Nadir Jeevanjee

An Introduction to Tensors and Group Theory for Physicists

物理学家用的张量和群论导论

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To My Parents

Preface

This book is composed of two parts: Part I (Chaps. 1 through 3) is an introduction to tensors and their physical applications, and Part II (Chaps. 4 through 6) introduces group theory and intertwines it with the earlier material. Both parts are written at the advanced-undergraduate/beginning-graduate level, although in the course of Part II the sophistication level rises somewhat. Though the two parts differ somewhat in flavor, I have aimed in both to fill a (perceived) gap in the literature by connecting the component formalisms prevalent in physics calculations to the abstract but more conceptual formulations found in the math literature. My firm belief is that we need to see tensors and groups in coordinates to get a sense of how they work, but also need an abstract formulation to understand their essential nature and organize our thinking about them.

My original motivation for the book was to demystify tensors and provide a unified framework for understanding them in all the different contexts in which they arise in physics. The word tensor is ubiquitous in physics (stress tensor, moment-of-inertia tensor, field tensor, metric tensor, tensor product, etc.) and yet tensors are rarely defined carefully, and the definition usually has to do with transformation properties, making it difficult to get a feel for what these objects *are*. Furthermore, physics texts at the beginning graduate level usually only deal with tensors in their component form, so students wonder what the difference is between a second rank tensor and a matrix, and why new, enigmatic terminology is introduced for something they have already seen. All of this produces a lingering unease, which I believe can be alleviated by formulating tensors in a more abstract but conceptually much clearer way. This coordinate-free formulation is standard in the mathematical literature on differential geometry and in physics texts on General Relativity, but as far as I can tell is not accessible to undergraduates or beginning graduate students in physics who just want to learn what a tensor is *without* dealing with the full machinery of tensor analysis on manifolds.

The irony of this situation is that a proper understanding of tensors does not require much more mathematics than what you likely encountered as an undergraduate. In Chap. 2 I introduce this additional mathematics, which is just an extension of the linear algebra you probably saw in your lower-division coursework. This material sets the stage for tensors, and hopefully also illuminates some of the more enigmatic objects from quantum mechanics and relativity, such as bras and kets, covariant and contravariant components of vectors, and spherical harmonics. After laying the necessary linear algebraic foundations, we give in Chap. 3 the modern (component-free) definition of tensors, all the while keeping contact with the coordinate and matrix representations of tensors and their transformation laws. Applications in classical and quantum physics follow.

In Part II of the book I introduce group theory and its physical applications, which is a beautiful subject in its own right and also a nice application of the material in Part I. There are many good books on the market for group theory and physics (see the references), so rather than be exhaustive I have just attempted to present those aspects of the subject most essential for upper-division and graduate-level physics courses. In Chap. 4 I introduce abstract groups, but quickly illustrate that concept with myriad examples from physics. After all, there would be little point in making such an abstract definition if it did not subsume many cases of interest! We then introduce Lie groups and their associated Lie algebras, making precise the nature of the symmetry 'generators' that are so central in quantum mechanics. Much time is also spent on the groups of rotations and Lorentz transformations, since these are so ubiquitous in physics.

In Chap. 5 I introduce representation theory, which is a mathematical formalization of what we mean by the 'transformation properties' of an object. This subject sews together the material from Chaps. 3 and 4, and is one of the most important applications of tensors, at least for physicists. Chapter 6 then applies and extends the results of Chap. 5 to a few specific topics: the perennially mysterious 'spherical' tensors, the Wigner–Eckart theorem, and Dirac bilinears. The presentation of these later topics is admittedly somewhat abstract, but I believe that the mathematically precise treatment yields insights and connections not usually found in the usual physicist's treatment of the subjects.

This text aims (perhaps naively!) to be simultaneously intuitive and rigorous. Thus, although much of the language (especially in the examples) is informal, almost all the definitions given are precise and are the same as one would find in a pure math text. This may put you off if you feel less mathematically inclined; I hope, however, that you will work through your discomfort and develop the necessary mathematical sophistication, as the results will be well worth it. Furthermore, if you can work your way through the text (or at least most of Chap. 5), you will be well prepared to tackle graduate math texts in related areas.

As for prerequisites, it is assumed that you have been through the usual undergraduate physics curriculum, including a "mathematical methods for physicists" course (with at least a cursory treatment of vectors and matrices), as well as the standard upper-division courses in classical mechanics, quantum mechanics, and relativity. Any undergraduate versed in those topics, as well as any graduate student in physics, should be able to read this text. To undergraduates who are eager to learn about tensors but have not yet completed the standard curriculum, I apologize; many of the examples and practically all of the motivation for the text come from those courses, and to assume no knowledge of those topics would preclude discussion of the many applications that motivated me to write this book in the first place.

However, if you are motivated and willing to consult the references, you could certainly work through this text, and would no doubt be in excellent shape for those upper-division courses once you take them.

Exercises and problems are included in the text, with exercises occurring within the chapters and problems occurring at the end of each chapter. The exercises in particular should be done as they arise, or at least carefully considered, as they often flesh out the text and provide essential practice in using the definitions. Very few of the exercises are computationally intensive, and many of them can be done in a few lines. They are designed primarily to test your conceptual understanding and help you internalize the subject. Please do not ignore them!

Besides the aforementioned prerequisites I have also indulged in the use of some very basic mathematical shorthand for brevity's sake; a guide is below. Also, be aware that for simplicity's sake I have set all physical constants such as c and $\hbar$ equal to 1. Enjoy!

Berkeley, USA

Nadir Jeevanjee

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Contents

Notation

Some Mathematical Shorthand

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of positive and negative integers
$\in$	"is an element of", "an element of", i.e. $2 \in \mathbb{R}$ reads "2 is an element of the real numbers"
$\notin$	"is not an element of"
$\forall$	"for all"
$\subset$	"is a subset of", "a subset of"
$\equiv$	Denotes a definition
$f : A \to B$	Denotes a map f that takes elements of the set A into elements of the set B
$f : a \mapsto b$	Indicates that the map f sends the element a to the element b
$\circ$	Denotes a composition of maps, i.e. if $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$ is given by $(g \circ f)(a) \equiv g(f(a))$
$A \times B$	The set $\{(a, b)\}$ of all ordered pairs where $a \in A$, $b \in B$. Referred to as the *cartesian product* of sets A and B. Extends in the obvious way to n-fold products $A_1 \times \cdots \times A_n$
$\mathbb{R}^n$	$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$
$\mathbb{C}^n$	$\underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n \text{ times}}$
$\{A \mid Q\}$	Denotes a set A subject to condition Q. For instance, the set of all even integers can be written as $\{x \in \mathbb{R} \mid x/2 \in \mathbb{Z}\}$
$\square$	Denotes the end of a proof or example

Dirac Dictionary[1]

Standard Notation	Dirac Notation
Vector $\psi \in \mathcal{H}$	$\|\psi\rangle$
Dual Vector $L(\psi)$	$\langle\psi\|$
Inner Product (ψ, ϕ)	$\langle\psi\|\phi\rangle$
$A(\psi)$, $A \in \mathcal{L}(\mathcal{H})$	$A\|\psi\rangle$
$(\psi, A\phi)$	$\langle\psi\|A\|\phi\rangle$
$T_i{}^j e^i \otimes e_j$	$\sum_{i,j} T_{ij}\|j\rangle\langle i\|$
$e_i \otimes e_j$	$\|i\rangle\|j\rangle$ or $\|i, j\rangle$

[1] We summarize here all of the translations given in the text between quantum-mechanical Dirac notation and standard mathematical notation.

Part I
Linear Algebra and Tensors

Chapter 1
A Quick Introduction to Tensors

The reason tensors are introduced in a somewhat ad-hoc manner in most physics courses is twofold: first, a detailed and proper understanding of tensors requires mathematics that is slightly more abstract than the standard linear algebra and vector calculus that physics students use everyday. Second, students do not necessarily *need* such an understanding to be able to manipulate tensors and solve problems with them. The drawback, of course, is that many students feel uneasy whenever tensors are discussed, and they find that they can use tensors for computation but do not have an intuitive feel for what they are doing. One of the primary aims of this book is to alleviate those feelings. Doing that, however, requires a modest investment (about 30 pages) in some abstract linear algebra, so before diving into the details we will begin with a rough overview of what a tensor is, which hopefully will whet your appetite and tide you over until we can discuss tensors in full detail in Chap. 3.

Many older books define a tensor as a collection of objects which carry indices and which 'transform' in a particular way specified by those indices. Unfortunately, this definition usually does not yield much insight into what a tensor is. One of the main purposes of the present text is to promulgate the more modern definition of a tensor, which is equivalent to the old one but is more conceptual and is in fact already standard in the mathematics literature. This definition takes a tensor to be a *function* which eats a certain number of vectors (known as the ***rank*** r of the tensor) and produces a number. The distinguishing characteristic of a tensor is a special property called ***multilinearity***, which means that it must be linear in each of its r arguments (recall that linearity for a function with a single argument just means that $T(v+cw)=T(v)+cT(w)$ for all vectors v and w and numbers c). As we will explain in a moment, this multilinearity enables us to express the value of the function on an *arbitrary* set of r vectors in terms of the values of the function on r *basis* vectors like $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. These values of the function on basis vectors are nothing but the familiar ***components*** of the tensor, which in older treatments are usually introduced first as part of the definition of the tensor.

To make this concrete, consider a rank 2 tensor T, whose job it is to eat two vectors v and w and produce a number which we will denote as $T(v, w)$. For such a tensor, multilinearity means

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$$T(v_1 + cv_2, w) = T(v_1, w) + cT(v_2, w) \tag{1.1}$$

$$T(v, w_1 + cw_2) = T(v, w_1) + cT(v, w_2) \tag{1.2}$$

for any number c and all vectors v and w. This means that if we have a coordinate basis for our vector space, say $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, then T is determined entirely by its values on the basis vectors, as follows: first, expand v and w in the coordinate basis as

$$\begin{aligned} v &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}} \\ w &= w_x\hat{\mathbf{x}} + w_y\hat{\mathbf{y}} + w_z\hat{\mathbf{z}}. \end{aligned}$$

Then by (1.1) and (1.2) we have

$$\begin{aligned} T(v, w) &= T(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}, w_x\hat{\mathbf{x}} + w_y\hat{\mathbf{y}} + w_z\hat{\mathbf{z}}) \\ &= v_x T(\hat{\mathbf{x}}, w_x\hat{\mathbf{x}} + w_y\hat{\mathbf{y}} + w_z\hat{\mathbf{z}}) + v_y T(\hat{\mathbf{y}}, w_x\hat{\mathbf{x}} + w_y\hat{\mathbf{y}} + w_z\hat{\mathbf{z}}) \\ &\quad + v_z T(\hat{\mathbf{z}}, w_x\hat{\mathbf{x}} + w_y\hat{\mathbf{y}} + w_z\hat{\mathbf{z}}) \\ &= v_x w_x T(\hat{\mathbf{x}}, \hat{\mathbf{x}}) + v_x w_y T(\hat{\mathbf{x}}, \hat{\mathbf{y}}) + v_x w_z T(\hat{\mathbf{x}}, \hat{\mathbf{z}}) + v_y w_x T(\hat{\mathbf{y}}, \hat{\mathbf{x}}) \\ &\quad + v_y w_y T(\hat{\mathbf{y}}, \hat{\mathbf{y}}) + v_y w_z T(\hat{\mathbf{y}}, \hat{\mathbf{z}}) + v_z w_x T(\hat{\mathbf{z}}, \hat{\mathbf{x}}) + v_z w_y T(\hat{\mathbf{z}}, \hat{\mathbf{y}}) \\ &\quad + v_z w_z T(\hat{\mathbf{z}}, \hat{\mathbf{z}}). \end{aligned}$$

If we then abbreviate our notation as

$$\begin{aligned} T_{xx} &\equiv T(\hat{\mathbf{x}}, \hat{\mathbf{x}}) \\ T_{xy} &\equiv T(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \\ T_{yx} &\equiv T(\hat{\mathbf{y}}, \hat{\mathbf{x}}) \end{aligned} \tag{1.3}$$

and so on, we have

$$\begin{aligned} T(v, w) = v_x w_x T_{xx} + v_x w_y T_{xy} + v_x w_z T_{xz} + v_y w_x T_{yx} + v_y w_y T_{yy} \\ + v_y w_z T_{yz} + v_z w_x T_{zx} + v_z w_y T_{zy} + v_z w_z T_{zz} \end{aligned} \tag{1.4}$$

which may look familiar from discussion of tensors in the physics literature. In that literature, the above equation is often part of the *definition* of a second rank tensor; here, though, we see that its form is really just a consequence of multilinearity. Another advantage of our approach is that the ***components*** $\{T_{xx}, T_{xy}, T_{xz}, \ldots\}$ of T have a meaning beyond that of just being coefficients that appear in expressions like (1.4); from (1.3), we see that **components are the values of the tensor when evaluated on a given set of basis vectors**. This fact is crucial in getting a feel for tensors and what they mean.

Another nice feature of our definition of a tensor is that it allows us to *derive* the tensor transformation laws which historically were taken as the definition of a tensor. Say we switch to a new set of basis vectors $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$ which are related to the old basis vectors by

$$\begin{aligned} \hat{\mathbf{x}}' &= A_{x'x}\hat{\mathbf{x}} + A_{x'y}\hat{\mathbf{y}} + A_{x'z}\hat{\mathbf{z}} \\ \hat{\mathbf{y}}' &= A_{y'x}\hat{\mathbf{x}} + A_{y'y}\hat{\mathbf{y}} + A_{y'z}\hat{\mathbf{z}} \\ \hat{\mathbf{z}}' &= A_{z'x}\hat{\mathbf{x}} + A_{z'y}\hat{\mathbf{y}} + A_{z'z}\hat{\mathbf{z}}. \end{aligned} \tag{1.5}$$