold to m-dimensional manifold with boundary.

A topological m-dimensional manifold with boundary is again a Hausdorff topological space W whose topology has a countable basis, but which is now assumed to be locally homeomorphic to either $\mathbb{R}^m$ or $[0, \infty) \times \mathbb{R}^{m-1}$; i.e. each point $x \in W$ has an open neighborhood homeomorphic to either $\mathbb{R}^m$ or $[0, \infty) \times \mathbb{R}^{m-1}$. Those points $x \in W$ which have no neighborhood homeomorphic to $\mathbb{R}^m$ form the boundary ∂W of W. It can be shown that either $\partial W = \emptyset$ or ∂W is an $(m-1)$-dimensional manifold (without boundary; i.e. $\partial(\partial W) = \emptyset$) and $W - \partial W$ is an m-dimensional manifold. Furthermore the collar neighborhood theorem says that ∂W has an open neighborhood C, "a collar of ∂W in W" such that C is homeomorphic to $\partial W \times [0, \infty)$ and via a homeomorphism

$$f\colon \partial W \times [0, \infty) \to C \subseteq W$$

such that $f(x, 0) = x$.

A differential structure Φ on a topological m-manifold W with boundary is again defined by giving a collection of charts

$$\psi_x\colon V_x \to U_x$$

for each $x \in W$, as before in defining a smooth structure on a topological m-manifold (without boundary), except now U_x is an open subset of $[0, \infty) \times \mathbb{R}^{m-1}$.

If Φ is a smooth structure on a topological m-manifold W with boundary then ∂W inherits, in a natural way, a smooth structure $\Phi_{\partial W}$ from Φ. And the collar map

$$f \colon \partial W \times [0, \infty) \to C$$

can be chosen to be a diffeomorphism.

Example 2.2. $\mathbb{D}^n = \{x \in \mathbb{R}^n | \, |x| \leq 1\}$ is a smooth n-dimensional manifold with $\partial\mathbb{D}^n = S^{n-1}$. A collar C in this case can be taken to be $\mathbb{D}^n - 0$. We refer to $\mathbb{D}^n$ as the "n-disc" or "n-ball".

A *cobordism* is a manifold W with boundary where ∂W is expressed as the disjoint union of two non-empty open subsets $\partial^- W$ and $\partial^+ W$ (which are now both $(\dim W - 1)$-dimensional manifolds); W is said to be a cobordism from $\partial^- W$ to $\partial^+ W$.

Example 2.3. Here is a 2-dimensional example.

In the pictured example (Figure 2.1), two small (open) discs have been deleted from the 2-dimensional torus $T^2 = S^1 \times S^1$.

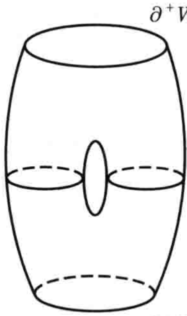


Figure 2.1

Definition 2.4. *An* h-cobordism *is a compact cobordism* W *such that both inclusion maps* $\partial^- W \subseteq W$ *and* $\partial^+ W \subseteq W$ *are homotopy equivalences.*

The cobordism W in the example above is not an h-cobordism since $\pi_1(\partial^- W)$ and $\pi_1(W)$ are not isomorphic; $\pi_1(\partial^- W) \cong \mathbb{Z}$ is an abelian group while $\pi_1(W) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ is not abelian.

But there is a canonical example of an h-cobordism; namely $W = M \times [0, 1]$, the cylinder with base a closed manifold M (Figure 2.2).

It is hard to think of another example. In fact that is the gist of Smale's Theorem.

h-Cobordism Theorem. *Let* W^m *be a smooth* h*-cobordism. If* $\pi_1(W) = 1$ *and* $m \geq 6$*, then* W *is diffeomorphic to* $\partial^- W \times [0, 1]$.

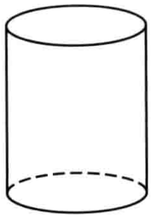


Figure 2.2

Addendum 2.5. *The diffeomorphism* $f\colon \partial^- W \times [0,1] \to W$ *can be chosen so that* $f(x, 0) = x$.

We now give Smale's deduction of the Generalized Poincaré Conjecture (GPC) from his h-Cobordism Theorem. Let D^+ and D^- be a pair of disjoint m-dimensional balls nicely embedded in M (say inside domains of disjoint charts) as shown in Figure 2.3. Let W be M with $\mathring{D}^+ \cup \mathring{D}^-$ deleted. Here $\mathring{D}^+$ and $\mathring{D}^-$ are the interiors of D^+ and D^- respectively. It is easy to see that W is a smooth cobordism from $\partial^- W$ to $\partial^+ W$ where $\partial^- W = \partial D^-$ and $\partial^+ W = \partial D^+$.

Figure 2.3

Lemma 2.6. *W is a simply-connected smooth h-cobordism.*

Before verifying this lemma, we use it to complete Smale's proof of GPC. By his h-Cobordism Theorem, W is diffeomorphic to $S^{m-1} \times [0,1]$ since $\partial D^- = S^{m-1}$; hence the picture above is really as drawn below in Figure 2.4.

The only ambiguity is how the three known pieces in this picture fit together. By Addendum 2.5, the bottom piece fits canonically into the middle piece; hence $W \cup D^-$ can be identified with $\mathbb{D}^m$. Therefore M^m is diffeomorphic to $\mathbb{D}^m \cup_f \mathbb{D}^m$ where $f\colon S^{m-1} \to S^{m-1}$ is a self diffeomorphism; i.e. M^m is a twisted double of $\mathbb{D}^m$ (Figure 2.5).

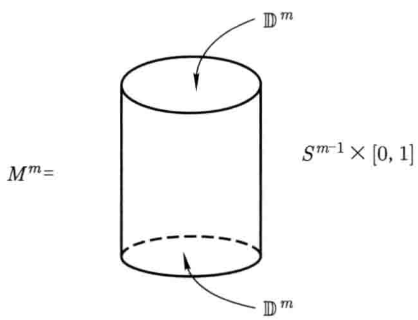


Figure 2.4

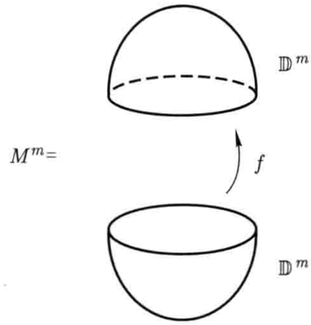


Figure 2.5

Remark 2.7. This is a quite important fact of independent interest which is fundamental to the Kervaire-Milnor analysis of "exotic spheres" to be described later.

Note that if $f = \text{id}$, then $\mathbb{D}^m \cup_{\text{id}} \mathbb{D}^m = S^m$. But we know, by the result of Milnor in 1956 mentioned in Chapter 1, that M^m is in general *not* diffeomorphic to S^m. But we can construct a homeomorphism $S^m \to M^m$ as illustrated below in Figure 2.6.

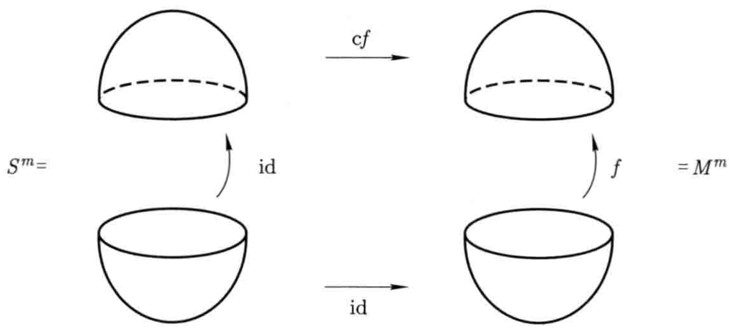


Figure 2.6

In this illustration c$f\colon \mathbb{D}^m \to \mathbb{D}^m$ is the "cone" of f which is defined analytically by

$$\text{c}f(x) = \begin{cases} |x| f(\frac{x}{|x|}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

where $x \in \mathbb{D}^m$; i.e. $|x| \leq 1$. But more informative is its pictorial description below in Figure 2.7.

Thus the proof of GPC is complete once Lemma 2.6 is verified.