2004恒隆数学奖
获奖论文集

区国强 吴恭孚 丘成桐 主编

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數學獎
HANG LUNG
MATHEMATICS AWARDS

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Hang Lung Mathematics Awards
Collection of Winning Papers 2004

恒隆數學獎
HANG LUNG MATHEMATICS AWARDS

數學科學研究所
The Institute of Mathematical Sciences

香港中文大學數學系
Department of Mathematics, The Chinese University of Hong Kong

恒隆地產
HANG LUNG PROPERTIES

Preface

It gives us great pleasure to present the first book comprised of the research papers written by the winners of the 2004 Hang Lung Mathematics Awards (HLMA). 2004 HLMA marked the inaugural competition jointly organized by the Department of Mathematics at The Chinese University of Hong Kong and Hang Lung Properties Ltd. in Hong Kong in December 2004. It was a pioneering collaboration between the education sector (including the strong support from the Hong Kong Education City) and the business sector in the launching of a research-based mathematics competition in Hong Kong.

HLMA came from a long-held dream we have shared: we saw the need to establish a platform where young students could be given the opportunity to discover and nurture their potentials in high-level mathematics research beyond the normal school curriculum. It is our hope that HLMA can help cultivate an independent learning environment in Hong Kong, stimulate critical thinking and encourage creativity among students.

This book has compiled the research papers of the winners of the 2004 HLMA which have displayed a high level of academic standards in terms of methodology, originality and scholarship. We were impressed by the students' thirst for knowledge and their passion for mathematics throughout the competition.

We hope that through HLMA, students would be encouraged to explore the world of mathematics and science, and to continue to cultivate their strengths and realize their potentials to the fullest. We wish to take this opportunity to thank the editors for their tireless efforts and devotion in the editing process. Through the publication of this book, we hope we will continue to inspire a future generation of mathematicians, scientists and scholars.

Professor Shing-Tung Yau	Mr. Ronnie C. Chan
Director, The Institute of Mathematical Sciences The Chinese University of Hong Kong	Chairman Hang Lung Properties Ltd.

October 2012

Acknowledgement

The publishing of this book has been made possible by donations from Hang Lung Properties Ltd.

The contribution to mathematics education from the initiators of Hang Lung Mathematics Awards, Mr. Ronnie C. Chan and Professor Shing-Tung Yau, will be marked in history. Professor Yau provides the academic leadership and vision for the Awards and he is also one of the editors of this volume. Hang Lung Properties Ltd. has been providing the financial support for the Awards and additional funding for the editorial and publishing cost of this volume.

The reputation of an academic award is always established upon its rigorous process of assessment and review. The Scientific Committee is the cornerstone of this intellectual excellence. The Screening Panel would like to thank Dah-Ming Chiu, Kai-Seng Chou, Sydney Chu, Dejun Feng, Jianguo Huang, Jiaxin Hu, Frank Hwang, Hui Rao, Man-Keung Siu, Baorui Song, Guobiao Zhou for their professional judgment.

The Steering Committee has contributed greatly to the popularity of the Awards in society. Many students, teachers, and schools have participated in the competition. The Executive Committee, chaired by Thomas Au, is indebted to the secretariat led by Serena Yip and the assistance of Yue-Hung Tam. The cooperation of Susan Wong and Joyce Leung from Hang Lung Properties has been invaluable. The Hong Kong Education City has been an important partner in this inaugural year, especially on publicity and event organization.

This volume will never be perfect without the commitment of Kung-Fu Ng who have led the editorial work. He would like to express his sincere gratitude to Xinhan Dong and Liren Huang for their careful reading and comment on the winning papers. The technical work of Yue-Hung Tam and the coordination of Mavis Chan had also played a crucial role in the editorial process.

Hang Lung Mathematics Awards

Introduction The Hang Lung Mathematics Awards (HLMA), co-organized by Hang Lung Properties Ltd. and the Department of Mathematics at The Chinese University of Hong Kong, is a biennial research-based mathematics competition for secondary school students in Hong Kong. Founded in 2004 by Mr. Ronnie C. Chan, Chairman of Hang Lung Properties Ltd., and world-renowned mathematician Professor Shing-Tung Yau, a 1982 Fields Medalist and 2010 Wolf Prize recipient, the competition aims to stimulate creativity and to encourage intellectual discovery in mathematics and science among secondary school students in Hong Kong.

Schools are invited to form teams and, under the supervision of a lead teacher, the teams design and carry out a mathematics research project. Each team submits a project report summarizing the findings, which is evaluated by the Scientific Committee in a multi-step process similar to that for the selection of publication in a scientific journal. Short-listed teams are invited to participate in an oral defense of their project before members of the Scientific Committee. This final stage is modeled after a doctoral degree defense and comprises two parts: a public presentation of the research project followed by a closed-door inquiry. The winners of the HLMA will be decided after the oral defense.

Hang Lung Properties Ltd. donated over HK$2 million to each competition. The Department of Mathematics at The Chinese University of Hong Kong provides tuition scholarships to the teachers of the winning schools. The Department of Mathematics handles all administrative, operational and educational aspects of the competition.

Participation Up to five students of the same secondary school may form a team to participate in the competition. The team shall be led by a teacher of the school. After a simple registration process which started in early 2004, each team performed their study and research on a topic selected by the team. In this inaugural year of the HLMA, students had only 6 months to work on the research and had to submit their reports in September 2004. There were 81 teams from over 50 schools participated and 44 of those teams submitted research reports.

Assessment To decide the winners, there are two stages of assessment: research report review and oral defense. The assessment in each stage is the jurisdiction of the Scientific Committee, with the support of the Screening Panel and external experts from the international mathematics community. In the reviewing stage, each research report has to pass an initial screening. Then it is sent to at least two referees of external experts. A shortlist will be selected by the Scientific Committee to proceed to the second stage of assessment, which is the oral defense. In this stage, each team will make a brief presentation of their research in front of the Scientific Committee. The presentation is open to the public. It is then followed by a closed door inquiry. The winners will be decided afterwards.

Awards Up to eight awards are presented to the mathematics projects that meet the highest academic standard in terms of research methodology and originality: Gold, Silver, Bronze, and up to five Honorable Mentions. Each award consists of four components: a "Student Education Award" to be shared equally among team members and applied towards their university studies; a "Teacher Leadership Award" for the supervising teacher; a "School Development Award" to promote mathematics education at the school; and a "Tuition Scholarship" for any teacher at the winning schools to earn a Master of Science (MSc) in Mathematics from The Chinese University of Hong Kong. The winning students and teachers also received a crystal trophy and a certificate while the school was presented with a crystal trophy.

Organization

The two principal committees of the Hang Lung Mathematics Awards are the Scientific Committee and the Steering Committee.

Scientific Committee, 2004

The Scientific Committee comprises world renowned mathematicians and is the academic and adjudicating body of the Hang Lung Mathematics Awards. The Scientific Committee oversees the whole process of assessment, including the review of the research project reports and conducting the oral defense. The committee upholds the academic integrity and high standard of the mathematics research competition as well as the awards.

List of members (affiliations at the time of the event)
Chair: Professor Shing-Tung Yau, The Chinese University of Hong Kong and Harvard University
Professor John H. Coates, Cambridge University
Professor Tony F. Chan, University of California, Los Angeles
Professor Shiu Yuen Cheng, The Hong Kong University of Science and Techology
Professor Ravi S. Kulkarni, Harish-Chandra Research Institute
Professor Ka-Sing Lau, The Chinese University of Hong Kong
Professor Jun Li, Stanford University
Professor Chang-Shou Lin, National Chung Cheng University
Professor Ngaiming Mok, The University of Hong Kong
Professor John Morgan, Columbia University
Professor Duong H. Phong, Columbia University
Professor Dan Stroock, Massachusetts Institute of Technology
Professor Tom Yau-Heng Wan, The Chinese University of Hong Kong
Professor Lo Yang, Chinese Academy of Sciences
Professor Andrew Chi-Chih Yao, Tsing Hua University

Screening Panel is a subcommittee of the Scientific Committee. It handles the initial screening of each report, supervises external review process, and serves as a bridge between all referees and members of Scientific Committee.

Chair: Professor Tom Yau-Heng Wan, The Chinese University of Hong Kong
Professor Wing Sum Cheung, The University of Hong Kong
Professor Conan Nai Chung Leung, The Chinese University of Hong Kong

Steering Committee, 2004

The Steering Committee comprises mathematicians and representatives from different sectors of society and serves as the advisory body. The committee also includes mathematics department heads of major Hong Kong universities. Some members from the Scientific Committee and Executive Committee also serve in the Steering Committee so that it has an overall perspective of all the aspects.

List of members (affiliations at the time of the event)

Chair: Professor Shing-Tung Yau, The Institute of Mathematical Sciences, The Chinese University of Hong Kong and Harvard University

Professor Thomas Kwok-Keung Au, Mathematics, The Chinese University of Hong Kong

Professor Shiu Yuen Cheng, Chairman of Mathematics Department, The Hong Kong University of Science and Technology

Mr. Bankee Kwan, Chairman, Celestial Asia Securities Holdings Ltd.

Professor Ka-Sing Lau, Chairman of Mathematics Department, The Chinese University of Hong Kong

Mr. Siu-Leung Ma, CEO, Fung Kai Public School

Mr. Chun-Kau Poon, Former principal, St. Paul Co-educational College

Professor Man Keung Siu, Chairman of Mathematics Department, University of Hong Kong

Ms. Susan Wong, Chairman's Office, Hang Lung Properties Ltd.

Professor Hung Hsi Wu, Mathematics, UC Berkeley

Professor Lo Yang, Professor, Chinese Academy of Sciences

Mr. Chee-Tim Yip, Principal, Pui Ching Middle School

Executive Committee is a subcommittee of the Steering Committee. It is in charge of the administration, promotion, team registration, communication, and organization of events including the oral defense.

Chair: Professor Thomas Kwok-Keung Au, The Chinese University of Hong Kong

Dr. Leung-Fu Cheung, The Chinese University of Hong Kong

Dr. Ka-Luen Cheung, The Chinese University of Hong Kong

Secretariat: Ms. Serena Wing-Hang Yip, The Chinese University of Hong Kong

Gold, Silver, and Bronze

GOLD

Team member(s): Edward Sin Tsun Fan
School: Sha Tin Government Secondary School
Team Teacher: Ms. Fei Wong
Topic: *Marked Ruler as a Tool for Geometric Constructions – from angle trisection to n-sided polygon*

SILVER

Team member(s): Fan Fan Lam, Ho Fung Tang, Ho Yin Poon, Lok Hin Yim, Yiu Tak Wong
School: Munsang College (Hong Kong Island)
Team Teacher: Mr. Wai Man Chu
Topic: *Further Investigation on Buffon's needle problem*

BRONZE

Team member(s): Ting Fai Man, Hoi Kwan Lau, Shek Yeung, Man Kit Ho
School: Sha Tin Government Secondary School
Team Teacher: Ms. Fei Wong
Topic: *Ghost Leg*

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MARKED RULER AS A TOOL FOR GEOMETRIC CONSTRUCTIONS - FROM ANGLE TRISECTION TO N-SIDED POLYGON

TEAM MEMBER
Sin-Tsun Edward Fan[1]

SCHOOL
Sha Tin Government Secondary School

Abstract. The ancient Greeks raised the famous problem of trisecting an arbitrary angle with a compass and an unmarked ruler, which was proved impossible. Such a construction is possible if a marked ruler is used instead. In this article, the possible geometric constructions by a compass and a marked ruler are studied.[2]

1. Introduction

Trisecting an arbitrary angle with a compass and straightedge was one of the famous ancient Greeks unsolved construction problems. Together with duplicating the cube, these problems have been pending to be resolved for more than 2000 years.

Plato (427–347 BC) defined clearly the rules of ruler and compass construction, which implies that the marks or scales in the ruler should not be relevant to the geometric construction. Many learned people tried employing different tools and methods to tackle the problem, in particular, the interesting and simple construction algorithm proposed by Archimedes (287–212 BC) who had employed a marked ruler and compass to solve the trisecting problem, which was very close to the Platos rules.

In 19th Century, Pierre L. Wantzel (1814–1848) proved in 1837 that based upon Plato's criteria, it is impossible to trisect an arbitrary angle. The

[1]This work is done under the supervision of the author's teacher, Ms. Fei Wong.
[2]The abstract is added by the editor.

problem became even more interesting after it was proved to be possible because of the "magic" marked ruler, which have opened a new area for the study of geometric constructions with marked ruler and compass.

Theorem 1.1. *If we have a marked ruler and a compass, then it is possible to trisect an arbitrary angle.*

Proof. Archimedes proved this theorem by giving a construction algorithm. As shown in figure 1.1 below, let $\angle AOB$ be the angle being trisected and the lengths $|OA| = |OB| = 1$, which is the distance between the two marks on the ruler. Draw a semicircle centered at O from B through A. If we mark C and D such that C is on the semi-circle and D is the intersection of the lines OB and AC with $|CD| = 1$, then $\angle AOB = 3\angle ADB$.

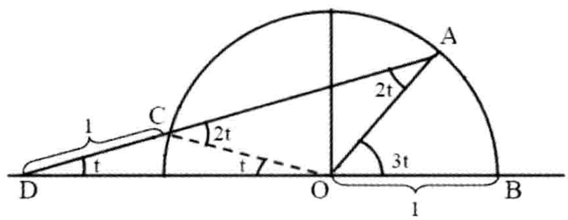

FIGURE 1.1

Let $\angle ADB = t$. Then $\angle COD = t$ and $\angle OCA = \angle OAC = 2t$ (base angles of isosceles triangle).

By the interior angle sum of triangle, $\angle AOC = \pi - 4t$. Hence,

$$\angle AOB = \angle COD.\angle AOC = \pi - t - (\pi - 4t) = 3t = 3\angle ADB. \qquad \square$$

In theorem 1.1, Archimedes made use of the so-called marked ruler instead of a typical straight edge in the construction. Some people criticized that Archimedes did not respect the conventional definition of ruler and his approach was not strict enough hence it was not commonly accepted.

In spite of this, it was quite natural, when compared with using conics, trisectrix or some other strange curves to give a solution to angle trisection, marked ruler was an easy available tool in real life. One should appreciate why adding two marks on a ruler makes the impossible becomes possible. In this project, we try to give some terminology of marked ruler and clarify which types of geometric constructions are possible by using marked ruler and compass.

Definition 1.2. *A ruler or more precisely a straight edge with two notches on it is called a marked ruler. Without loss of generality, the distance between the two notches is taken to be* 1.

From definition 1.2, we notice that the marked ruler introduces the concept of unit length into the system of geometric construction. It allows us to cut off equal distance on a straight line in particular, and we will show that the marked ruler is much more useful with the help of compass in the subsequent sections. In order to study geometric construction algebraically, we will introduce a rectangular coordinate system on the two dimensional Euclidean Space. Moreover, by the end of this report, we will study how the problem of constructing regular n-sided polygon is related to the construction by using marked ruler and compass.

Now, let us define the meaning of constructible points and constructible curves.

Definition 1.3. *A constructible curve is a curve constructed from given quantities such as points, lengths, etc, which are provided by given points and constructible points. A constructible point is a point of intersection of two constructible curves.*

Our task is getting clearer that we treat construction as drawing the constructible curves. If we know what curves marked ruler and compass can draw, we will know the properties of the constructible points. Before going deep into our main goal, lets take a brief review on the general construction.

2. Classification of Construction

Up to this moment, our understanding on the term "construction" is too vague for a mathematical theory to build on. In the following sections, we will give a clear definition of "construction", then classify different types of construction and their relative field of extension.

Definition 2.1. *A construction C is defined to be a finite set of constructible points $\{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_n\}$, where $0 = (0,0)$, $\mathbf{u} = (1,0)$ and $A_0 = (0,1)$, such that A_{n+1} is a point of intersection of any two of the constructible curves γ_i constructed from the points in the sub-construction $C_k = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_k\}$ where $k = 1, 2, \ldots, n$ under specific construction rules.*

Definition 2.2. *Let $\mathcal{C} = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_n\}$ be a construction of n steps and $\mathcal{C}_k = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_k\}$ where $k \leqslant n$ be a sub-construction of C. Also, let $z_1, z_2, \ldots, z_n$ be the complex numbers that represent the points $A_1, A_2, \ldots, A_n$ respectively. Then, $\mathrm{K}[\mathcal{C}] = \mathbb{Q}(\mathbf{i}, z_1, z_2, \ldots, z_n)$ is defined to be the field of extension of $\mathbb{Q}$ by construction $\mathcal{C}$. Note that $\mathrm{K}[\mathcal{C}]$ is the smallest field that contains $\mathbf{i}, z_1, \ldots, z_n$ and we have*

$$\mathrm{K}[\mathcal{C}_k] = \mathrm{K}[\mathcal{C}_{k-1}](z_k) \text{ for } k = 1, 2, \ldots, n.$$

Remark: Since $0 = (0, 0)$, $\mathbf{u} = (1, 0)$ and $A_0 = (0, 1)$, it is easy to see that $\mathrm{K}[\mathcal{C}_0] = \mathbb{Q}(\mathbf{i})$.

Definition 2.3. *A construction is called plane if it can be solved by using ruler and compass only.*
A construction is called solid if it can be solved by using conic sections only.
A construction is called higher dimensional if it is not plane or solid.

Remark: This classification was introduced by Pappus, but I replace the term "linear" by "higher dimensional" since it will be more appropriate.

Definition 2.4. *For plane constructions, a constructible straight line is a line, which passes through two constructible points; and a constructible circle is a circle centered at a constructible point, which passes through another constructible point.*

Before we state the well-known theorem for ruler and compass construction, we give a lemma, which is used to prove this theorem.

Lemma 2.5. *If two circles intersect or a circle and a straight line intersects, where the coefficients of the equations of the circles and straight lines are in field K, then the coordinates of the point of intersection lie in a field of quadratic extension over K.*

Proof. Firstly, let $y = mx + c$ and $x^2 + y^2 + dx + ey + f = 0$ be the equations of a straight line and a circle with m, c, d, e, $f \in \mathrm{K}$ respectively. Then, by solving the two equations, we have

$$x^2 + (mx + c)^2 + dx + e(mx + c) + f = 0$$
$$\Rightarrow (1 + m^2)x^2 + (2mc + d + me)x + c^2 + ce + f = 0$$

Note that the coefficients of the above equation are in K, its roots lie in a field of quadratic extension of K. Also from $y = mx + c$, y is linear to x and so the coordinates of the points of intersection lie in a field of quadratic extension over K.

Secondly, let $x^2 + y^2 + d_1x + e_1y + f_1 = 0$ and $x^2 + y^2 + d_2x + e_2y + f_2 = 0$ be the equations of two distinct circles with d_1, d_2, e_1, e_2, f_1, $f_2 \in$ K. Then by subtracting the two equations, it yields a straight line $(d_2 - d_1)x + (e_2 - e_1)y + (f_2 - f_1) = 0$ with its coefficients in K. Hence, by the above argument, the coordinates of the points of intersection lie in a field of quadratic extension over K. □

Theorem 2.6. *A point (x, y) has a plane construction if and only if $x+y\mathbf{i} \in \mathbb{C}$ lies in a sub-field* K *of $\mathbb{C}$ such that $\exists K_i$, $i = 0, 1, \ldots, n$, satisfying that*

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$ or 2 for $j = 1, 2, \ldots, n$.

Proof. Let $\mathcal{C} = \{0, \mathbf{u}, A_0, A_1, \ldots, A_{n-1}\}$ be a plane construction with $A_{n-1} = (x, y)$. It is clear that when two constructible lines intersect, no extension of field is needed. Also note that the extension $\mathbb{Q}(\mathbf{i})$ is of degree 2. Then, by lemma 2.5, we must have $[\mathrm{K}[\mathcal{C}_{k+1}] : \mathrm{K}[\mathcal{C}_k]] = 1$ or 2, where $k = 1, 2, \ldots, n-2$.

Hence, by letting $\mathrm{K}_{j+1} = \mathrm{K}[\mathcal{C}_j]$, we have

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}[\mathcal{C}] = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1, 2$ for $j = 1, 2, \ldots, n$.

Conversely, given a tower of fields

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

where the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$ or 2 for $j = 1, 2, \ldots, n$. It suffices to verify that there is a plane construction associated to each step of field extension. If $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$, then $\mathrm{K}_j = \mathrm{K}_{j-1}$ and the result is trivial. If $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2$, let $z_j = x_j + y_j\mathbf{i}$ such that $\mathrm{K}_j = \mathrm{K}_{j-1}[z_j]$ where $z_j \notin \mathrm{K}_{j-1}$ for $j = 1, 2, \ldots, n$. Since the degree of extension is 2, both x_j and y_j are roots of certain quadratic equations with coefficients in K_{j-1}, say, $x^2 + ax + b = 0$ and $y^2 + py + q = 0$ respectively. Then by constructing the circle $x^2 + y^2 + ax + b = 0$ and the line $y = 0$, we solve x_j as the x-coordinate. Similarly, y_j can be obtained as the y-coordinate of the intersection of the circle $x^2 + y^2 + px + q = 0$ and the line $x = 0$. Hence $z_j = x_j + y_j\mathbf{i}$ has a plane construction. □