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Marc Levine
Fabien Morel

Algebraic Cobordism

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To Rebecca, Anna and Ute–M.L.
To Juliette, Elise and Mymy–F.M.

Introduction

Motivation. This work grew out of our attempt to understand the analog in algebraic geometry of the fundamental paper of Quillen on the cobordism of differentiable manifolds [30]. In this paper, Quillen introduced the notion of a *(complex) oriented cohomology theory* on the category of differentiable manifolds, which basically means that the cohomology theory is endowed with suitable Gysin morphisms, and in particular gives the cohomology theory the additional structure of Chern classes for complex vector bundles. Quillen then observed that the complex cobordism theory $X \mapsto MU^*(X)$ is the universal such cohomology theory.

This new point of view allowed him to shed some new light on classical computations in cobordism theory. He made more precise the computation by Milnor and Novikov of the complex cobordism ring MU^* as a polynomial ring: it is in fact the Lazard ring $\mathbb{L}$, the coefficient ring of the universal formal group law defined and studied in [16]. The isomorphism

$$\mathbb{L} \cong MU^*$$

is obtained via the formal group law $F_{MU}(u, v)$ on MU^* defined as the expression of the Chern class $c_1(L \otimes M)$ of a tensor product of line bundles as a power series in $c_1(L)$ and $c_1(M)$ by the formula

$$c_1(L \otimes M) = F_{MU}(c_1(L), c_1(M)).$$

This result of Quillen is in fact a particular case of his main theorem obtained in [30]: for any differentiable manifold X, the $\mathbb{L}$-module $MU^*(X)$ is generated by the elements of non-negative degrees. We observe that this is highly non-trivial as the elements of $\mathbb{L}$, in the cohomological setting, are of negative degree!

Quillen's notion of oriented cohomology extends formally to the category $\mathbf{Sm}_k$ of smooth quasi-projective k-schemes, with k a fixed field, see section 1.1. Our main achievement here is to construct the universal oriented cohomology theory Ω^* on $\mathbf{Sm}_k$, which we call algebraic cobordism, and to prove the exact analogs of Quillen's theorems in this setting, at least over a field of characteristic zero. The computation

$$\mathbb{L} \cong \Omega^*(\mathrm{Spec}\, k)$$

is done in section 4.3, and the theorem asserting that $\Omega^*(X)$ is generated by elements of non-negative degrees is proved in section 4.4. Surprisingly, this result can be precisely reformulated, in algebraic geometry, as the *generalized degree formula* conjectured by Rost. We will give on the way other applications and examples, explaining for instance the relationship between our Ω^* and the K_0 functor of Grothendieck or the Chow ring functor CH^*.

It is fascinating to see that in the introduction of his paper Quillen acknowledges the influence of Grothendieck's philosophy of motives on his work. Here the pendulum swings back: our work is strongly influenced by Quillen's ideas, which we try to bring back to the "motivic" world. In some sense, this book is the result of putting Quillen's work [30] together with Grothendieck's work on the theory of Chern classes [11]. Indeed, if one relaxes the axiom from the paper of Grothendieck that the first Chern class $c_1 : \mathrm{Pic}(X) \to A^1(X)$ is a group homomorphism, then in the light of Quillen's work, one has to discover algebraic cobordism.

Overview. Most of the main results in this book were announced in [20, 21], and appeared in detailed form in the preprint [18] by Levine and Morel and the preprint [19] by Levine. This book is the result of putting these works together[1].

The reader should notice that we have made a change of convention on degrees from [20, 21]; there our cohomology theories were assumed to be take values in the category of graded commutative rings, and the push-forward maps were assumed to increase the degree by 2 times the codimension. This had the advantage of fitting well with the notation used in topology. But as is clear from our constructions, we only deal with the even part, and for notational simplicity we have divided the degrees by 2.

This book is organized as follows. In order to work in greater generality as in [9], instead of dealing only with cohomology theories on smooth varieties, we will construct Ω^* as an oriented Borel-Moore homology theory $X \mapsto \Omega_*(X)$ on the category of a finite type k-schemes.

In chapter 1, we introduce the notion of an oriented cohomology theory and state our main results. In chapter 2, we construct algebraic cobordism over any field as the universal "oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type" on the category of finite type k-schemes. Our construction is not merely an existence theorem, we define algebraic cobordism by giving explicit generators and relations.

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[1] The second author wishes to thank the first author very much for incorporating his part, and for his work combining the two parts into a whole

An oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type has by definition projective push-forward, smooth pull-back, external products and 1st Chern class operators for line bundles, satisfying some natural axioms. However, this structure is not sufficient for our purposes, as one needs in addition a projective bundle formula and an extended homotopy property. In chapter 3 we establish our fundamental technical result: the localization theorem 1.2.8, when k admits resolution of singularities. The rest of the chapter 3 deduces from this theorem the projective bundle formula and extended homotopy invariance for algebraic cobordism.

Chapter 4 introduces the dual notions of oriented weak cohomology theories and oriented Borel-Moore weak homology theories. We develop the theory of Chern classes for these theories, give some applications, and then prove all the theorems announced in the introduction. One should note however that theorems 1.2.2 and 1.2.6 are only proven here in the weaker form where one replaces the notion of oriented cohomology theory by the notion of weak oriented cohomology theory. However, the proofs of the other theorems such as theorem 1.2.3, the various degree formulas and theorem 1.2.7 require only those weak forms.

Chapters 5 and 6 of this work deal with pull-backs. The essential difference between an oriented cohomology theory and an oriented weak cohomology theory is that the latter have only pull-backs for smooth morphisms while the former have pull-backs for any morphism between smooth k-schemes. It is convenient to work with the dual notion of an oriented Borel-Moore homology theory on the category of finite type k-schemes, which is introduced in chapter 5. Our main task in this setting is to construct pull-back maps for any local complete intersection morphism, which is done in chapter 6 (assuming k admits resolution of singularities). We conclude in chapter 7 by finishing the proofs of theorems 1.2.2 and 1.2.6, and extending many of our results on the oriented cohomology of smooth schemes to the setting of Borel-Moore homology of local complete intersection schemes.

Notations and conventions. We denote by $\mathbf{Sch}_S$ the category of separated schemes of finite type over S and by $\mathbf{Sm}_S$ its full subcategory consisting of schemes smooth and quasi-projective over S. For an S-scheme X, we shall denote by $\pi_X : X \to S$ the structural morphism. By a smooth morphism, we will always mean a smooth and quasi-projective morphism. In particular, a smooth S-scheme will always be assumed to be quasi-projective over S.

Throughout this paper, we let k be an arbitrary field, unless otherwise stated. We will usually, but not always, take $S = \operatorname{Spec} k$.

We denote by $\mathcal{O}_X$ the structure sheaf of a scheme X and by O_X, or simply O when no confusion can arise, the trivial line bundle over X. Given a Cartier divisor $D \subset X$ we let $\mathcal{O}_X(D)$ denote the invertible sheaf determined by D and $O_X(D)$ the line bundle whose $\mathcal{O}_X$-module of sections is $\mathcal{O}_X(D)$. For a vector bundle $E \to X$, we write $\mathcal{O}_X(E)$ for the sheaf of (germs of) sections of E. In general, we will pass freely between vector bundles over X and the corresponding locally free coherent sheaves of $\mathcal{O}_X$ modules.

For a locally free coherent sheaf $\mathcal{E}$ on a scheme X, we let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle $\mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E}))$, and $q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}$ the canonical quotient invertible sheaf. For a vector bundle $E \to X$, we write $\mathbb{P}(E)$ for $\mathbb{P}(\mathcal{O}(E))$, and $q^*E \to O(1)_E$ for the canonical quotient line bundle. For $n > 0$, O_X^n will denote the trivial vector bundle of rank n over X, and we write γ_n for the line bundle $O(1)_{O_X^{n+1}}$ on $\mathbb{P}^n_X$.

For a an element of a commutative ring R, we write a for the R-valued point $(1 : a)$ of $\mathbb{P}^1_R := \mathrm{Proj}_R R[X_0, X_1]$, and ∞ for the point $(0 : 1)$. Similarly, we use the coordinate $x := X_1/X_0$ to identify $\mathbb{P}^1_R \setminus \infty$ with $\mathbb{A}^1_R$. For a functor F defined on a sub-category of $\mathbf{Sch}_k$ we will usually write $F(k)$ instead of $F(\mathrm{Spec}\, k)$.

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Beside our obvious debt to Quillen, the reader will not fail to notice our repeated reliance on the ideas in Fulton's book [9]. In fact, one can view a large portion of this book as a revision of [9], replacing cycles with "cobordism cycles" and adding a liberal dash of Hironaka's resolution of singularities.

Marc Levine: Much of what went in to this book came out of discussions with Fabien Morel during my visit in the summer of 2000 to the Université de Paris 7, with subsequent work taking place during a number of visits to the Universität Duisburg–Essen. I would like thank both universities for their support and hospitality. Thanks are also due to Northeastern University for encouraging and supporting my research. Finally, I am grateful for support from the NSF via the grants DMS-987629, DMS-0140445 and DMS-0457195, and the Humboldt Foundation through the Wolfgang Paul program.

Boston, Essen, München,
August, 2006

Marc Levine
Fabien Morel

Contents

1

Cobordism and oriented cohomology

In this chapter, we introduce the axiomatic framework of oriented cohomology theories, and state our main results.

1.1 Oriented cohomology theories

Fix a base scheme S. For $z \in Z \in \mathbf{Sm}_S$ we denote by $\dim_S(Z, z)$ the dimension over S of the connected component of Z containing z.

Let $d \in \mathbb{Z}$ be an integer. A morphism $f : Y \to X$ in $\mathbf{Sm}_S$ has *relative dimension* d if, for each $y \in Y$, we have $\dim_S(Y, y) - \dim_S(X, f(y)) = d$. We shall also say in that case that f has relative codimension $-d$.

For a fixed base-scheme S, $\mathcal{V}$ will usually denote a full subcategory of $\mathbf{Sch}_S$ satisfying the following conditions

1. S and the empty scheme are in $\mathcal{V}$.
2. If $Y \to X$ is a smooth quasi-projective morphism in $\mathbf{Sch}_S$ with $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.
3. If X and Y are in $\mathcal{V}$, then so is the product $X \times_S Y$.
4. If X and Y are in $\mathcal{V}$, so is $X \coprod Y$.

(1.1)

In particular, $\mathcal{V}$ contains $\mathbf{Sm}_S$. We call such a subcategory of $\mathbf{Sch}_S$ *admissible*.

Definition 1.1.1. *Let $f : X \to Z$, $g : Y \to Z$ be morphisms in an admissible subcategory $\mathcal{V}$ of* $\mathbf{Sch}_S$. *We say that f and g are* transverse in $\mathcal{V}$ *if*

1. $\mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ *for all* $q > 0$.
2. *The fiber product $X \times_Z Y$ is in $\mathcal{V}$.*

If $\mathcal{V} = \mathbf{Sm}_S$ we just say f and g are transverse; if $\mathcal{V} = \mathbf{Sch}_S$, we sometimes say instead that f and g are Tor-independent.

We let $\mathbf{R}^*$ denote the category of *commutative, graded rings with unit*. Observe that a commutative graded ring is not necessarily *graded commutative*. We say that a functor $A^* : \mathbf{Sm}_S^{\text{op}} \to \mathbf{R}^*$ is *additive* if $A^*(\emptyset) = 0$ and for any pair $(X, Y) \in \mathbf{Sm}_S^2$ the canonical ring map $A^*(X \amalg Y) \to A^*(X) \times A^*(Y)$ is an isomorphism.

The following notion is directly taken from Quillen's paper [30]:

Definition 1.1.2. *Let $\mathcal{V}$ be an admissible subcategory of* $\mathbf{Sch}_S$. *An* oriented cohomology theory *on $\mathcal{V}$ is given by*

(D1). An additive functor $A^* : \mathcal{V}^{\text{op}} \to \mathbf{R}^*$.
(D2). For each projective morphism $f : Y \to X$ in $\mathcal{V}$ of relative codimension d, a homomorphism of graded $A^*(X)$-modules:

$$f_* : A^*(Y) \to A^{*+d}(X)$$

Observe that the ring homomorphism $f^* : A^*(X) \to A^*(Y)$ gives $A^*(Y)$ the structure of an $A^*(X)$-module.

These satisfy

(A1). One has $(\text{Id}_X)_* = \text{Id}_{A^*(X)}$ for any $X \in \mathcal{V}$. Moreover, given projective morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$, with f of relative codimension d and g of relative codimension e, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X).$$

(A2). Let $f : X \to Z$, $g : Y \to Z$ be transverse morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Suppose that f is projective of relative dimension d (thus so is f'). Then $g^* f_* = f'_* g'^*$.
(PB). Let $E \to X$ be a rank n vector bundle over some X in $\mathcal{V}$, $O(1) \to \mathbb{P}(E)$ the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\xi \in A^1(\mathbb{P}(E))$ by

$$\xi := s^*(s_*(1)).$$

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module, with basis

$$(1, \xi, \ldots, \xi^{n-1}).$$

(EH). Let $E \to X$ be a vector bundle over some X in $\mathcal{V}$, and let $p : V \to X$ be an E-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

A morphism of oriented cohomology theories on $\mathcal{V}$ is a natural transformation of functors $\mathcal{V}^{\mathrm{op}} \to \mathbf{R}^*$ which commutes with the maps f_*.

The morphisms of the form f^* are called *pull-backs* and the morphisms of the form f_* are called *push-forwards.* Axiom (PB) will be referred to as the *projective bundle formula* and axiom (EH) as the *extended homotopy property.*

We now specialize to $S = \operatorname{Spec} k$, $\mathcal{V} = \mathbf{Sm}_k$, k a field. Given an oriented cohomology theory A^*, one may use Grothendieck's method [11] to define Chern classes $c_i(E) \in A^i(X)$ of a vector bundle $E \to X$ of rank n over X as follows: Using the notations of the previous definition, axiom (PB) implies that there exists unique elements $c_i(E) \in A^i(X)$, $i \in \{0, \dots, n\}$, such that $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i \, c_i(E) \, \xi^{n-i} = 0.$$

One can check all the standard properties of Chern classes as in [11] using the axioms listed above (see §4.1.7 for details). Moreover, these Chern classes are characterized by the following properties:

1) For any line bundle L over $X \in \mathbf{Sm}_k$, $c_1(L)$ equals $s^* s_*(1) \in A^1(X)$, where $s : X \to L$ denotes the zero section.
2) For any morphism $Y \to X \in \mathbf{Sm}_k$, and any vector bundle E over X, one has for each $i \geq 0$

$$c_i(f^* E) = f^*(c_i(E)).$$

3) Whitney product formula: if

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles, then one has for each integer $n \geq 0$:

$$c_n(E) = \sum_{i=0}^{n} c_i(E') \, c_{n-i}(E'').$$

Sometime, to avoid confusion, we will write $c_i^A(E)$ for the Chern classes of E computed in the oriented cohomology theory A^*.

The fundamental insight of Quillen in [30], and the main difference with Grothendieck's axioms in [11], is that it is not true in general that one has the formula

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

for line bundles L and M over the same base. In other words the map

$$\begin{aligned} c_1 : \operatorname{Pic}(X) &\to A^1(X) \\ L &\mapsto c_1(L), \end{aligned}$$

is not assumed to be a group homomorphism, but is only a natural transformation of *pointed sets.* In fact, a classical remark due to Quillen [30, Proposition 2.7] describes the way c_1 is not additive as follows (see proposition 5.2.4 for a proof of this lemma):

Lemma 1.1.3. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Then for any line bundle L on $X \in \mathbf{Sm}_k$ the class $c_1(L)^n$ vanishes for n large enough*[1]*. Moreover, there is a unique power series*

$$F_A(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A^*(k)[[u,v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$, such that, for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L, M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

In addition, the pair $(A^*(k), F_A)$ is a commutative formal group law of rank one.

Recall from [16] that a commutative formal group law of rank one with coefficients in A is a pair (A, F) consisting of a commutative ring A and a formal power series

$$F(u,v) = \sum_{i,j} a_{i,j}\, u^i\, v^j \in A[[u,v]]$$

such that the following holds:

1. $F(u,0) = F(0,u) = u \in A[[u]]$.
2. $F(u,v) = F(v,u) \in A[[u,v]]$.
3. $F(u, F(v,w)) = F(F(u,v), w) \in A[[u,v,w]]$.

These properties of F_A reflect the fact that, for line bundles L, M, N on $X \in \mathbf{Sm}_k$, one has:

1′. $L \otimes O_X = O_X \otimes L = L \in \mathrm{Pic}(X)$.
2′. $L \otimes M = M \otimes L \in \mathrm{Pic}(X)$.
3′. $L \otimes (M \otimes N) = (L \otimes M) \otimes N \in \mathrm{Pic}(X)$.

Lazard pointed out in [16] that there exists a universal commutative formal group law of rank one $(\mathbb{L}, F_{\mathbb{L}})$ and proved that the ring $\mathbb{L}$ (now called the *Lazard ring*) is a polynomial ring with integer coefficients on a countable set of variables x_i, $i \geq 1$. The construction of $(\mathbb{L}, F_{\mathbb{L}})$ is rather easy. Set $\tilde{\mathbb{L}} := \mathbb{Z}[\{A_{i,j} |\ (i,j) \in \mathbb{N}^2\}]$, and $\tilde{F}(u,v) = \sum_{i,j} A_{i,j}\, u^i\, v^j \in \tilde{\mathbb{L}}[[u,v]]$. Then define $\mathbb{L}$ to be the quotient ring of $\tilde{\mathbb{L}}$ by the relations obtained by imposing the relations (1), (2) and (3) above to $\tilde{F}$, and let

$$F_{\mathbb{L}} = \sum_{i,j} a_{i,j}\, u^i\, v^j \in \mathbb{L}[[u,v]]$$

denote the image of F by the homomorphism $\tilde{\mathbb{L}} \to \mathbb{L}$. It is clear that the pair $(\mathbb{L}, F_{\mathbb{L}})$ is the universal commutative formal group law of rank one, which

[1] In fact we will prove later on that $n > \dim_k(X)$ suffices; this follows from theorem 2.3.13 and proposition 5.2.4.

means that to define a commutative formal group law of rank one (F, A) on A is equivalent to giving a ring homomorphism $\Phi_F : \mathbb{L} \to A$.

The Lazard ring can be graded by assigning the degree $i + j - 1$ to the coefficient $a_{i,j}$. We denote by $\mathbb{L}_*$ this commutative graded ring. We could as well have graded it by assigning the degree $1 - i - j$ to the coefficient $a_{i,j}$, in which case we denote by $\mathbb{L}^*$ the corresponding commutative graded ring. For instance $\mathbb{L}^0 = \mathbb{L}_0 = \mathbb{Z}$ and $\mathbb{L}^{-n} = \mathbb{L}_n = 0$ if $n < 0$.

One can then check that for any oriented cohomology theory A^* the homomorphism of rings induced by the formal group law given by lemma 1.1.3 is indeed a homomorphism of graded rings

$$\Phi_A : \mathbb{L}^* \to A^*(k)$$

Example 1.1.4. The Chow ring $X \mapsto \mathrm{CH}^*(X)$ is a basic example of an oriented cohomology theory on $\mathbf{Sm}_k$; this follows from [9]. In that case, the formal group law obtained on $\mathbb{Z} = \mathrm{CH}^*(k)$ by lemma 1.1.3 is the *additive* formal group law $F_a(u, v) = u + v$.

Example 1.1.5. Another fundamental example of oriented cohomology theory is given by the Grothendieck K^0 functor $X \mapsto K^0(X)$, where for X a smooth k-scheme, $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves on X. For $\mathcal{E}$ a locally free sheaf on X we denote by $[\mathcal{E}] \in K^0(X)$ its class. The tensor product of sheaves induces a unitary, commutative ring structure on $K^0(X)$. In fact we rather consider the graded ring $K^0(X)[\beta, \beta^{-1}] := K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$, where $\mathbb{Z}[\beta, \beta^{-1}]$ is the ring of Laurent polynomial in a variable β of degree -1.

It is endowed with pull-backs for any morphism $f : Y \to X$ by the formula:

$$f^*([\mathcal{E}] \cdot \beta^n) := [f^*(\mathcal{E})] \cdot \beta^n$$

for $\mathcal{E}$ a locally free coherent sheaf on X and $n \in \mathbb{Z}$. We identify $K^0(X)$ with the Grothendieck group $G_0(X)$ of all coherent sheaves on X by taking a finite locally free resolution of a coherent sheaf (X is assumed to be regular). This allows one to define push-forwards for a projective morphism $f : Y \to X$ of pure codimension d by the formula

$$f_*([\mathcal{E}] \cdot \beta^n) := \Sigma_{i=0}^{\infty}(-1)^i [R^i f_*(\mathcal{E})] \cdot \beta^{n-d} \in K_0(X)[\beta, \beta^{-1}]$$

for $\mathcal{E}$ a locally free sheaf on Y and $n \in \mathbb{Z}$. One can easily check using standard results that this is an oriented cohomology theory.

Moreover, for a line bundle L over X with with projection $\pi : L \to X$, zero section $s : X \to L$ and sheaf of sections $\mathcal{L}$, one has

$$s^*(s_*(1_X)) = s^*([\mathcal{O}_{s(X)}]\beta^{-1}) = s^*(1 - [\pi^*(\mathcal{L})^\vee])\beta^{-1} = (1 - [\mathcal{L}^\vee])\beta^{-1}$$

so that $c_1^K(L) := (1 - [\mathcal{L}^\vee])\beta^{-1}$. We thus find that the associated power series F_K is the *multiplicative formal group law*

$$F_m(u,v) := u + v - \beta uv$$

as this follows easily from the relation

$$(1 - [(\mathcal{L} \otimes \mathcal{M})^\vee]) = (1 - [\mathcal{L}^\vee]) + (1 - [\mathcal{M}^\vee]) - (1 - [\mathcal{L}^\vee])(1 - [\mathcal{M}^\vee])$$

in $K^0(X)$, where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X.

1.2 Algebraic cobordism

Definition 1.2.1. *Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$ with associated formal group law F_A.*

1) We shall say that A^* is ordinary *if $F_A(u,v)$ is the additive formal group law.*
2) We shall say that A^* is multiplicative *if $F_A(u,v) = u + v - buv$ for some (uniquely determined) $b \in A^{-1}(k)$; we shall say moreover that A^* is* periodic *if b is a unit in $A^*(k)$.*

Our main results on oriented cohomology theories are the following three theorems. In each of these statements, A^* denoted a fixed oriented cohomology theory on $\mathbf{Sm}_k$:

Theorem 1.2.2. *Let k be a field of characteristic zero. If A^* is ordinary then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^{\mathrm{CH}} : \mathrm{CH}^* \to A^*.$$

Theorem 1.2.3. *Let k be a field. If A^* is multiplicative and periodic then there exists one and only one morphism of oriented cohomology theories*

$$\vartheta_A^K : K^0[\beta, \beta^{-1}] \to A^*.$$

Theorem 1.2.2 says that, in characteristic zero, the Chow ring functor is the universal ordinary oriented cohomology theory on $\mathbf{Sm}_k$. It seems reasonable to conjecture that this statement still holds over any field. Theorem 1.2.3 says that $K^0[\beta, \beta^{-1}]$ is the universal multiplicative and periodic oriented cohomology theory on $\mathbf{Sm}_k$.

Remark 1.2.4. The classical Grothendieck-Riemann-Roch theorem can be easily deduced from theorem 1.2.3, see remark 4.2.11.

Remark 1.2.5. Using theorem 1.2.3 and the fact that for any smooth k-scheme the Chern character induces an isomorphism

$$ch : K^0(X) \otimes \mathbb{Q} \cong \mathrm{CH}(X) \otimes \mathbb{Q}$$

(where CH denotes the ungraded Chow ring), it is possible to prove $\mathbb{Q}$-versions of theorem 1.2.2 and theorem 1.2.6 below over *any* field.