John H. Hubbard,
Barbara Burke Hubbard

Vector Calculus, Linear Algebra, and Differential Forms

A Unified Approach

3rd Edition

向量微积分、线性代数和微分形式

第3版

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Preface

Joseph Fourier (1768–1830)

Fourier was arrested during the French Revolution and threatened with the guillotine, but survived and later accompanied Napoleon to Egypt; in his day he was as well known for his studies of Egypt as for his contributions to mathematics and physics. He found a way to solve linear partial differential equations while studying heat diffusion. An emphasis on computationally effective algorithms is one theme of this book.

... The numerical interpretation ... is however necessary. ... So long as it is not obtained, the solutions may be said to remain incomplete and useless, and the truth which it is proposed to discover is no less hidden in the formulae of analysis than it was in the physical problem itself.

–Joseph Fourier, *The Analytic Theory of Heat*

Chapters 1 through 6 of this book cover the standard topics in multivariate calculus and a first course in linear algebra. The book can also be used for a course in analysis, using the proofs in the appendix.

The organization and selection of material differs from the standard approach in three ways, reflecting the following principles.

First, we believe that at this level linear algebra should be more a convenient setting and language for multivariate calculus than a subject in its own right. The guiding principle of this unified approach is that locally, a nonlinear function behaves like its derivative.

Thus when we have a question about a nonlinear function we will answer it by looking carefully at a linear transformation: its derivative. In this approach, everything learned about linear algebra pays off twice: first for understanding linear equations, then as a tool for understanding nonlinear equations.

We discuss abstract vector spaces in section 2.6, but the emphasis is on $\mathbb{R}^n$, as we believe that most students find it easiest to move from the concrete to the abstract.

Second, we emphasize computationally effective algorithms, and we prove theorems by showing that these algorithms work.

We feel this better reflects the way mathematics is used today, in both applied and pure mathematics. Moreover, it can be done with no loss of rigor.

For linear equations, row reduction is the central tool from which everything else follows; we use row reduction to prove all the standard results about dimension and rank. For nonlinear equations, the cornerstone is Newton's method, the best and most widely used method for solving nonlinear equations; we use it both as a computational tool and in proving the inverse and implicit function theorems. We include a section on numerical methods of integration, and we encourage the use of computers both to reduce tedious calculations and as an aid in visualizing curves and surfaces.

Third, we use differential forms to generalize the fundamental theorem of calculus to higher dimensions.

The great conceptual simplification gained by doing electromagnetism in the language of forms is a central motivation for using forms. We apply the language of forms to electromagnetism in section 6.9, and we will expand on this discussion in a subsequent volume.

In our experience, differential forms can be taught to freshmen and sophomores if forms are presented geometrically, as integrands that take an oriented piece of a curve, surface, or manifold, and return a number. We are aware that students taking courses in other fields need to master the language of vector calculus, and we devote three sections of chapter 6 to integrating the standard vector calculus into the language of forms.

Other ways this book differs from standard texts include

The treatment of eigenvectors and eigenvalues
Rules for computing Taylor polynomials
Lebesgue integration

Eigenvectors and eigenvalues In keeping with our prejudice in favor of computationally effective algorithms, we provide in section 2.7 a theory of eigenvectors and eigenvalues that bypasses determinants, which are more or less uncomputable for large matrices. The treatment we give is also stronger theoretically: theorem 2.7.6 gives an "if and only if" statement for the existence of eigenbases. In addition, our emphasis on defining an eigenvector $\mathbf{v}$ as satisfying $A\mathbf{v} = \lambda\mathbf{v}$ has the advantage of working when A is a linear transformation between infinite-dimensional vector spaces, whereas the definition in terms of roots of the characteristic polynomial does not. However, in section 4.8 we define the characteristic polynomial of a matrix, connecting eigenvalues and eigenvectors to determinants.

Rules for computing Taylor polynomials Even good graduate students are often unaware of the rules that make computing Taylor polynomials in higher dimensions palatable. We give these in section 3.4, with proofs in appendix A.11.

In our experience, students are less resistant to approaching the Lebesgue integral via the Riemann integral than they are to the standard approach.

Lebesgue integration We give a new approach to Lebesgue integration, tying it much more closely to Riemann integrals, and completely avoiding the standard σ-algebras of measurable sets. We were motivated by two considerations. First, integrals over unbounded domains and integrals of unbounded functions are really important, for instance in physics and probability, and students will need to know about such integrals before they take a course in analysis. Second, there simply does not appear to be a successful theory of improper multiple integrals.

What's different in the third edition

Anyone familiar with the second edition (published by Prentice Hall) will recognize the third edition, but there are significant changes, including:

1. We have corrected all errors on the errata list for the second edition, and some errors that never made it on that list.
2. Section 2.7 on eigenvectors, eigenvalues, and diagonalization is new.
3. Section 5.4 on integration and curvature is new.
4. Section 6.9 on electromagnetism is new.
5. We have a new and stronger statement about the equality of crossed partials (theorem 3.3.9).
6. New examples have been added, notably example 3.7.12 on checking boundary points when looking for critical points of functions.
7. We have added more than a dozen figures, including figures 2.7.1, 3.7.9, 3.7.12, 6.8.3, 6.9.4, A17.1, and A25.1.
8. We have shortened a number of proofs. For example, the proof of proposition 6.4.8 on orientation-preserving parametrizations went from more than a page to less than half a page.
9. Section 4.8 contains new material on the determinant and eigenvalues, including material needed for proving the Cayley-Hamilton theorem.
10. New exercises have been added, including exercise A5.1 proving a version of Kantorovich's theorem with a weaker continuity condition on the derivatives (with implications for the inverse and implicit function theorems).
11. We prove that the matrix $\tilde{A}$ in row echelon form, obtained from a matrix A by row operations, is unique (theorem 2.1.7).
12. The index has grown from 7 to 11 pages.

To incorporate so many additions while keeping the page count the same, we moved the programs that had been in Appendix B to the Matrix Editions web site, MatrixEditions.com. We also put the exercises into smaller type.

Practical information

Chapter 0 and back cover Chapter 0 is intended as a resource. Students should not feel that they need to read it before beginning chapter 1. Another resource is the inside back cover, which lists some useful formulas.

Errata Errata for the third edition will be posted at

http://www.MatrixEditions.com

Exercises Exercises are given at the end of each section; chapter review exercises are given at the end of each chapter, except chapter 0 and the appendix. They range from very easy exercises intended to make students familiar with vocabulary, to quite difficult ones. The hardest exercises are marked with an asterisk (in rare cases, two asterisks).

A student solution manual, with solutions to odd-numbered exercises, is available from Matrix Editions.

Notation Mathematical notation is not always uniform. For example, $|A|$ can mean the length of a matrix A or the determinant of A. Different notations for partial derivatives also exist. This should not pose a problem for readers who begin at the beginning and end at the end, but for those who are using only selected chapters, it could be confusing. Notations used in the book are listed on the front inside cover, along with an indication of where they are first introduced.

Numbering Theorems, lemmas, propositions, corollaries, and examples share the same numbering system: proposition 2.3.7 is not the seventh proposition of section 2.3; it is the seventh numbered item of that section.

We often refer back to theorems, examples, and so on, and believe this numbering makes them easier to find.

Figures and tables share their own numbering system; figure 4.5.2 is the second figure or table of section 4.5. Virtually all displayed equations are numbered, with the numbers given at right; equation 4.2.3 is the third equation of section 4.2. When an equation is displayed a second time, it keeps its original number, but the number is in parentheses.

Programs The three programs used in this book – NEWTON.M (used in section 2.8), MONTE CARLO (section 4.6), and DETERMINANT (section 4.8) – are posted at http://MatrixEditions.com/Programs.html. NEWTON.M is a MATLAB program; the others are written in Pascal. Readers are welcome to propose additional programs (or translations of the above programs into other programming languages); if interested, please write jhh8@cornell.edu.

Symbols We use △ to mark the end of an example or remark, and □ to mark the end of a proof. Sometimes we specify what proof is being ended: for instance, □ corollary 1.6.15 means end of the proof of corollary 1.6.15.

Using this book as a calculus text or as an analysis text

The SAT test used to have a section of analogies; the "right" answer sometimes seemed contestable. In that spirit,

Calculus is to analysis as a sonata is to a Chopin etude (playing a sonata is more fun, the etude will do more to improve your technique).

Calculus is to analysis as playing a sonata is to composing one (composing requires a better understanding of music theory).

Calculus is to analysis as performing in a ballet or modern dance is to choreographing it.

These are meant to suggest that generally, calculus is more fun, and it is challenging – rote learning isn't enough. Analysis involves more painstaking technical work, which at times may seem like drudgery, but it provides a level of mastery that calculus alone cannot give.

This book can be used at different levels of rigor. Chapters 1 through 6 contain material appropriate for a course in linear algebra and multivariate calculus. Appendix A contains the technical, rigorous underpinnings appropriate for a course in analysis. It includes proofs of those statements not proved in the main text, and a painstaking justification of arithmetic.

In deciding what to include in this appendix, and what to put in the main text, we used the analogy that learning calculus is like learning to drive a car with standard transmission – acquiring the understanding and intuition to shift gears smoothly when negotiating hills, curves, and the stops and starts of city streets. Analysis is like designing and building a car. To use this book to "learn how to drive," appendix A should be omitted.

Most of the proofs included in this appendix are more difficult than the proofs in the main text, but difficulty was not the only criterion; many students find the proof of the fundamental theorem of algebra (section 1.6) quite difficult. But we find this proof qualitatively different from the proof of the Kantorovich theorem, for example. A professional mathematician who has understood the proof of the fundamental theorem of algebra should be able to reproduce it. A professional mathematician who has read through the proof of the Kantorovich theorem, and who agrees that each step is justified, might well want to refer to notes in order to reproduce it. In this sense, the first proof is more conceptual, the second more technical.

One-year courses

At Cornell University this book is used for the honors courses Math 223 (fall semester) and 224 (spring semester). Students are expected to have a 5 on the Advanced Placement BC Calculus exam, or the equivalent. When John Hubbard teaches the course, he typically gets to the middle of chapter 4 in the first semester, sometimes skipping section 3.8 on the geometry of curves and surfaces, and going through sections 4.2–4.4 rather rapidly, in order to get to section 4.5 on Fubini's theorem and begin to compute integrals. In the second semester he gets to the end of chapter 6 and goes on to teach some of the material that will appear in a sequel volume, in particular differential equations.[1]

Jean Dieudonné (1906–1992)

Dieudonné, one of the founding members of "Bourbaki," a group of young mathematicians who published collectively under the pseudonym Nicolas Bourbaki, and whose goal was to put modern mathematics on a solid footing, was the personification of rigor in mathematics. Yet in his book *Infinitesimal Calculus*, he put the harder proofs in small type, saying "a beginner will do well to accept plausible results without taxing his mind with subtle proofs."

One could also spend a year on chapters 1–6. Some students might need to review chapter 0; others may be able to include some proofs from the appendix.

Semester courses

1. A one-semester course for students who have studied neither linear algebra nor multivariate calculus.

For such a course, we suggest covering only the first four chapters. Topics that could be omitted include the proof of the fundamental theorem of algebra in section 1.6, criteria for differentiability in section 1.9, the part of section 2.9 concerning a stronger version of the Kantorovich theorem, section 3.8 on the geometry of curves and surfaces, section 4.4 on measure 0 (if Lebesgue integration is to be skipped), the proof of theorem 4.9.1, and the discussion in section 4.11 on Fourier and Laplace transforms.

Sections 4.2 (integrals and probability) and 4.6 (numerical methods of integration) could also be skipped, but we feel these topics are generally given too little attention. If section 4.2 is skipped, then one should also skip the discussion of Monte Carlo methods in section 4.6.

2. A course for students who have had some exposure to either linear algebra or multivariate calculus, but who are not ready for a course in analysis.

We used an earlier version of this text with students who had taken a course in linear algebra, and feel they gained a great deal from seeing how linear algebra and multivariate calculus mesh. Such students could be expected to cover chapters 1–6, possibly omitting some material. For a less fast-paced course, the book could also be covered in a year, possibly including some proofs from the appendix.

[1]Eventually, he would like to take three semesters to cover chapters 1–6 of the current book and the material of the sequel (referred to as "Volume 2" in this text), including differential equations, inner products (with Fourier analysis and wavelets), and advanced topics in differential forms.

3. A one-semester analysis course.

In one semester one could hope to cover all six chapters and some or most of the proofs in the appendix. This could be done at varying levels of difficulty; students might be expected to follow the proofs, for example, or they might be expected to understand them well enough to construct similar proofs.

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John H. Hubbard (BA Harvard University, PhD University of Paris) is professor of mathematics at Cornell University and at the University of Provence in Marseille; he is the author of several books on differential equations (with Beverly West), a book on Teichmüller theory, and a two-volume book in French on scientific computing (with Florence Hubert). His research mainly concerns complex analysis, differential equations, and dynamical systems. He believes that mathematics research and teaching are activities that enrich each other and should not be separated.

Barbara Burke Hubbard (BA Harvard University) is the author of *The World According to Wavelets*, which was awarded the prix d'Alembert by the French Mathematical Society in 1996.

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Preliminaries

Allez en avant, et la foi vous viendra
(Keep going; faith will come.)—Jean d'Alembert (1717–1783), to those questioning calculus

0.0 Introduction

This chapter is intended as a resource. You may be familiar with its contents, or there may be topics you never learned or that you need to review. You should *not* feel that you need to read chapter 0 before beginning chapter 1; just refer back to it as needed. (A possible exception is section 0.7 on complex numbers.)

We have included reminders in the main text; for example, in section 1.5 we write, "You may wish to review the discussion of quantifiers in section 0.2." So you may skip ahead to chapter 1 without being afraid that you will miss something really important.

In section 0.1 we share some guidelines that in our experience make reading mathematics easier, and discuss a few specific issues like sum notation.

Section 0.2 analyzes the rather tricky business of negating mathematical statements. (To a mathematician, the statement "All eleven-legged alligators are orange with blue spots" is an obviously true statement, not an obviously meaningless one.) We first use this material in section 1.5.

Set theory notation is discussed in section 0.3. The "eight words" of set theory are used beginning in section 1.1. The discussion of Russell's paradox is not necessary; we include it because it is fun and not hard.

Section 0.4 defines the word "function" and discusses the relationship between a function being "onto" or "one to one" and the existence and uniqueness of solutions. This material is first needed in section 1.3.

Real numbers are discussed in section 0.5, in particular, least upper bounds, convergence of sequences and series, and the intermediate value theorem. This material is first used in sections 1.5 and 1.6.

Most of this text concerns real numbers, but we think that anyone beginning a course in multivariate calculus should know what complex numbers are and be able to compute with them.

The discussion of countable and uncountable sets in section 0.6 is optional; again, we include it because it is fun and not hard.

In our experience, most students studying vector calculus for the first time are comfortable with complex numbers, but a sizable minority have either never heard of complex numbers or have forgotten everything they once knew. If you are among them, we suggest reading at least the first few pages of section 0.7 and doing some of the exercises.

0.1 Reading mathematics

The most efficient logical order for a subject is usually different from the best psychological order in which to learn it. Much mathematical writing is based too closely on the logical order of deduction in a *subject, with too many definitions without, or before, the examples which motivate them, and too many answers before, or without, the questions they address.*—William Thurston

Many students do well in high school mathematics courses without reading their texts. At the college level you are expected to read the book. Better yet, read ahead. If you read a section before listening to a lecture on it, the lecture will be more comprehensible, and if there is something in the text you don't understand, you will be able to listen more actively and ask questions.

Reading mathematics is different from other reading. We think the following guidelines can make it easier. First, keep in mind that there are two parts to understanding a theorem: understanding the statement, and understanding the proof. *The first is more important than the second.*

What if you don't understand the statement? If there's a symbol in the formula you don't understand, perhaps a δ, look to see whether the next line continues, "where δ is such and such." In other words, read the whole sentence before you decide you can't understand it.

If you're still having trouble, *skip ahead to examples.* This may contradict what you have been told – that mathematics is sequential, and that you must understand each sentence before going on to the next. In reality, although mathematical writing is necessarily sequential, mathematical understanding is not: you (and the experts) never understand perfectly up to some point and not at all beyond. The "beyond," where understanding is only partial, is an essential part of the motivation and the conceptual background of the "here and now." You may often (perhaps usually) find that when you return to something you left half-understood, it will have become clear in the light of the further things you have studied, even though the further things are themselves obscure.

Many students are very uncomfortable in this state of partial understanding, like a beginning rock climber who wants to be in stable equilibrium at all times. To learn effectively one must be willing to leave the cocoon of equilibrium. So *if you don't understand something perfectly, go on ahead and then circle back.*

In particular, an example will often be easier to follow than a general statement; you can then go back and reconstitute the meaning of the statement in light of the example. Even if you still have trouble with the general statement, you will be ahead of the game if you understand the examples. We feel so strongly about this that we have sometimes flouted mathematical tradition and given examples before the proper definition.

Read with pencil and paper in hand, making up little examples for yourself as you go on.

Some of the difficulty in reading mathematics is notational. A pianist who has to stop and think whether a given note on the staff is A or F will not be able to sight-read a Bach prelude or Schubert sonata. The temptation, when faced with a long, involved equation, may be to give up. You need to take the time to identify the "notes."

The Greek Alphabet

Greek letters that look like Roman letters are not used as mathematical symbols; for example, A is capital a, not capital α. The letter χ is pronounced "kye," to rhyme with "sky"; φ, ψ, and ξ may rhyme with either "sky" or "tea."

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ	E	epsilon
ζ	Z	zeta
η	H	eta
θ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ	P	rho
σ	Σ	sigma
τ	T	tau
υ	Υ	upsilon
φ, ϕ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

Learn the names of Greek letters – not just the obvious ones like alpha, beta, and pi, but the more obscure psi, xi, tau, omega. The authors know a mathematician who calls all Greek letters "xi" (ξ), except for omega (ω), which he calls "w." This leads to confusion. Learn not just to recognize these letters, but how to pronounce them. Even if you are not reading mathematics out loud, it is hard to think about formulas if ξ, ψ, τ, ω, φ are all "squiggles" to you.

Sum and product notation

Sum notation can be confusing at first; we are accustomed to reading in one dimension, from left to right, but something like

$$\sum_{k=1}^{n} a_{i,k} b_{k,j} \qquad 0.1.1$$

requires what we might call two-dimensional (or even three-dimensional) thinking. It may help at first to translate a sum into a linear expression:

$$\sum_{i=0}^{\infty} 2^i = 2^0 + 2^1 + 2^2 \dots \qquad 0.1.2$$

$$\text{or} \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j}. \qquad 0.1.3$$

In equation 0.1.3, the symbol $\sum_{k=1}^{n}$ says that the sum will have n terms. Since the expression being summed is $a_{i,k}b_{k,j}$, each of those nt terms will have the form ab.

Usually the quantity being summed has an index matching the index of the sum (for instance, k in equation 0.1.1). If not, it is understood that you add one term for every "whatever" that you are summing over. For example, $\sum_i^{10} 1 = 10$.

Two $\sum$ placed side by side do not denote the product of two sums; one sum is used to talk about one index, the other about another. The same thing could be written with one $\sum$, with information about both indices underneath. For example,

$$\sum_{i=1}^{3}\sum_{j=2}^{4}(i+j) = \sum_{\substack{i \text{ from 1 to 3,} \\ j \text{ from 2 to 4}}} (i+j)$$

$$= \left(\sum_{j=2}^{4} 1+j\right) + \left(\sum_{j=2}^{4} 2+j\right) + \left(\sum_{j=2}^{4} 3+j\right) \qquad 0.1.4$$

$$= \big((1+2)+(1+3)+(1+4)\big)$$
$$+ \big((2+2)+(2+3)+(2+4)\big)$$
$$+ \big((3+2)+(3+3)+(3+4)\big);$$

this double sum is illustrated in figure 0.1.1.

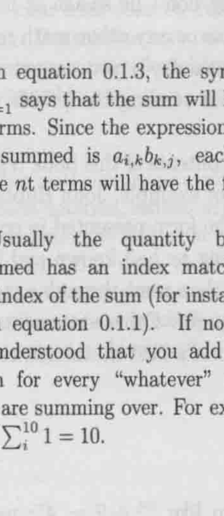

FIGURE 0.1.1.

In the double sum of equation 0.1.4, each sum has three terms, so the double sum has nine terms.

Rules for product notation $\prod$ are analogous to those for sum notation:

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n; \quad \text{for example,} \quad \prod_{i=1}^{n} i = n!. \qquad 0.1.5$$

Proofs

When Jacobi complained that Gauss's proofs appeared unmotivated, Gauss is said to have answered, *You build the building and remove the scaffolding.* Our sympathy is with Jacobi's reply: he likened Gauss to *the fox who erases his tracks in the sand with his tail.*

FIGURE 0.1.2.
Nathaniel Bowditch (1773–1838)

We said earlier that it is more important to understand a mathematical statement than to understand its proof. We have put some of the harder proofs in appendix A; these can safely be skipped by a student studying multivariate calculus for the first time. We urge you, however, to read the proofs in the main text. By reading many proofs you will learn what a proof is, so that (for one thing) you will know when you have proved something and when you have not.

In addition, a good proof doesn't just convince you that something is true; it tells you why it is true. You presumably don't lie awake at night worrying about the truth of the statements in this or any other math textbook. (This is known as "proof by eminent authority"; you assume the authors know what they are talking about.) But reading the proofs will help you understand the material.

If you get discouraged, keep in mind that the contents of this book represent a cleaned-up version of many false starts. For example, John Hubbard started by trying to prove Fubini's theorem in the form presented in equation 4.5.1. When he failed, he realized (something he had known and forgotten) that the statement was in fact false. He then went through a stack of scrap paper before coming up with a correct proof. Other statements in the book represent the efforts of some of the world's best mathematicians over many years.

0.2 Quantifiers and negation

According to a contemporary, the French mathematician Laplace (1749–1827) wrote *il est aisé à voir* ("it's easy to see") whenever he couldn't remember the details of a proof.

"I never come across one of Laplace's '*Thus it plainly appears*' without feeling sure that I have hours of hard work before me to fill up the chasm and find out and show *how* it plainly appears," wrote Bowditch.

Forced to leave school at age 10 to help support his family, the American Bowditch taught himself Latin in order to read Newton, and French in order to read French mathematics. He made use of a scientific library captured by a privateer and taken to Salem. In 1806 he was offered a professorship at Harvard but turned it down.

Interesting mathematical statements are seldom like "$2 + 2 = 4$"; more typical is the statement "every prime number such that if you divide it by 4 you have a remainder of 1 is the sum of two squares." In other words, most interesting mathematical statements are about infinitely many cases; in the case above, it is about all those prime numbers such that if you divide them by 4 you have a remainder of 1 (there are infinitely many such numbers).

In a mathematical statement, every variable has a corresponding quantifier, either implicit or explicitly stated. There are two such quantifiers: "for all" (the *universal quantifier*), written symbolically $\forall$, and "there exists" (the *existential quantifier*), written $\exists$. Above we have a single quantifier, "every." More complicated statements have several quantifiers, for example, the statement, "For all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$." This true statement says that the squaring function is continuous.

The order in which these quantifiers appears matters. If we change the order of quantifiers in the preceding statement about the squaring function to "For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$," we have a meaningful mathematical sentence but it is false. (It claims that the squaring function is uniformly continuous, which it is not.)