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北京航空航天大学“研究生英文教材”系列丛书

Introduction to Matrix Theory

矩阵理论引论

Li Hongyi　Zhao Di

李红裔　　赵　迪

北京航空航天大学出版社

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Abstract

This textbook contains six chapters, covering reviews on linear algebra; matrix functions; matrix decompositions such as singular value decompositions and spectral decompositions; generalized inverses; tensor product and nonnegative matrices. Each chapter includes many examples and problems to help students master the presented material . There are no prerequisites except for some basic knowledge on linear algebra.

This book aims to provide the material for a basic matrix theory course to senior undergraduates or postgraduates in science and engineering, and can be used as a self-contained reference for a variety of readers.

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Li Hongyi　Zhao Di
李红裔　赵　迪
责任编辑　宋淑娟
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Preface

As a branch of mathematics, matrix theory has long been a fundamental tool in other mathematical disciplines such as numerical analysis, probability theory, statistics, optimization, and has fertile applications in computer science, machine learning, economics, signal processing, etc.

Throughout this book, we tend to pay more attention on the computational aspects of matrix theory, despite that we also have presented detailed proofs of some important results. Thus, this book emphasizes the basic concepts, practical formulas, and a wide variety of examples.

The material of this textbook is standard in that the topics covered are linear algebra; matrix functions; matrix decompositions such as ***QR*** decomposition, full rank decomposition, singular value decomposition, spectral decomposition; generalized inverses; tensor product; nonnegative matrices. These contents are divided into six chapters, of which each corresponds to a topic.

In this textbook, we provide basic concepts and classical results of matrix theory. The book is designed to help the senior undergraduate students, or junior graduate students in science and engineering to master the material of a standard matrix theory course, and of course could also be used as a self-contained reference for a variety of readers. There are no prerequisites other than some basic knowledge of linear algebra such as linear space, linear transformation, the Jordan canonical form and so on, which, however, will be reviewed in the first chapter.

We appreciate our colleague Prof. Chen Zuming for contributing valuable reviews and suggestions, and we are grateful to students who have taken a lot of time to convey their reactions to the class while preparing this book.

Li Hongyi　Zhao Di
Beihang University
January, 2014

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Chapter 1
Introduction to Linear Algebra

The aim of this chapter is to review some basic knowledge about linear algebra.

1.1 The linear space

1.1.1 Fields and mappings

In discussing mathematical problems, it is necessary to determine the scope of numbers considered. For instance, when solving a single-variable quadratic equation, if we consider in the scope of real numbers, the equation may have no solutions. However, there always exist complex solutions. That is to say, considering the same problem in different scopes may yield different solutions. On the other hand, different sets of numbers may have common properties. In order to unify these common properties, we will introduce a general concept.

Definition 1.1.1 Let K be a set of numbers containing at least one nonzero number. If the sum, difference, product and quotient of any two numbers in K is still a number in K, then K is called a **field.**

It is not difficult to verify that the sets of all rational numbers $\mathbf{Q}$, real numbers $\mathbf{R}$ and complex numbers $\mathbf{C}$ are fields, which could be called the rational filed, real filed and complex field, respectively. While, neither of the sets of natural numbers and integers can be called a field. A slightly more difficult instance is the set described as the following:

$$a + b\sqrt{2}, \quad (a, b \in \mathbf{Q})$$

which also constitutes a filed. Besides these fields, we can still find many (infinite exactly) fields.

Definition 1.1.2 Let S and S' be two sets. For a given rule σ, if for any $\alpha \in S$, there exists a certain corresponding $\beta \in S'$ by this rule, then σ is called a **mapping** from S to S', which could be denoted by $\sigma: S \to S'$. The correspondence of α to β can be denoted by $\sigma(\alpha) = \beta$. β is called the **image** of α with respect to σ, and α is called the **preimage** of β with respect to σ. Correspondingly, the set S is called the **domain** of σ, and the set $R(\sigma) = \{\sigma(\alpha) \mid \alpha \in S\} \subset S'$ is the **range** of σ. If for any $\alpha_1, \alpha_2 \in S, \alpha_1 \neq \alpha_2$ implies $\sigma(\alpha_1) \neq \sigma(\alpha_2)$, then σ is a **injection.** If for any $\beta \in S'$, there exists $\alpha \in S$ such that $\sigma(\alpha)=\beta$ or $R(\sigma)=S'$, then σ is a **surjection.** If a mapping σ is a injection as well as a surjection, then σ is called a **bijection.** Particularly, when $S=S'$, the mapping σ is a **transformation.**

Next we shall see some examples.

Example 1.1.1 $\mathbf{Z}$ is the set of all integers, and $\mathbf{Z}'$ is the set of all even integers. We define the following mapping

$$\sigma(n) = 2n \quad (n \in \mathbf{Z}).$$

Then σ is a bijection from $\mathbf{Z}$ to $\mathbf{Z}'$.

Example 1.1.2 Let $K^{n\times n}$ be the set of all square matrices on the field of K with order n. Given the following two mappings

$$\sigma(\boldsymbol{A}) = \det \boldsymbol{A} \quad (\boldsymbol{A} \in K^{n\times n}), \quad \tau(a) = a\boldsymbol{I}_n \quad (a \in K),$$

where σ is a mapping from $K^{n\times n}$ to K, and τ is also a mapping from K to $K^{n\times n}$. From Definition 1.1.2, σ is a surjection but not a injection, and τ is a injection but not a surjection.

Example 1.1.3 We denote $P[t]$ as the set of all polynomials on the field K, and define the following mapping by derivative

$$\sigma[f(t)] = f'(t) \quad \{f(t) \in P[t]\}.$$

Then σ is a transformation in the set $P[t]$. It is a surjection, but not a injection. If we denote $P[t]_n$ as the set of all polynomials with degrees less than n on the field K, then the aforementioned σ is a transformation in the set $P[t]_n$, and it is neither a surjection nor a injection.

Given two mappings, we can define the equality and their product.

Definition 1.1.3 Let σ_1 and σ_2 be two mappings from S to S'. If $\sigma_1(a)=\sigma_2(a)$ holds for any $a \in S$, then σ_1 is equal to σ_2, denoted by $\sigma_1=\sigma_2$. Let σ be a mapping from S to S' and τ be a mapping from S' to S''. The product of σ and τ, denoted by $\tau\sigma$, is a mapping from S to S'' defined as the following

$$(\tau\sigma)(a) = \tau[\sigma(a)] \quad (a \in S).$$

The product of mappings is a generalization of the concept of composite functions. However, not all mapping pairs have products. From Definition 1.1.3, the necessary and sufficient condition that σ and τ have a product is that the domain of τ contains the range of σ.

For example, the two mappings σ and τ defined in Example 1.1.2 have the products

$$(\tau\sigma)(\boldsymbol{A}) = \tau[\sigma(\boldsymbol{A})] = \tau(\det \boldsymbol{A}) = (\det \boldsymbol{A})\boldsymbol{I}_n \quad (\boldsymbol{A} \in K^{n\times n}),$$
$$(\sigma\tau)(a\boldsymbol{I}_n) = \sigma[\tau(a)] = \sigma(a\boldsymbol{I}_n) = \det(a\boldsymbol{I}_n) = a^n \quad (a \in K).$$

$\tau\sigma$ is a transformation in $K^{n\times n}$, and $\sigma\tau$ is a transformation in K. This example also shows that the product of two mappings is not communicative, i. e., $\tau\sigma \neq \sigma\tau$. It is not difficult to prove that the product is associative, i. e., $(\omega\tau)\sigma=\omega(\tau\sigma)$, where σ is a mapping from S to S', τ is a mapping from S' to S'', and ω is a mapping from S'' to S'''.

1.1.2 Definition of the linear space

The linear space is one of the basic concepts in linear algebra, which is a further abstraction and generalization of the n-dimensional real vector space. It has a well defined addition and scalar multiplication operation satisfying some particular properties. The linear space is a special algebraic system. In order to introduce its definition, we shall first introduce some preliminary concepts.

Definition 1.1.4 Let V be a nonempty set. If there exists a operation "+" called the addition such that for any $u, v \in V$, there exists a unique element in V correspondingly, which is called the sum of u, v and denoted by $u+v$. The addition operation should satisfy the following laws:

① Associate: $u+v=v+u$, $\forall u, v \in V$.

② Communicative: $(u+v)+w=u+(v+w)$, $\forall u, v, w \in V$.

③ There exists an element $\theta \in V$ called **"zero"** such that $u+\theta=u$, for $\forall u \in V$.

④ For $\forall u \in V$, there exists a unique element $-u \in V$ such that $u+(-u)=\theta$.

In this case, V is called a **additive group** under the addition, denoted by $(V, +)$.

Example 1.1.4 Under the ordinary addition operation on real numbers, the sets of all integers, rational numbers, real numbers, complex numbers are all additive groups, which are denoted by $(\mathbf{Z}, +)$, $(\mathbf{Q}, +)$, $(\mathbf{R}, +)$, $(\mathbf{C}, +)$, respectively.

Example 1.1.5 With the natural multiplication operation on numbers, the set of all nonzero rational numbers is an additive group, denoted by $(\mathbf{Q}\backslash\{0\}, +)$. Similarly, $(\mathbf{R}\backslash\{0\}, +)$, $(\mathbf{C}\backslash\{0\}, +)$ are both additive groups, while $(\mathbf{Z}\backslash\{0\}, +)$ is not an additive group.

Definition 1.1.5 Assuming that $(V, +)$ is an additive group and K is a field, if for any $\lambda \in K$ and $v \in V$, there exists a unique element $\lambda v \in V$ satisfying

① $\lambda(u+v)=\lambda u+\lambda v$, for $\forall \lambda \in K, u, v \in V$.

② $(\lambda+\mu)u=\lambda u+\mu u$, for $\forall \lambda, \mu \in K, u \in V$.

③ $\lambda(\mu u)=(\lambda\mu)u$, for $\forall \lambda, \mu \in K, u \in V$.

④ $1u=u, u \in V$.

Then V is called a **linear space** or **vector space** on the field K. Any element in V is called a **vector**, and element in K is called a **scalar**.

Example 1.1.6 The geometry space (e.g., the line $\mathbf{R}$, the planar $\mathbf{R}^2$, the 3-dimensional space $\mathbf{R}^3$) is a linear space on the field $\mathbf{R}$ with respect to the natural addition for vectors.

More generally, let

$$V = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}, x_i \in \mathbf{R}, i = 1, \cdots, n\},$$

and $K=\mathbf{R}$. For any $\lambda \in \mathbf{R}$, $\boldsymbol{y}=(y_1, y_2, \cdots, y_n)^{\mathrm{T}}$, we define $\boldsymbol{x}=\boldsymbol{y} \Leftrightarrow x_i=y_i$ for $i=1, 2, \cdots, n$ and

$$\boldsymbol{x}+\boldsymbol{y} = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n)^{\mathrm{T}},$$

$$\lambda\boldsymbol{x} = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)^{\mathrm{T}}, \quad \boldsymbol{\theta} = (0, 0, \cdots, 0)^{\mathrm{T}}.$$

It is not difficult to verify that V is a linear space on $\mathbf{R}$, which is called a n-dimensional real vector space, and denoted by $\mathbf{R}^n$.

Example 1.1.7 The set $K^{m\times n}$ consisting of all $m\times n$ matrices on K is a linear space according to the addition and scalar multiplication for matrices. $K^{m\times n}$ is usually called the matrix space.

Example 1.1.8 The set $P[t]_n$ of all polynomials with degrees less than $n+1$ on a field K is a linear space, according to the natural addition and multiplication for polynomials, which is usually called the polynomial space.

Example 1.1.9 The set $C[a,b]$ consisting of all the continuous real functions defined in $[a,b]$ is a linear space, according to the natural addition and scalar multiplication for functions.

A more complicated example is the following.

Example 1.1.10 Let $\boldsymbol{A}\in\mathbf{C}^{m\times n}$, and $W=\{\boldsymbol{x}\in\mathbf{C}^n \mid \boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}$. It is not difficult to verify that W is a linear space on the field $\mathbf{C}$. In many occasions, W is alternatively called the null (or kernel) space of $\boldsymbol{A}$, and denoted by $N(\boldsymbol{A})$.

1.1.3 Basis and dimension

We have known some properties of vectors in $\mathbf{R}^n$ such as the linear combination, linear dependence, linear independence and so on. For a general linear space, there are similar concepts and properties.

Definition 1.1.6 Let V be a linear space over K, $\boldsymbol{\alpha}_i\in V(i=1,2,\cdots,n)$, $\boldsymbol{\alpha}\in V$. If there exist numbers $k_i\in K(i=1,2,\cdots,n)$ such that

$$\boldsymbol{\alpha}=k_1\boldsymbol{\alpha}_1+k_2\boldsymbol{\alpha}_2+\cdots+k_n\boldsymbol{\alpha}_n,$$

then $\boldsymbol{\alpha}$ is called a **linear combination** of vectors $\boldsymbol{\alpha}_i\ (i=1,2,\cdots,n)$, or $\boldsymbol{\alpha}$ can be **linearly represented** by vectors $\boldsymbol{\alpha}_i(i=1,2,\cdots,n)$.

If there exist n numbers $k_i\in K(i=1,2,\cdots,n)$ which are not all zeros such that

$$k_1\boldsymbol{\alpha}_1+k_2\boldsymbol{\alpha}_2+\cdots+k_n\boldsymbol{\alpha}_n=\boldsymbol{\theta},$$

then $\boldsymbol{\alpha}_i\in V(i=1,2,\cdots,n)$ are **linear dependent**. While, if the equation above holds if and only if $k_1=k_2=\cdots=k_n=0$, then $\boldsymbol{\alpha}_i\in V(i=1,2,\cdots,n)$ are **linear independent**.

Definition 1.1.7 Let V be a linear space over K, $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_r\}$ and $\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_s\}$ are two vector sets in V. If for all $i=1,2,\cdots,r$, $\boldsymbol{\alpha}_i$ is a linear combination of $\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_s\}$, then we say $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}$ can be **linearly represented** by $\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_s\}$. If conversely $\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_s\}$ can be linearly represented by $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_t\}$, then the vector set $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}$ is called equivalent to the vector set $\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\cdots,\boldsymbol{\beta}_s\}$. If a vector set $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{a}_m\}$ has t linear independent vectors $\{\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\cdots,\boldsymbol{\varepsilon}_t\}\in\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}$, and $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}$ can be linearly represented by $\{\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\cdots,\boldsymbol{\varepsilon}_t\}$, then we say the vector set $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}$ has rank t, and $\{\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\cdots,\boldsymbol{\varepsilon}_t\}$ is a maximal linear independent vector set of $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}$. Denote $\operatorname{rank}\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\cdots,\boldsymbol{\alpha}_m\}=t$.

Definition 1.1.7 can be seen as a copy of the relevant concepts in $\mathbf{R}^n$. We should note that the "addition" and "scalar multiplication" are replaced by the addition and multiplication defined for V, and the "zero" is replaced by $\boldsymbol{\theta}$, which is the **zero element** in V.

Many properties for vectors in $\mathbf{R}^n$ still hold for the more general linear spaces. We list some of them as the following.

Conclusion 1.1.1 A single vector is linear dependent, if and only if it is the zero element. A vectors set consisting of more than one element is linear dependent if and only if there exists at least one vector that can be linearly represented by other vectors.

Conclusion 1.1.2 If $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ is linear independent, and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r, \boldsymbol{\beta}$ is linear dependent, then $\boldsymbol{\beta}$ can be uniquely linearly represented by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$.

Conclusion 1.1.3 Any two maximal linear independent vector sets of a fixed vector set are equivalent.

Conclusion 1.1.4 Equivalent vector sets have the same rank.

Example 1.1.11 In the linear space $K^{m\times n}$, let $\boldsymbol{E}_{ij}$ $(i=1,2,\cdots,m; j=1,2,\cdots,n)$ be an $m\times n$ matrix, of which the element lying in the i-th row and j-th column is 1, and the others are zeros. Suppose that there exist numbers k_{ij} such that $\sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij}\boldsymbol{E}_{ij} = \boldsymbol{0}$, it can be easily deduced that $k_{ij}=0$ for $i=1,2,\cdots,m, j=1,2,\cdots,n$. This shows that $\boldsymbol{E}_{ij}$ $(i=1,2,\cdots,m; j=1, 2,\cdots,n)$ are linear independent. On the other hand, for any $m\times n$ matrix $\boldsymbol{A}=(a_{ij})_{m\times n}$, it can be linearly represented by $\{\boldsymbol{E}_{ij}\}$ from the fact that $\boldsymbol{A}=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\boldsymbol{E}_{ij}$.

Example 1.1.12 In the linear space $P[t]$, the polynomials $1, t, t^2, \cdots, t^N$ are linear independent. The reason relies on that if there exist numbers $k_0, k_1, k_2, \cdots, k_N \in K$ such that $k_0+k_1t+k_2t^2+\cdots+k_Nt^N=0$, then the following equations hold

$$\begin{cases} k_1 + 2k_2t + \cdots + Nk_Nt^{N-1} = 0, \\ 2k_2 + \cdots + N(N-1)k_Nt^{N-2} = 0, \\ \qquad\vdots \\ N!k_N = 0. \end{cases}$$

The unique solution of equations above are $k_0=k_1=k_2=\cdots=k_N=0$.

Definition 1.1.8 Let V be a linear space. If there exist n elements $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ such that

① $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ are linear independent,

② any element $\boldsymbol{\alpha}$ can be linearly represented by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$,

then $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ is called a **basis** of V, and n is called the **dimension** of V, denoted by dim V. The linear space V with dimension n is called a n dimensional linear space, denoted by V^n, which is also called a finite dimensional linear space. If for any given positive integer N, we can always find N linear independent elements, then the space V is a infinite dimensional linear space. If there exist no linear independent vectors, then the dimension of V is zero.

Let $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ be a basis of V^n, then V^n can be denoted by

$$V^n = \{k_1\boldsymbol{\alpha}_1 + k_2\boldsymbol{\alpha}_2 + \cdots + k_n\boldsymbol{\alpha}_n \mid k_1, k_2, \cdots, k_n \in K\},$$

which clearly depicts the structure of V^n.

Example 1.1.13 From Example 1.1.11, $K^{m\times n}$ is an mn dimensional linear space, and $\{\boldsymbol{E}_{ij}\}$ $(i=1,2,\cdots,m; j=1,2,\cdots,n)$ is a basis.

Example 1.1.14 In $P[t]_n$, any polynomial $f(t)=a_0+a_1t+\cdots+a_nt^n$ can be represented by the linear independent polynomial set consisting of $1, t, \cdots, t^n$, which is a basis of $P[t]_n$ and $P[t]_n$ is $n+1$ dimensional. From Example 1. 1. 12, $P[t]$ is an infinite dimensional linear space.

From Definition 1. 1. 8, it can be proved that any vector set consisting of n linear independent elements is a basis. That is to say, the basis of a linear space is not unique. For instance, both of the following two sets of vectors are bases of the 4 dimensional space $K^{2\times 2}$

$$\boldsymbol{E}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\boldsymbol{G}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{G}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{G}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{G}_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Throughout this chapter, we will mainly focus on finite dimensional spaces.

1.1.4 Coordinate

In the study on the analytic geometry, the coordinate is a very important tool, which plays the same role as in a linear space.

Definition 1. 1. 9 Let V be an n dimensional linear space over a field K, and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ be its basis. Then any element α can be uniquely linearly represented by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$:

$$\boldsymbol{\alpha} = x_1\boldsymbol{\alpha}_1 + x_2\boldsymbol{\alpha}_2 + \cdots + x_n\boldsymbol{\alpha}_n \quad (x_1, x_2, \cdots, x_n \in K).$$

$x_1, x_2, \cdots, x_n$ is then called the **coordinate** of $\boldsymbol{\alpha}$ with respect to $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$, denoted by $(x_1, x_2, \cdots, x_n)^{\mathrm{T}}$.

Example 1. 1. 15 In the n dimensional linear space K^n, we choose the following basis

$$\boldsymbol{e}_1 = (1,0,\cdots,0), \boldsymbol{e}_2 = (0,1,\cdots,0), \cdots, \boldsymbol{e}_n = (0,0,\cdots,1).$$

Then the coordinate of vector $\boldsymbol{b}=(b_1, b_2, \cdots, b_n)$ with respect to the basis above is $(b_1, b_2, \cdots, b_n)^{\mathrm{T}}$. While with respect to the following basis

$$\boldsymbol{a}_1 = (1,0,\cdots,0), \boldsymbol{a}_2 = (1,1,\cdots,0), \cdots, \boldsymbol{a}_n = (1,1,\cdots,1),$$

the coordinate of $\boldsymbol{b}$ is $(b_1 - b_2, b_2 - b_3, \cdots, b_{n-1} - b_n, b_n)^{\mathrm{T}}$.

Example 1. 1. 16 In the $n+1$ dimensional linear space $P[t]_n$, with respect to the basis $1, t, t^2, \cdots, t^n$, the coordinate of the polynomial $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ is $(a_0, a_1, \cdots, a_n)^{\mathrm{T}}$.

Example 1. 1. 17 In the 4 dimensional linear space $K^{2\times 2}$, find the coordinates of the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$ with respect to the following two bases

$$\boldsymbol{E}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.1.1}$$

$$\boldsymbol{G}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{G}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{G}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{G}_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.1.2}$$

Solution: Since $\boldsymbol{A} = 0\boldsymbol{E}_{11} + 1\boldsymbol{E}_{12} + 2\boldsymbol{E}_{21} - 3\boldsymbol{E}_{22}$, then the coordinate of $\boldsymbol{A}$ with respect to equation(1. 1. 1) is $(0,1,2,-3)^{\mathrm{T}}$. Assuming that

$$\boldsymbol{A} = x_1\boldsymbol{G}_1 + x_2\boldsymbol{G}_2 + x_3\boldsymbol{G}_3 + x_4\boldsymbol{G}_4,$$

the following linear equations hold

$$\begin{cases} x_2 + x_3 + x_4 = 0, \\ x_1 \quad + x_3 + x_4 = 1, \\ x_1 + x_2 \quad + x_4 = 2, \\ x_1 + x_2 + x_3 \quad = -3. \end{cases}$$

The solution to this equation is $x_1 = 0, x_2 = -1, x_3 = -2, x_4 = 3$. Thus the coordinate of **A** with respect to equation(1.1.2) is $(0, -1, -2, 3)^{\mathrm{T}}$.

With the help of coordinates, not only the abstract element $\boldsymbol{\alpha}$ in V^n can be connected to the concrete vector $(x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ in K^n, but also the abstract linear operations such as "addition ", "scalar product" in V^n can be connected to the addition and scalar product operations in K^n.

Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ be a basis of V^n, and for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in V^n$

$$\boldsymbol{\alpha} = x_1\boldsymbol{\alpha}_1 + x_2\boldsymbol{\alpha}_2 + \cdots + x_n\boldsymbol{\alpha}_n, \quad \boldsymbol{\beta} = y_1\boldsymbol{\alpha}_1 + y_2\boldsymbol{\alpha}_2 + \cdots + y_n\boldsymbol{\alpha}_n,$$

then

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (x_1 + y_1)\boldsymbol{\alpha}_1 + (x_2 + y_2)\boldsymbol{\alpha}_2 + \cdots + (x_n + y_n)\boldsymbol{\alpha}_n,$$
$$k\boldsymbol{\alpha} = kx_1\boldsymbol{\alpha}_1 + kx_2\boldsymbol{\alpha}_2 + \cdots + kx_n\boldsymbol{\alpha}_n,$$

i.e., the coordinate of $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is

$$(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)^{\mathrm{T}} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} + (y_1, y_2, \cdots, y_n)^{\mathrm{T}},$$

and the coordinate of $k\alpha$ is

$$(kx_1, kx_2, \cdots, kx_n)^{\mathrm{T}} = k(x_1, x_2, \cdots, x_n)^{\mathrm{T}}.$$

We can find that element in V^n and the vector in K^n have the following relationships

$$\boldsymbol{\alpha} \leftrightarrow (x_1, x_2, \cdots, x_n)^{\mathrm{T}}, \quad \boldsymbol{\beta} \leftrightarrow (y_1, y_2, \cdots, y_n)^{\mathrm{T}},$$
$$\boldsymbol{\alpha} + \boldsymbol{\beta} \leftrightarrow (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)^{\mathrm{T}}, \quad k\boldsymbol{\alpha} \leftrightarrow k(x_1, x_2, \cdots, x_n)^{\mathrm{T}}.$$

That is to say, V^n and K^n have the same structure, or V^n and K^n are isomorphic.

Correspondingly, results about the linear relations in V^n, such as linear dependent, linear independent, rank, the maximal linear independent vector set and so on, can be deduced in K^n by using coordinates.

Example 1.1.18 Check that whether the polynomials in $P[t]_3$, i.e., $f_1(t) = t^3 - 2t^2 + 4t + 1$, $f_2(t) = 2t^3 - 3t^2 + 9t - 1$, $f_3(t) = t^3 + 6t - 5$, $f_4(t) = 2t^3 - 5t^2 + 7t + 5$ are linear dependent or not.

Solution: Let the basis in $P[t]_3$ be $t^3, t^2, t, 1$. The coordinates of $f_1(t), f_2(t), f_3(t)$, $f_4(t)$ are

$$\boldsymbol{a}_1 = (1, -2, 4, 1)^{\mathrm{T}}, \boldsymbol{a}_2 = (2, -3, 9, -1)^{\mathrm{T}}, \boldsymbol{a}_3 = (1, 0, 6, -5)^{\mathrm{T}}, \boldsymbol{a}_4 = (2, -5, 7, 5)^{\mathrm{T}}.$$

It is not difficult to deduce that $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ are linear dependent. Correspondingly, the polynomials $f_1(t), f_2(t), f_3(t), f_4(t)$ are linear dependent.

Example 1.1.19 Find the rank and the maximal linear independent vector set of

$$A_1 = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}.$$

Solution: We take the elements shown in equation (1.1.1) as a basis of $K^{2\times 2}$. Coordinates of $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4$ are

$$\boldsymbol{\alpha}_1 = (2,1,-1,3)^{\mathrm{T}}, \boldsymbol{\alpha}_2 = (1,0,2,0)^{\mathrm{T}}, \boldsymbol{\alpha}_3 = (3,1,1,3)^{\mathrm{T}}, \boldsymbol{\alpha}_4 = (1,1,-3,3)^{\mathrm{T}}.$$

Thus the rank of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$ is 2, and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ consist the maximal linear independent vector set. Also rank$\{\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4\} = 2$, and $\boldsymbol{A}_1, \boldsymbol{A}_2$ constitute the maximal linear independent set.

1.1.5 Transformations of bases and coordinates

In this subsection, we will review the relations between different bases and coordinates of a certain vector with respect to different bases.

Definition 1.1.10 Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be two bases in V, and

$$y_i = a_{1i}x_1 + \cdots + a_{1n}x_n = (x_1, x_2, \cdots, x_n)\begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}, \quad i = 1, \cdots, n.$$

We introduce the matrix denotation $(y_1, y_2, \cdots, y_n) = (x_1, x_2, \cdots, x_n)\boldsymbol{A}$, where $\boldsymbol{A} = (a_{ij})_{n\times n} \in F^{n\times n}$. $\boldsymbol{A}$ is called the **transition matrix** from $x_1, x_2, \cdots, x_n$ to $y_1, y_2, \cdots, y_n$.

From the knowledge of linear equations, it is not difficult to see that $\boldsymbol{A}$ is invertible. Therefore, $(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_n)\boldsymbol{A}^{-1}$, i.e., $\boldsymbol{A}^{-1}$ is the transition matrix from $y_1, y_2, \cdots, y_n$ to $x_1, x_2, \cdots, x_n$.

Let $x \in V$, and $x = \sum_{i=1}^{n} \xi_i x_i = \sum_{i=1}^{n} \eta_i y_i$, then

$$x = (x_1, x_2, \cdots, x_n)\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = (y_1, y_2, \cdots, y_n)\left(\boldsymbol{A}^{-1}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}\right) = (y_1, y_2, \cdots, y_n)\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}.$$

From the uniqueness of coordinate, we can deduce the following

$$(\eta_1, \cdots, \eta_n)^{\mathrm{T}} = \boldsymbol{A}^{-1}(\xi_1, \cdots, \xi_n)^{\mathrm{T}},$$

i.e.

$$(\xi_1, \cdots, \xi_n)^{\mathrm{T}} = \boldsymbol{A}(\eta_1, \cdots, \eta_n)^{\mathrm{T}}.$$

The equation above reflects the transformation from the coordinate of x with respect to $y_1, y_2, \cdots, y_n$ to the coordinate with respect to $x_1, x_2, \cdots, x_n$.

Example 1.1.20 Given the following two bases of $P[t]_3$

$$\begin{cases} f_1(t) = t^3 + 2t^2 - t, \\ f_2(t) = t^3 - t^2 + t + 1, \\ f_3(t) = -t^3 + 2t^2 + t + 1, \\ f_4(t) = -t^3 - t^2 + 1, \end{cases} \qquad \begin{cases} g_1(t) = 2t^3 + t^2 + 1, \\ g_2(t) = t^2 + 2t + 2, \\ g_3(t) = -2t^3 + t^2 + t + 2, \\ g_4(t) = t^3 + 3t^2 + t + 2, \end{cases}$$

find the transition matrix from $f_1(t), f_2(t), f_3(t), f_4(t)$ to $g_1(t), g_2(t), g_3(t), g_4(t)$.

Solution: In order to simplify the computation, we select an intermediate basis $t^3, t^2, t, 1$.

The transition matrix from $t^3, t^2, t, 1$ to $f_1(t), f_2(t), f_3(t), f_4(t)$ is

$$(f_1(t), f_2(t), f_3(t), f_4(t)) = (t^3, t^2, t, 1)\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

and the transition matrix from $t^3, t^2, t, 1$ to $g_1(t), g_2(t), g_3(t), g_4(t)$ is

$$(g_1(t), g_2(t), g_3(t), g_4(t)) = (t^3, t^2, t, 1)\begin{pmatrix} 2 & 0 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ -1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix}.$$

Then the transition matrix from $f_1(t), f_2(t), f_3(t), f_4(t)$ to $g_1(t), g_2(t), g_3(t), g_4(t)$ is

$$(g_1(t), g_2(t), g_3(t), g_4(t)) =$$

$$(f_1(t), f_2(t), f_3(t), f_4(t))\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 0 & -2 & 1 \\ 1 & 1 & 1 & 3 \\ -1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{pmatrix} =$$

$$(f_1(t), f_2(t), f_3(t), f_4(t))\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

which implies that the transition matrix from $f_1(t), f_2(t), f_3(t), f_4(t)$ to $g_1(t), g_2(t), g_3(t), g_4(t)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

1.1.6 Subspace and the dimension theorem for vector spaces

We discuss subspaces of a linear space.

Definition 1.1.11 Let V be a linear space on the field K, and $W \subseteq V$ be a nonempty set. If W is linear space with respect to the addition and scalar product operations defined for V, then W is called a **subspace** of V.

Remark: The set $\{\theta\}$ and V are two special subspace of V, which are called the trivial subspace and false subspace, respectively. A subspace that is neither a trivial subspace nor a false subspace is called the proper subspace.

To judge whether a nonempty subset is a subspace, we can directly use the definition of the linear space. While, there exists a more convenient alternative method.

Theorem 1.1.1 A nonempty subset W of a linear space V over K is a linear subspace, if and only if the addition and scalar product operation defined in V is **closed** in W, i. e. , $\boldsymbol{x}+\boldsymbol{y}\in W$, $k\boldsymbol{x}\in W$, for any $\boldsymbol{x},\boldsymbol{y}\in W$, $k\in K$.

Example 1. 1. 21 In $\mathbf{R}^3$, any line or plane containing the origin is a linear subspace of $\mathbf{R}^3$.

Example 1. 1. 22 $P[t]_n$ is a linear subspace of $P[t]$.

Example 1. 1. 23 Given a set of vectors $\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n\in V$, the following set

$$W=\operatorname{span}\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n\}=\left\{\sum_{i=1}^{n}\lambda_i\boldsymbol{x}_i \mid \lambda_i\in F\right\}\subseteq V$$

is a linear subspace of V, called the **span** of $\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n$.

Proof Since $\boldsymbol{x}_1\in V$, W is nonempty. For any $\boldsymbol{\alpha},\boldsymbol{\beta}\in V$

$$\boldsymbol{\alpha}=k_1\boldsymbol{x}_1+k_2\boldsymbol{x}_2+\cdots+k_n\boldsymbol{x}_n,\quad \boldsymbol{\beta}=l_1\boldsymbol{x}_1+l_2\boldsymbol{x}_2+\cdots+l_n\boldsymbol{x}_n,$$

and $s\in K$, it can be easily deduced that

$$\boldsymbol{\alpha}+\boldsymbol{\beta}=(k_1+s_1)\boldsymbol{x}_1+(k_2+l_2)\boldsymbol{x}_2+\cdots+(k_n+l_n)\boldsymbol{x}_n\in W,$$

$$s\boldsymbol{\beta}=(sk_1)\boldsymbol{x}_1+(sk_2)\boldsymbol{x}_2+\cdots+(sk_n)\boldsymbol{x}_n\in W,$$

which implies that W is a linear subspace of V. ■

From Example 1. 1. 23, we can obtain a method for generating linear subspaces.

Theorem 1. 1. 2 Let V be a linear space in the field K, and $\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n$ be n elements of V. We construct the following subset

$$W=\{k_1\boldsymbol{x}_1+k_2\boldsymbol{x}_2+\cdots+k_n\boldsymbol{x}_n \mid k_1,k_2,\cdots,k_n\in K\},$$

then W is a linear subspace of V, which could be called the **linear span** of $\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n$ and denoted by **span** $\{\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n\}$.

Theorem 1. 1. 2 implies that any finite dimensional linear space is a linear span of its basis.

Example 1. 1. 24 Let $\boldsymbol{A}\in\mathbf{R}^{m\times n}$, and

$$N(\boldsymbol{A})=\{\boldsymbol{x}\in\mathbf{R}^n \mid \boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\},\quad R(\boldsymbol{A})=\{\boldsymbol{y}\in\mathbf{R}^n \mid \boldsymbol{y}=\boldsymbol{A}\boldsymbol{x},\boldsymbol{x}\in\mathbf{R}^n\},$$

then $N(\boldsymbol{A})$, $R(\boldsymbol{A})$ are both linear subspaces in $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively.

The linear subspaces in Example 1. 1. 24 are two important subsets related to a given matrix $\boldsymbol{A}$.

Definition 1. 1. 12 Let $\boldsymbol{A}\in K^{m\times n}$, and $\boldsymbol{a}_i\,(i=1,2,\cdots,n)$ is the i-th column vector of $\boldsymbol{A}$. The linear subspace $\operatorname{span}\{\boldsymbol{a}_1,\boldsymbol{a}_2,\cdots,\boldsymbol{a}_n\}$ is called the **range** or **column space** of $\boldsymbol{A}$, denoted by $R(\boldsymbol{A})$. The set $\{\boldsymbol{x}\in K^n \mid \boldsymbol{A}\boldsymbol{x}=\boldsymbol{0}\}$ is called the **kernel** or **null space** of $\boldsymbol{A}$, denoted by $N(\boldsymbol{A})$. Similarly, suppose that $\boldsymbol{b}_j\,(j=1,2,\cdots,m)$ is the j-th row vector of $\boldsymbol{A}$, the linear subspace $\operatorname{span}\{\boldsymbol{b}_1,\boldsymbol{b}_2,\cdots,\boldsymbol{b}_n\}$ is called the **row space** of $\boldsymbol{A}$, denoted by $R(\boldsymbol{A}^{\mathrm{T}})$. The set $\{\boldsymbol{y}\in K^m \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{0}\}$ is called the **left null space** of $\boldsymbol{A}$, denoted by $N(\boldsymbol{A}^{\mathrm{T}})$.

Remark: Alternative representations of $R(\boldsymbol{A})$ and $R(\boldsymbol{A}^{\mathrm{T}})$ are

$$R(\boldsymbol{A})=\{\boldsymbol{A}x \mid x\in K^n\},\quad R(\boldsymbol{A}^{\mathrm{T}})=\{\boldsymbol{A}^{\mathrm{T}}y \mid y\in K^m\}.$$

It is not difficult to prove that

$$\dim R(\boldsymbol{A})=\operatorname{rank}(\boldsymbol{A})=\operatorname{rank}(\boldsymbol{A}^{\mathrm{T}})=\dim R(\boldsymbol{A}^{\mathrm{T}}),$$

and $N(\boldsymbol{A})$, $N(\boldsymbol{A}^{\mathrm{T}})$ are the solution spaces of the homogenous linear equations $\boldsymbol{A}x=\boldsymbol{0}$ and

$A^T x = \mathbf{0}$, respectively. Thus,

$$\dim N(A) = n - \text{rank}(A) \quad \text{and} \quad \dim N(A^T) = m - \text{rank}(A),$$

which implies the following well known theorem.

Theorem 1.1.3 (Rank-nullity theorem) Let $A \in K^{m \times n}$, then the following equations hold

$$\dim N(A) = n - \dim R(A) \quad \text{and} \quad \dim N(A^T) = m - \dim R(A^T).$$

The intersection and union operations for subspaces have the following properties.

Theorem 1.1.4 Let V be a linear space on the field K, and V_1 and V_2 be two linear subspaces, then $V_1 \cap V_2$ is also a linear subspace of V.

Proof Since $\boldsymbol{\theta} \in V_1$ and $\boldsymbol{\theta} \in V_2$, then $\boldsymbol{\theta} \in V_1 \cap V_2$, implying that $V_1 \cap V_2$ is a nonempty set. For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in V_1 \cap V_2$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in V_1$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in V_2$. Since W_1 and W_2 are two linear subspaces, $\boldsymbol{\alpha} + \boldsymbol{\beta} \in V_1$ and $\boldsymbol{\alpha} + \boldsymbol{\beta} \in V_2$, i.e., $\boldsymbol{\alpha} + \boldsymbol{\beta} \in V_1 \cap V_2$. Similarly, for any $k \in K$, $k\boldsymbol{\alpha} \in V_1 \cap V_2$. Thus from Theorem 1.1.1, $V_1 \cap V_2$ is a linear subspace of V. ■

Remark: The union of two subspace is generally not a subspace.

Next we introduce the sum of subspaces.

Definition 1.1.13 Let V_1 and V_2 be two linear subspaces of the linear space V. The following set

$$V_1 + V_2 = \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1 \in V_1, \boldsymbol{\alpha}_2 \in V_2\}$$

is called the **sum** of V_1 and V_2.

Example 1.1.25 In $\mathbf{R}^3$, V_1 and V_2 are two lines along the x axis and y axis, respectively, which are subspaces of $\mathbf{R}^3$. Then $V_1 + V_2$ is the plane consisting of the origin O, x axis and y axis, which is obviously a subspace of $\mathbf{R}^3$.

The sum of subspaces satisfies the following theorem.

Theorem 1.1.5 Let V be a linear space over the field K, and V_1 and V_2 be two linear subspaces, then $V_1 + V_2$ is also a linear subspace of V.

Proof Since $\boldsymbol{\theta} = \boldsymbol{\theta} + \boldsymbol{\theta} \in V_1 + V_2$, $V_1 + V_2$ is a nonempty set. For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in V_1 + V_2$ and $k \in K$, we can assume that

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, \quad \boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 \quad (\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1 \in V_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2 \in V_2).$$

Thus,

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (\boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1) + (\boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2) \in V_1 + V_2, \quad k\boldsymbol{\alpha} = (k\boldsymbol{\alpha}_1) + (k\boldsymbol{\beta}_1) \in V_1 + V_2,$$

which implies that $V_1 + V_2$ is a linear subspace of V. ■

Remark: From Theorem 1.1.4 and Theorem 1.1.5, besides the linear span, we can generate new subspaces by using the intersection and sum operation on two existing subspaces.

For the dimensions of intersections and sums of subspaces, we have the following important theorem.

Theorem 1.1.6 (Dimension theorem) Let V be a linear space over the field K, and V_1 and V_2 be two linear subspaces, then

$$\dim V_1 + \dim V_2 = \dim(V_1 + V_2) + \dim(V_1 \cap V_2).$$

Proof We assume that