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Representations of Compact Lie Groups

紧李群的表示

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Representations of Compact Lie Groups

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Auch ging es mir, wie jedem, der reisend oder lebend mit Ernst gehandelt, daß ich in dem Augenblicke des Scheidens erst einigermaßen mich wert fühlte, hereinzutreten. Mich trösteten die mannigfaltigen und unentwickelten Schätze, die ich mir gesammlet.

G.

Preface

This book is based on several courses given by the authors since 1966. It introduces the reader to the representation theory of compact Lie groups.

We have chosen a geometrical and analytical approach since we feel that this is the easiest way to motivate and establish the theory and to indicate relations to other branches of mathematics. Lie algebras, though mentioned occasionally, are not used in an essential way. The material as well as its presentation are classical; one might say that the foundations were known to Hermann Weyl at least 50 years ago.

Prerequisites to the book are standard linear algebra and analysis, including Stokes' theorem for manifolds. The book can be read by German students in their third year, or by first-year graduate students in the United States.

Generally speaking the book should be useful for mathematicians with geometric interests and, we hope, for physicists.

At the end of each section the reader will find a set of exercises. These vary in character: Some ask the reader to verify statements used in the text, some contain additional information, and some present examples and counterexamples. We advise the reader at least to read through the exercises.

The book is organized as follows. There are six chapters, each containing several sections. A reference of the form III, (6.2) refers to Theorem (Definition, etc.) (6.2) in Section 6 of Chapter III. The roman numeral is omitted whenever the reference concerns the chapter where it appears. References to the Bibliography at the end of the book have the usual form, e.g. Weyl [1].

Naturally, we would have liked to write in our mother tongue. But we hope that our English will be acceptable to a larger mathematical community, although any personal manner may have been lost and we do not feel competent judges on matters of English style.

Arunas Liulevicius, Wolfgang Lück, and Klaus Wirthmüller have read the manuscript and suggested many improvements. We thank them for their generous help. We are most grateful to Robert Robson who translated part of the German manuscript and revised the whole English text.

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CHAPTER I

Lie Groups and Lie Algebras

In this chapter we explain what a Lie group is and quickly review the basic concepts of the theory of differentiable manifolds. The first section illustrates the notion of a Lie group with classical examples of matrix groups from linear algebra. The spinor groups are treated in a separate section, §6, but the presentation of the general theory of representations in this book presupposes no knowledge of spinor groups. They only appear as examples which, although important, may be skipped. In §§2, 3, and 4 we construct the exponential map and exploit it to obtain elementary information about the structure of subgroups and quotients, and in §5 we explain how to construct an invariant integral using differential forms. We quote Stokes' theorem to get a result about mapping degrees which we shall use in Chapter IV.

1. The Concept of a Lie Group and the Classical Examples

The concept of a Lie group arises naturally by merging the algebraic notion of a group with the geometric notion of a differentiable manifold. However, the classical examples, as well as the methods of investigation, show the theory of Lie groups to be a significant geometric extension of linear algebra and analytic geometry.

(1.1) Definition. A ***Lie group*** is a differentiable manifold G which is also a group such that the group multiplication

$$\mu \colon G \times G \to G$$

(and the map sending g to g^{-1}) is a differentiable map. A ***homomorphism of Lie groups*** is a differentiable group homomorphism between Lie groups.

For us the word ***differentiable*** means infinitely often differentiable. Throughout this book we use the words differentiable, ***smooth***, and C^∞ as synonymous.

The identity map on a Lie group is a homomorphism, and composing homomorphisms yields a homomorphism—Lie groups and homomorphisms form a category. One may define the usual categorical notions: in particular, an ***isomorphism*** (denoted by $\cong$) is an invertible homomorphism.

We will use e or 1 to denote the identity element of G, although we will sometimes use E when considering a matrix group and 0 when considering an additive abelian group.

The reader should know what a group is, and the concept of a differentiable manifold should not be new. Nonetheless, we review a few facts about manifolds.

(1.2) Definition. An ***n-dimensional*** (differentiable) ***manifold*** M^n is a Hausdorff topological space with a countable (topological) basis, together with a maximal ***differentiable atlas***. This atlas consists of a family of ***charts*** $h_\lambda: U_\lambda \to U'_\lambda \subset \mathbb{R}^n$, where the ***domains of the charts***, $\{U_\lambda\}$, form an open cover of M^n, the U'_λ are open in $\mathbb{R}^n$, the charts (***local coordinates***) h_λ are homeomorphisms, and every ***change of coordinates*** $h_{\lambda\mu} = h_\mu \circ h_\lambda^{-1}$ is differentiable on its domain of definition $h_\lambda(U_\lambda \cap U_\mu)$.

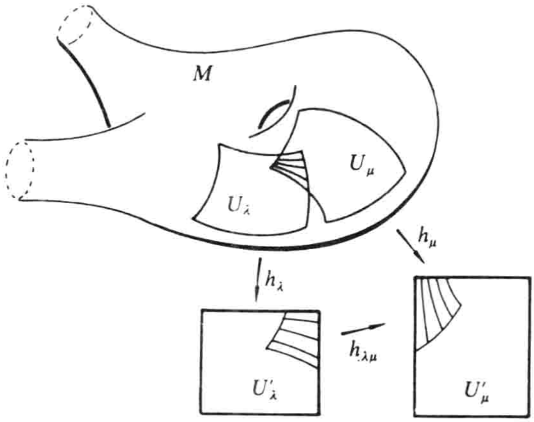


Figure 1

The atlas is maximal in the sense that it cannot be enlarged to another differentiable atlas by adding more charts, so any chart which could be added to the atlas in a consistent fashion is already in the atlas.

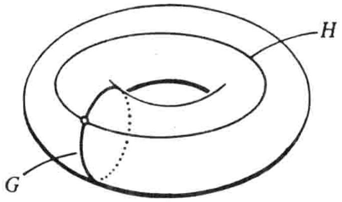
A continuous map $f: M \to N$ of differentiable manifolds is called ***differentiable*** if, after locally composing with the charts of M and N, it induces a differentiable map of open subsets of Euclidean spaces.

The reader may find an elementary introduction to the basic concepts of differentiable manifolds in the books by Bröcker and Jänich [1] or Guillemin and Pollak [1], but we will assume little in the way of background. We now turn to the examples which, as previously mentioned, one more or less knows from linear algebra.

(1.3) Every finite-dimensional vector space with its additive group structure is a Lie group in a canonical way. Thus, up to isomorphism, we get the groups $\mathbb{R}^n$, $n \in \mathbb{N}_0$.

(1.4) The ***torus*** $\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n \cong (S^1)^n$ is a Lie group. Here $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle viewed as a multiplicative subgroup of $\mathbb{C}$, and the isomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ is induced by $t \mapsto e^{2\pi i t}$. The n-fold product of the circle with itself has the structure of an abelian Lie group due to the following general remark:

(1.5) If G and H are Lie groups, so is $G \times H$ with the direct product of the group and manifold structures on G and H.



$G \times H$
Figure 2

It will turn out that every connected abelian Lie group is isomorphic to the product of a vector space and a torus (3.6).

(1.6) Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. The set $\mathrm{Aut}(V)$ of linear automorphisms of V is an open subset of the finite-dimensional vector space $\mathrm{End}(V)$ of linear maps $V \to V$, because $\mathrm{Aut}(V) = \{A \in \mathrm{End}(V) \mid \det(A) \neq 0\}$ and the determinant is a continuous function. Thus $\mathrm{Aut}(V)$ has the structure of a differentiable manifold. After the introduction of coordinates, the group operation of $\mathrm{Aut}(V)$ is matrix multiplication, which is algebraic and hence differentiable. Therefore $\mathrm{Aut}(V)$ has a canonical structure as a Lie group, and we get the groups

$$\mathrm{GL}(n, \mathbb{R}) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n) \quad \text{and} \quad \mathrm{GL}(n, \mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n).$$

Linear maps $\mathbb{R}^n \to \mathbb{R}^k$ may be described by $(k \times n)$-matrices, and, in particular, $\mathrm{GL}(n, \mathbb{R})$ is canonically isomorphic to the group of invertible $(n \times n)$-matrices. Thus we will think of $\mathrm{GL}(n, \mathbb{R})$, its classical subgroups $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, ..., and $\mathrm{GL}(n, \mathbb{C})$ as matrix groups.

The group $\mathrm{GL}(n, \mathbb{R})$ has two connected components on which the sign of the determinant is constant. Automorphisms with positive determinant form an open and closed subgroup $\mathrm{GL}^+(n, \mathbb{R})$. It is connected because performing elementary row and column operations which do not involve multiplication by a negative scalar does not change components.

These linear groups yield many others once one knows, as we will show in (3.11) and (4.5), that a closed subgroup of a Lie group and the quotient of a Lie group by a closed normal subgroup inherit Lie group structures.

(1.7) As a result we get the groups

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}, \quad \text{and}$$
$$\mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \det(A) = 1\},$$

the ***special linear groups*** over $\mathbb{R}$ and $\mathbb{C}$. We also get the ***projective groups***

$$\mathrm{PGL}(n, \mathbb{R}) = \mathrm{GL}(n, \mathbb{R})/\mathbb{R}^* \quad \text{and} \quad \mathrm{PGL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^*,$$

where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ are embedded as the subgroups of scalar multiples of the identity matrix. The projective groups are groups of transformations of projective spaces, see (1.16), Ex. 11.

In this book, however, we are primarily interested in compact groups, so we recall the following closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ from linear algebra:

(1.8) The ***orthogonal groups*** $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid {}^tA \cdot A = E\}$, where ${}^tA$ denotes transpose and E is the identity matrix. Analogously there is the ***unitary group*** $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid {}^*A \cdot A = E\}$, where ${}^*A = {}^t\bar{A}$ is the conjugate transpose of A. Elements of $\mathrm{O}(n)$ are called ***orthogonal*** and elements of $\mathrm{U}(n)$ are called ***unitary***. On $\mathbb{R}^n$ there is an ***inner product***, the ***standard Euclidean scalar product***

$$\langle x, y \rangle = \sum_{\nu=1}^{n} x_\nu \cdot y_\nu,$$

and on $\mathbb{C}^n$ one has the ***standard Hermitian product***

$$\langle x, y \rangle = \sum_{\nu=1}^{n} x_\nu \cdot \bar{y}_\nu.$$

$\mathrm{O}(n)$ (resp. $\mathrm{U}(n)$) consists of those automorphisms which preserve the inner product on $\mathbb{R}^n$ (resp. $\mathbb{C}^n$), i.e., those automorphisms A for which

$$\langle Ax, Ay \rangle = \langle x, y \rangle.$$

$\mathrm{O}(n)$ is also split into two connected components by the values ± 1 of the determinant, and one of these is the ***special orthogonal group***

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det(A) = 1\}.$$

The connectedness of SO(n) follows from (4.7), but one may also, for example, join $A \in \mathrm{SO}(n)$ to E by an arc in $\mathrm{GL}^+(n, \mathbb{R})$ and apply Gram–Schmidt orthogonalization to this arc (see Lang [2], VI, §2).

The ***special unitary group*** is defined analogously:

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}.$$

These groups are compact, being closed and bounded in the finite-dimensional vector space End(V).

(1.9) ***Quaternions.*** There is up to isomorphism only one proper finite field extension of $\mathbb{R}$, namely the field $\mathbb{C}$ of complex numbers. There is, however, a skew field containing $\mathbb{C}$ of complex dimension 2 and real dimension 4, called the ***quaternion algebra*** $\mathbb{H}$, which may be described as follows: The $\mathbb{R}$-algebra $\mathbb{H}$ is the algebra of (2×2) complex matrices of the form

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},$$

with matrix addition and multiplication.

If such a matrix is nonzero, its determinant, $|a|^2 + |b|^2$, is nonzero, and its inverse is another matrix of the same form. Thus every nonzero $h \in \mathbb{H}$ has a multiplicative inverse, so $\mathbb{H}$ is a ***division algebra*** (also called skew field). We consider $\mathbb{C}$ as a subfield of $\mathbb{H}$ via the canonical embedding $\mathbb{C} \to \mathbb{H}$ given by

$$c \mapsto \begin{bmatrix} c & 0 \\ 0 & \bar{c} \end{bmatrix},$$

so we may think of $\mathbb{C}$, and therefore also $\mathbb{R}$, as subfields of $\mathbb{H}$.

The field $\mathbb{R}$ is the center of $\mathbb{H}$. For the center, $Z = \{z \in \mathbb{H} \mid zh = hz \text{ for all } h \in \mathbb{H}\}$, certainly contains $\mathbb{R}$, and, were Z larger than $\mathbb{R}$, then Z as a proper finite field extension of $\mathbb{R}$, would be isomorphic to $\mathbb{C}$. But $Z \neq \mathbb{H}$, so choosing $x \in \mathbb{H}$ with $x \notin Z$ we get a proper finite (commutative!) field extension $Z(x) \cong \mathbb{C}(x)$, which is impossible; see also (1.16), Ex. 14.

The algebra $\mathbb{H}$ is a complex vector space, $\mathbb{C}$ acting by left multiplication. As such it has a ***standard basis*** comprised of two elements

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

with the rules for multiplication

$$zj = j\bar{z} \quad \text{for } z \in \mathbb{C} \text{ and } j^2 = -1.$$

This basis gives the ***standard isomorphism*** of complex vector spaces

$$\mathbb{C}^2 \to \mathbb{H}, \qquad (a, b) \mapsto a + bj = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}.$$

The quaternion algebra $\mathbb{H}$ has a ***conjugation anti-automorphism***

$$\iota\colon \mathbb{H} \to \mathbb{H}, \qquad h = a + bj \mapsto \iota(h) = \bar{h} = \bar{a} - bj, \qquad a, b \in \mathbb{C}.$$

Viewing h as a complex matrix, $\iota(h) = {}^*h$, where ${}^*h$ is the adjoint matrix. Conjugation is $\mathbb{R}$-linear, coincides with complex conjugation on $\mathbb{C}$, and obeys the laws

$$\iota(h \cdot k) = \iota(k) \cdot \iota(h) \quad \text{and} \quad \iota^2 = \mathrm{id}.$$

The ***norm*** on $\mathbb{H}$ is defined analogously to the complex norm by

$$\mathrm{N}(h) = h \cdot \bar{h} = \bar{h} \cdot h.$$

Note that $\mathrm{N}(a + bj) = |a|^2 + |b|^2$ is real and nonnegative, and that $\mathrm{N}(h) = 0$ precisely if $h = 0$. As with the complex numbers, the multiplicative inverse of $h \in \mathbb{H}$ is $\bar{h} \cdot \mathrm{N}(h)^{-1}$, and if $h \in \mathbb{C}$, $\mathrm{N}(h) = |h|^2$. If one views h as a (2×2) complex matrix, $\mathrm{N}(h) = \det(h)$.

As a real vector space $\mathbb{H}$ has a ***standard basis*** consisting of the four elements

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

with rules for multiplication

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad \text{and} \quad ki = -ik = j.$$

The quaternions $ai + bj + ck$, $a, b, c \in \mathbb{R}$, are called ***pure*** quaternions, and, as a real vector space, $\mathbb{H}$ splits into $\mathbb{R}$ and the space of pure quaternions isomorphic to $\mathbb{R}^3$. Each $h \in \mathbb{H}$ has unique expression as $h = r + q$ with $r \in \mathbb{R}$ and $q \in \mathbb{R}^3$ (pure). Conjugation may be expressed in this notation as

$$\iota(r + q) = r - q,$$

and therefore $\mathrm{N}(r + q) = r^2 - q^2$. Thus on the subspace $\mathbb{R}^3$ of pure quaternions, $\mathrm{N}(q) = -q^2$, so q^2 is a nonpositive real number. The pure quaternions may be characterized by this property using only the ring structure of $\mathbb{H}$. If $h = r + q$, $r \in \mathbb{R}$, q pure, then $h^2 = r^2 + q^2 + 2rq$ is real if and only if $r = 0$ or $q = 0$, and is nonpositive real if and only if $r = 0$.

With the ***standard isomorphism*** of real vector spaces $\mathbb{R}^4 \to \mathbb{H}$ sending (a, b, c, d) to $a + bi + cj + dk$, the norm on $\mathbb{H}$ corresponds to the Euclidean norm, the square of the Euclidean absolute value on $\mathbb{R}^4$. With the standard isomorphism $\mathbb{C}^2 \cong \mathbb{H}$, the quaternionic norm corresponds to the standard Hermitian norm on $\mathbb{C}^2$. The group

$$\mathrm{Sp}(1) = \{h \in \mathbb{H} \mid N(h) = 1\}$$

is called the ***quaternion group***, or group of unit quaternions. In matrix notation Sp(1) consists of the matrices

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \qquad a, b \in \mathbb{C}, \qquad |a|^2 + |b|^2 = 1,$$

and thus is the same as SU(2). The standard isomorphism $\mathbb{H} \cong \mathbb{R}^4$ identifies Sp(1) with the unit sphere, S^3. This group is the universal covering of the rotation group SO(3), see (6.17), (6.18), and plays an important role in theoretical physics. We will meet the quaternion algebra again in §6 in the guise of the Clifford algebra C_2.

(1.10) The $\mathbb{H}$-Linear Groups. The basic statements of linear algebra may also be formulated for skew fields. An endomorphism $\varphi\colon \mathbb{H}^n \to \mathbb{H}^n$, which is linear with respect to multiplication on the left by scalars from $\mathbb{H}$, may be described by an $(n \times n)$-matrix $(\varphi_{\lambda\nu})$ with coefficients in $\mathbb{H}$ as follows: If $e_\nu \in \mathbb{H}^n$ is the νth unit vector, then $\varphi_{\lambda\nu}$ is defined by $\varphi(e_\nu) = \sum_\lambda \varphi_{\lambda\nu} e_\lambda$. Thus if $h = (h_1, \ldots, h_n) \in \mathbb{H}^n$, we have

$$\varphi(h) = \varphi\left(\sum_\nu h_\nu e_\nu\right) = \sum h_\nu \varphi(e_\nu) = \sum_{\nu,\lambda} h_\nu \varphi_{\lambda\nu} e_\lambda,$$

and

$$\varphi(h)_\lambda = \sum_\nu h_\nu \varphi_{\lambda\nu}.$$

Consequently we may canonically identify the $\mathbb{H}$-linear group

$$\mathrm{GL}(n, \mathbb{H}) = \mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^n)$$

with the group of invertible $(n \times n)$-matrices with coefficients in $\mathbb{H}$, as we did with linear groups earlier. In this case matrices are multiplied as follows:

$$(\psi_{\mu\lambda}) \cdot (\varphi_{\lambda\nu}) = \left(\sum_\lambda \varphi_{\lambda\nu} \cdot \psi_{\mu\lambda}\right).$$

An $\mathbb{H}$-endomorphism of $\mathbb{H}^n$ is invertible precisely if it is invertible as an $\mathbb{R}$-linear map, so, as before, $\mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^n)$ is open in the $\mathbb{H}$-vector space $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ and $\mathrm{GL}(n, \mathbb{H})$ is a $4n^2$-dimensional Lie group.

The standard isomorphism $\mathbb{H} = \mathbb{C} + \mathbb{C}j = \mathbb{C}^2$ induces a standard isomorphism of complex vector spaces

$$\mathbb{H}^n = \mathbb{C}^n + \mathbb{C}^n \cdot j = \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^{2n},$$

and, accordingly, an $\mathbb{H}$-linear endomorphism φ of $\mathbb{H}^n$ may be thought of as a special kind of $\mathbb{C}$-linear endomorphism of $\mathbb{C}^{2n}$:

$$\mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^n + \mathbb{C}^n \cdot j \xrightarrow{\varphi} \mathbb{C}^n + \mathbb{C}^n \cdot j = \mathbb{C}^n \oplus \mathbb{C}^n,$$

namely, one which commutes with the $\mathbb{R}$-linear (but not $\mathbb{C}$-linear!) map

$$j\colon \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n,$$
$$(u, v) = u + vj \mapsto j(u + vj) = -\bar{v} + \bar{u}j = (-\bar{v}, \bar{u})$$

coming from left multiplication by j. The condition that φ commute with left multiplication by j is equivalent to the condition that, as an endomorphism of $\mathbb{C}^n \oplus \mathbb{C}^n$, the map φ is given by a matrix of the form

$$\begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix}, \qquad A, B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n).$$

Note that an $\mathbb{H}$-linear endomorphism may be represented uniquely in the form $A + Bj$, where A and B are complex $(n \times n)$-matrices.

(1.11) There is an inner product on $\mathbb{H}^n$, the ***standard symplectic scalar product***: If $h = (h_1, \ldots, h_n)$ and $k = (k_1, \ldots, k_n)$, then

$$\langle h, k \rangle = \sum_{\nu=1}^{n} h_\nu \bar{k}_\nu.$$

The corresponding norm is given by $\langle h, h \rangle = \sum_\nu h_\nu \bar{h}_\nu = \sum_\nu N(h_\nu) \geq 0$. The ***symplectic group***, $\mathrm{Sp}(n)$, is the group of norm-preserving automorphisms of $\mathbb{H}^n$:

$$\mathrm{Sp}(n) = \{\varphi \in \mathrm{GL}(n, \mathbb{H}) \mid N(\varphi(h)) = N(h) \text{ for all } h \in \mathbb{H}^n\}.$$

A norm-preserving automorphism leaves the inner product invariant ((1.16), Ex. 10). If we identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ as above, the standard norms on $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ correspond, so $\mathrm{Sp}(n)$ is identified with the subgroup of $\mathrm{U}(2n)$ of matrices of the form

$$\begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix} \in \mathrm{U}(2n), \qquad A, B \in \mathrm{End}(\mathbb{C}^n).$$

Thus we will view $\mathrm{Sp}(n)$ as a group of complex matrices. A complex $(2n \times 2n)$-matrix in $\mathrm{Sp}(n)$ is called a ***symplectic matrix***.

(1.12) The map $\mathbb{C}^{2n} = \mathbb{H}^n \xrightarrow{j} \mathbb{H}^n = \mathbb{C}^{2n}$ from (1.10), which sends $(u, v) = u + vj$ to $(-\bar{v}, \bar{u}) = j(u + vj)$ is not $\mathbb{C}$-linear. It is composed of the $\mathbb{C}$-linear map induced by right multiplication by j followed by complex conjugation $c\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, where $c(w) = \bar{w}$. Right multiplication by j may be written as

$$J\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \qquad (u, v) \mapsto (-v, u)$$

and expressed by the matrix

$$J = \begin{bmatrix} 0 & -E \\ E & 0 \end{bmatrix}, \qquad E = \text{identity matrix in } \mathrm{GL}(n, \mathbb{C}).$$

Hence a unitary matrix $A \in \mathrm{U}(2n)$ is symplectic if and only if $AcJ = cJA$. Since $Ac = c\bar{A}$, this means $c\bar{A}J = cJA$, and therefore $\bar{A}J = JA$. And because $A \in \mathrm{U}(2n)$, ${}^{t}A = \bar{A}^{-1}$, so we end up with

$${}^{t}AJA = J.$$

This equation expresses the fact that the linear transformation A fixes the bilinear form

$$(u, v) \mapsto {}^{t}uJv,$$

defined by the matrix J.

Dropping the condition that A be unitary gives the ***complex symplectic group***

$$\mathrm{Sp}(n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{C}) \mid {}^{t}AJA = J\}.$$

(1.13) As a matter of principle, one should always consider the three cases $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, and these are the only three finite-dimensional real division algebras. This is the content of the Frobenius theorem. For a proof see Jacobson [2], 7.7, p. 430. Further information and historical remarks on quaternions may be found in Chapters 6 and 7 by Koecher and Remmert in Ebbinghaus *et al.* [1].

We have defined subgroups

$$\mathrm{GL}(n, \mathbb{H}) \supset \mathrm{Sp}(n), \text{ symplectic scalar product,}$$

$$\mathrm{GL}(n, \mathbb{C}) \supset \mathrm{U}(n), \text{ Hermitian scalar product,}$$

$$\mathrm{GL}(n, \mathbb{R}) \supset \mathrm{O}(n), \text{ Euclidean scalar product,}$$

in a completely analogous fashion. We refer to each of the scalar products involved simply as ***inner product***.

More generally, to every bilinear map of a finite-dimensional real vector space V into a real vector space H

$$V \times V \to H, \qquad (v, w) \mapsto \langle v, w \rangle,$$

there belongs a Lie group $G = \{A \in \mathrm{Aut}(V) \mid \langle Av, Aw \rangle = \langle v, w \rangle$ for all $v, w \in V\}$. Many important Lie groups with a geometric flavor arise in this way, for example the ***Lorentz group***, which comes from the scalar product on $\mathbb{R}^4$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4.$$