变分法——理论与应用

Variational Methods — Theory and Applications

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中国科学技术大学研究生教材

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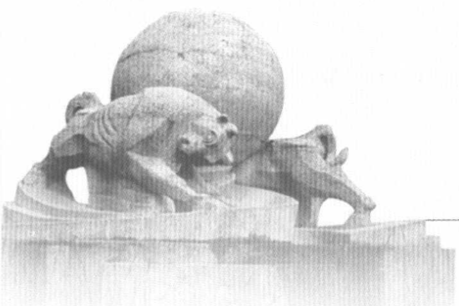

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内 容 简 介

本书不仅对变分法的基本概念、理论和方法作了严谨的介绍和论述，而且特别注重介绍变分法在解决椭圆型方程中的应用。本书中的许多证明都被有意识地分解成几个步骤，每个步骤都给出子目标，这样不仅利于读者理解证明思路和过程，而且更便于总结命题条件与结论之间的因果关系。本书在内容上尽量做到自封，只是在极少数地方引用了代数拓扑和泛函分析中的命题，也尽量给出参考文献，以便读者查阅。

本书可作为数学系分析类研究生专业教材，也可作为数学系高年级本科生选修课教材。

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前　言

自 1998 年以来，作者在中国科学技术大学数学系和哥伦比亚国立大学(Universidad Nacional de Colombia)数学系多次讲授研究生课程“变分法”。虽然在变分理论方面有好几部专著，如本书参考文献中所列，有 M. Struwe 编著的《Variational Methods》、M. Willem 编著的《Minimax Theorems》、张恭庆院士编著的《临界点理论及其应用》、陆文端编著的《微分方程中的变分法》、沈尧天和严树森编著的《拟线性椭圆型方程的变分法》等等，但是作者在科研和教学中总感觉到上述专著要么起点太高、内容没有自封性，要么体系庞大、内容过于烦杂。在教学中，为了做到理论体系相对完整、自封，证明脉络清晰，需要查阅许多文献资料；同时，为了使学生尽快接触到科研前沿，我们也有意识地增加最新的科研成果的介绍。经过几轮讲授，现将讲稿整理出版，希望对以后的教学有所帮助。

为了提高学生的专业英语水平和国际竞争力，教育部教高[2001]4 号文件明确指出：高校教育应采用双语教学，尤其是被列入“211”重点工程的各高校，要创造使用英语等外语进行公共课和专业课教学的气氛。结合现在研究生教学的实际情况，本书采用英语编写。

本书主要内容安排如下：

第 1 章概述变分理论的主要思想、基本概念和术语。通过广义解将微分方程的求解与泛函极值问题的求解联系起来，提出了现代变分理论的中心任务：通过寻求适当泛函的临界点求解微分方程的解。最后介绍了几个源自物理和几何的具体例子，首先利用物理定律或几何性质，给出适当的能量泛函，再导出欧拉-拉格朗日(Euler-Lagrange)方程。

第 2 章研究索伯列夫(Sobolev)空间的基本性质。索伯列夫空间已经成为应用泛函分析知识得到微分方程信息的最合适的工作空间，本章主要介绍后面内容将要用到的主要性质，如基本不等式、嵌入定理，以及各种意义下的强弱收敛关系。关于索伯列夫空间知识的完整介绍，可以参考 R. Adams 和 J. Fournier 编著的《Sobolev Spaces》第 2 版，以及 L. C. Evans 编著的《Partial Differential Equations》的第 5 章。

第 3 章介绍巴拿赫(Banach)空间中的微积分，给出了作为全微分和方向导数推广的弗雷歇(Fréchet)导数和加托(Gâteaux)导数的定义，以及它们之间的相互关系。考察了涅梅茨基(Nemytski)算子的连续性，这在验证泛函满足紧性条件——(PS)条件

时很有用。最后，为了在第 5 章中构造伪梯度向量场，我们介绍了巴拿赫空间中抽象函数初值问题的可解性。

第 4 章讨论求解极值问题的直接方法，以及它在不同情形时的推广。首先导出了求解极值问题最重要的概念——下半连续性。由下半连续性和埃克兰(Ekeland)变分原理，从下有界的泛函中很容易得到(PS)序列。为得到极小值点，我们需要一定的紧性条件，以从(PS)序列中抽取收敛的子列，这就是(PS)条件。(PS)条件在变分理论中起着重要的作用，局部地，它保证(PS)序列中收敛子列的存在性；整体地，它保证泛函水平集在含有临界值的水平之间可以作连续形变。

第 5 章介绍形变原理。为了方便学习，我们首先介绍希尔伯特(Hilbert)空间中的形变原理，因为这时可以运用梯度场这个概念。然后介绍一般巴拿赫空间中的形变原理，这要推广希尔伯特空间中的梯度场，提出伪梯度场这个概念。同时，我们分别介绍不带(PS)条件和带有(PS)条件两种不同情形时的形变原理，以便应用时对不同情形加以选择。

第 6 章介绍极小极大方法。首先，我们叙述一个一般性的极小极大原理，而将山路引理、指标理论和环绕方法作为其推论给出，这样就更能看出这几种方法的统一性。对于每种方法，我们都介绍几个应用实例，以说明如何针对具体问题选择合适的方法。

第 7 章介绍失去紧性的变分问题。我们首先介绍波霍扎叶夫(Pohozaev)型积分恒等式，并由此得出一些不存在性结果，特别是拟线性椭圆型方程在临界增长情形下解的不存在性结果。我们注意到，此时紧性的缺失是由于泛函在某些非紧群作用下所具有的不变性所导致的。为消除这些非紧群作用的影响，我们将求解空间加上某种对称性，从而重新获得紧性。但是，对于一般非紧泛函(可能没有对称性)，由 P. L. Lions 所发展起来的集中紧性原理刻画了泛函极小化序列的收敛性态：紧性、消失性或二分性。而关于一般(PS)序列的收敛性态，Struwe 等给出了全局紧性定理。

第 8 章研究源自水波方程的广义 Kadomtsev-Petviashvili 方程在高维空间中孤波解和稳态解的存在性。这部分综合应用了前面的各种理论方法。

第 9 章介绍索伯列夫不等式中最佳常数的达到函数。在第 7 章中，我们发现达到函数的显式形式在求解具有临界增长的方程时起着重要的作用。由于许多文献中直接给出其显式形式，但是没有具体推导过程，所以在此我们从 1 维情形的 Bliss 引理开始，推导出一般情形时达到函数的显式形式，并综述了它在求解 Yamabe 问题中的应用。

在附录中，我们简单介绍了椭圆型方程的正则性结果。为叙述方便起见，我们只考虑了简单的线性情形。

在行文中，我们力求叙述准确，重点突出。每个证明都被有意识地分解成几个小步骤，每步都明确写出子目标，这样更方便读者把握证明思想；同时，也能较清楚地看清条件和结论之间的因果关系。因为在不同的具体问题中，有的是思想框架

相似，局部细节不同；有的是局部细节可以应用到不同情形等情况，所以对证明的框架和细节都必须熟练掌握。在每个理论方法之后，我们都介绍几个应用实例，有的是直接验证定理中的条件，以显示如何去验证条件；有的是正文定理证明中某些与前面步骤类似的地方，留作习题，让读者自己补上，以强化理解。

本书适合高等院校数学系微分方程、微分几何以及数学物理等专业的研究生学习变分法理论，前半部分也可以作为数学系高年级本科生的选修课教材，特别是有关 Sobolev 空间的知识、Banach 空间微积分以及变分法的直接方法。

本书的编写得到国家自然科学青年基金（No. 10101024）和中国科学技术大学研究生教材出版基金的资助。在编写和教学过程中，班上同学曾提出过许多宝贵的意见和建议，在此一并致谢。由于作者学识所限而导致的错误和不足在所难免，望读者批评指正。

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Preface

This book is intended to be as a textbook for graduate students or undergraduate students for their last year. The guiding principle followed is that all definitions, theorems, etc., should be clearly and precisely stated. Proofs are given with the student in mind. Most are presented in detail and step by step, indicated the sub-goal for each step if possible. When this is not the case, the reader is told precisely what is missing and asked to fill in the gap as exercise which is meant to fix the ideas of the section in the reader's mind.

This book is organized as follows:

In Chapter 1, we give the fundamental questions and basic ideas of variational method.

In Chapter 2, we recall some basic definitions and useful inequalities of Sobolev spaces, which turn out to be the proper setting in which to apply ideas of functional analysis to get information concerning partial differential equations.

Chapter 3 is concerned with calculus in Banach space, various derivatives of functionals defined on Banach space and critical points of differentiable functionals regarded as the weak solutions of PDE are introduced.

In Chapter 4, we study the direct methods to solve extremal problems. Some variants of direct methods, such as Lagrange multiplier method, weak sub- and super-solutions method and Nehari manifolds, are presented with examples.

In Chapter 5, we established some deformation theorems with or without (PS) condition, which will be the base-stone for minimax theorems.

In Chapter 6, we first present a somewhat general minimax principle, then study the Mountain Pass Lemma and its variants, the Index Theory and the Linking Argument as the consequences of the general principle. Several applications to semilinear or quasilinear elliptic equations are given.

In Chapter 7, using the Pohozaev type variational identity and the concentration compactness principle, we study the nonexistence and existence of noncompact variational problems, which include problems involving critical Sobolev exponents and the problems on unbounded domain as examples.

Chapter 8 includes the newly obtained results about the existence and nonexistence of solitary solutions to the Generalized Kadomtsev-Petviashvili (GKP) equation in higher dimensional space and that of stationary solutions to the GKP equation in bounded domain. This chapter can be taken as applications of the Mountain Pass Lemma, the Pohozaev type variational identity.

Chapter 9 is concerned with the problem of best constants in Sobolev inequalities and its extremal functions (if exist), which had played very important role in the study of problems involving critical Sobolev exponents in bounded or unbounded domains. There are extremely many results in this direction, we make a review only for a small part of all the references concerning it.

In Appendix, we briefly discuss the well known De Giorgi-Nash-Moser regularity theory of weak solution to elliptic equations. In order to explain ideas clearly, we will only discuss the linear equations in divergence form, but the idea is easy to be extended to quasilinear elliptic equation, because that the linearity has no bearing in their arguments.

Contents

Chapter 1

Introduction

1.1 Basic ideas of variational methods

What is the variational method, or the calculus of variations? This is a method to select the best among a variety of objects, it is the following process:

(1) Gather all relevant objects into a space X; (For P.D.E., we will appeal to some appropriate Sobolev space. See Chapter 2 for Sobolev space.)

(2) Take an appropriate function (or functional) E on X. If E is appropriate for the purpose, then the minima or maxima of E in X are the best objects.

It has a long history and refreshes its face according to the developments of mathematics and other sciences. Laws in astronomy, mechanics, physics, all nature sciences and technologies as well as in the economy behavior obey variational principles. From the time of Newton, Leibnitz, Euler and Lagrange, the variational method has been carried out as follows:

(1) On the space X, one may consider the concept of the *differential* E' of E; (See Chapter 3, the calculus in Banach space.)

(2) If $x_0 \in X$ is *best*, then it should attain the maximum or minimum of E. So the *derivative* of E vanishes at x_0, i.e.,

$$E'(x_0) = 0.$$

(3) The point x_0 satisfying $E'(x_0) = 0$ (called a critical point of E) could be written and characterized in term of some differential equation (called the Euler-Lagrange equation).

(4) Thus, it remains only to solve this differential equation.

This is the classical process of variational method, which deals with minima or maxima by solving the Euler-Lagrange equation. Since the Euler-Lagrange equation is only a necessary condition for minima or maxima, a second order differential condition has to be derived in verifying the minimality or maximality. However, only very few Euler-Lagrange equations can be solved explicitly. The method is limited in applications.

Conversely, in his study of conformal mappings, Dirichlet introduced the so-called Dirichlet principle, i.e., finding a minimizing sequence $\{x_n\}_{n=1}^{\infty} \subset X$:

$$E(x_n) \to \inf\{E(x) \ : \ x \in X\},$$

if one can show that

$$x_n \to x_0 \in X \text{ in some sense such that}$$

$$E(x_n) \to E(x_0),$$

then x_0 solves the Euler-Lagrange equation. This provides a direct method which can be used to solve differential equations as follows:

(1) One should solve some differential equation which is important but difficult to solve;

(2) To solve this equation, one could consider a certain space X and a function (or functional) E on X in such a way that the Euler-Lagrange equation associated with E, i.e., $E'(x_0) = 0$ corresponds to the equation in question.

(3) Then one may only find a maximum or minimum of E on X.

For many interesting problems in mathematics and physics, one could formulate the calculus of variations in this way, but it often happens that both to find maxima and minima of E and to solve the corresponding Euler-Lagrange equation are very difficult.

Let's formally formulate the process of the variational method as follows. Many boundary value problems are equivalent to the following operator equation:

$$Au = 0, \tag{1.1.1}$$

where $A : X \to Y$ is a mapping between two Banach spaces. The problem (1.1.1) is called a variational problem, if there exists a differentiable functional $\phi : X \to \mathbb{R}$ such that $A = \phi'$, i.e.,

$$< Au, \ v >= \lim_{t \to 0} \frac{\phi(u + tv) - \phi(u)}{t}.$$

The space Y corresponds then to the topological dual X' of X and *Eq.* (1.1.1) is equivalent to $\phi'(u) = 0$, i.e.,

$$< \phi'(u), \ v >= 0, \ \forall v \in X. \tag{1.1.2}$$

A function $u \in X$ is called a critical point of ϕ, if it is a solution of (1.1.2), and then the value of $c = \phi(u)$ is called a critical value of ϕ if u is a critical point.

Question 1: How to find critical values and critical points?

When ϕ is bounded from below, the infimum

$$c := \inf_X \phi$$

is a natural candidate.

Question 2: Is the infimum always a critical value?

In 1895, Weierstrass [93] constructed a counter-example: To find a continuously differentiable function $u : [-1, 1] \to \mathbb{R}$ minimizing integral

$$I(u) = \int_{-1}^{1} |x \frac{\mathrm{d}u}{\mathrm{d}x}|^2 \mathrm{d}x$$

subject (for instance) to boundary conditions $u(\pm 1) = \pm 1$. Choosing

$$u_\varepsilon(x) = \frac{\arctan(\frac{x}{\varepsilon})}{\arctan(\frac{1}{\varepsilon})}, \ \varepsilon > 0,$$

Weierstrass was able to show that the infimum of I in the above class was 0 (Exercise); however, the value 0 is not attained, since this means that $u \equiv 0$ on $[-1, 1]$, which contradicts the boundary condition.

In 1900, Hilbert, in his speech at the International Congress in Paris, proposed his famous 23 problems - three of which devoted to questions related to the calculus of variations.

In 1970s, Ekeland discovered the so-called Ekeland Variational Principle which implies the existence of the minimizing sequence with some fine properties; In 1960s, Palais and Smale (cf. [71, 82]) introduced the concept of the so-called Palais-Smale condition (Abbr. (PS) condition), which will also guarantee the convergence not only for a minimizing sequence, but also for (PS) sequence. We shall show that if the (PS) condition holds and the corresponding function E is bounded below, then E attains a minimum, which gives the desired answer! Otherwise, the problems are very difficult. It so happens that many interesting problems from geometric problems and physical problems do not satisfy (PS) condition, but E has a minimum. (cf. Chapter 4 for details.)

Question 3: Does there exist any other critical point rather than the minimizer?

In 1929, Ljusternik and Schnirelman [60] obtained the existence of three distinct closed geodesics on any compact surface of genus zero. From then on, we no longer consider only minimizer (or maximizer) of variational integrals, but instead look at all their critical points — the calculus of variations in the large or global variational method. In 1930s, they [61] developed the so-called Ljusternik-Schnirelman theory.

The work of Ljusternik-Schnirelman revealed that much of the complexity of a function space is invariably reflected in the set of critical points of any variational integral defined on it.

In 1934, M. Morse [65] developed another approach towards a global theory of critical points. Morse's work also revealed a deep relation between the topology of a space and the number and types of critical points of any function defined on it.

In this part, we will study the Minimax Method, including the Mountain Pass Lemma, the Z_2 index theory, the Linking Argument and its applications in elliptic equations.

Usually, the Minimax Method consists of three steps:

(1) A-priori compactness condition, such as $(\mathrm{PS})_c$ condition;

(2) Deformation Lemma depending on this condition;

(3) Construction of a critical value.

1.2 Classical solution and generalized solution

Let us start from an example of the boundary value problem of linear elliptic equation:

$$\begin{cases} -\Delta u = f(x),\ x \in \Omega, \\ u = 0,\ x \in \partial\Omega, \end{cases} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded domain with smooth boundary, e.g, $\partial\Omega \in C^1$, $f \in C_0(\Omega)$.

$u(x) : \Omega \to \mathbb{R}$ is a classical solution of (1.2.1) if $u \in C(\bar{\Omega}) \cap C_0^2(\Omega)$ and satisfies problem (1.2.1). How about the existence of classical solutions of (1.2.1)? Since a classical solution is required to be second order continuously differentiable, the space of admissible functions is too small. In most cases, it is very difficult to find a classical solution.

In order to enlarge the space of admissible functions, let us multiply the equation in (1.2.1) by a smooth test function $v \in C_0^\infty(\Omega)$, and integrate over Ω by parts, to find

$$\int_\Omega \mathrm{D}u \cdot \mathrm{D}v \, \mathrm{d}x = \int_\Omega f(x) v \, \mathrm{d}x. \tag{1.2.2}$$

It is easy to verify that equality (1.2.2) is true for all $v \in C_0^\infty(\Omega)$ if u is a classical solution of (1.2.1); reversely, in order to make the integration of the left-hand side of (1.2.2) meaningful, one can just require that: for example, $u \in C(\bar{\Omega}) \cap C_0^1(\Omega)$ or even the Sobolev space $H_0^1(\Omega)$.

A generalized solution of (1.2.1) is a function u in some space of admissil functions, which makes the equality of (1.2.2) be true for all $v \in C_0^\infty(\Omega)$.

Next, consider an extremal problem of functional: For a given function $f \in C_0(\Omega)$, define functional $I: C_0^1(\Omega) \to \mathbb{R}$:

$$I(v) = \int_\Omega (\frac{1}{2}|\mathrm{D}v|^2 - fv)\,\mathrm{d}x. \tag{1.2.3}$$

The extremal problem (variational problem) is to find a minimizer $u \in C_0^1(\Omega)$ such that

$$\min_{v \in C_0^1(\Omega)} I(v) = I(u). \tag{1.2.4}$$

As we are familiar with the extremal problem of functions with single or several variables, we want to solve the extremal problem of functionals, using the results or the methods of that of functions. There are two ways: One is to define a function from the functional, and then solve the extremal problem of that function. We will show the idea below; The other is to use the methods or the ideas which are used to deal with the extremal problem of functions, such as derivatives, critical points and so on, we will make this clear in the following chapters.

Now, let us show the idea of the first way. Let $\varphi \in C_0^\infty(\Omega)$, $t \in \mathbb{R}$, since $u \in C_0^1(\Omega)$, $u + t\varphi \in C_0^1(\Omega)$, define a function with single variable $J: \mathbb{R} \to \mathbb{R}$:

$$J(t) = I(u + t\varphi) = \frac{t^2}{2}\int_\Omega |\mathrm{D}\varphi|^2\,\mathrm{d}x + t\int_\Omega (\mathrm{D}u \cdot \mathrm{D}\varphi - f\varphi)\,\mathrm{d}x + I(u). \tag{1.2.5}$$

If u is solution of the extremal problem (1.2.4), the function $J(t) \in C^\infty(\mathbb{R})$ attains its minimum at $t = 0$. From the theory of Calculus, the necessary condition under which the function $J(t)$ attains its minimum at $t = 0$ is

$$\frac{\mathrm{d}J(t)}{\mathrm{d}t}\Big|_{t=0} = 0. \tag{1.2.6}$$

Simple calculation shows

$$0 = \frac{\mathrm{d}J(t)}{\mathrm{d}t}\Big|_{t=0} = \int_\Omega (\mathrm{D}u \cdot \mathrm{D}\varphi - f\varphi)\,\mathrm{d}x,$$

which is equivalent to the equality (1.2.2).

1.3 First variation, Euler-Lagrange equation

In this section, we first state the fundamental theorem of calculus of variations, and then formally derive the Euler-Lagrange equation, finally illustrate the process by some concrete examples.

Let's recall the fundamental theorem of calculus of variations as follows.

Theorem 1.3.1 (Fundamental Theorem of Calculus of Variations) *If a function $u \in C(\Omega)$ satisfies*

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 0, \quad \forall\, \varphi \in C_0^\infty(\Omega),$$

then $u \equiv 0$ in Ω.

Proof. By contradiction, assume that there exists a point $x_0 \in \Omega$ such that $u(x_0) \neq 0$, say $u(x_0) > 0$. From the continuity of function u, there exists an open neighborhood $B_r(x_0) \subset \Omega$ of x_0, $r > 0$, such that $u(x) > u(x_0)/2 > 0$ for all $x \in B_r(x_0)$.

Choose the test function φ as:

$$\varphi(x) = \begin{cases} \exp\left\{-\frac{|x-x_0|^2}{r^2-|x-x_0|^2}\right\}, & \text{if } x \in B_r(x_0); \\ 0, & \text{if } x \in \Omega \setminus B_r(x_0). \end{cases}$$

It is easy to show that $\varphi \in C_0^\infty(\Omega)$, $\varphi(x) > 0$, $\forall\, x \in B_r(x_0)$. Thus, it follows that

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = \int_{B_r(x_0)} u(x)\varphi(x)\,\mathrm{d}x > \frac{u(x_0)}{2}\int_{B_r(x_0)} \varphi(x)\,\mathrm{d}x > 0,$$

which contradicts to the assumption, and implies the desired result. ■

Let $\Omega \subset \mathbb{R}^n$ is bounded domain with smooth boundary $\partial\Omega$, and we are given a smooth function

$$L:\ \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \to \mathbb{R}.$$

We call $L = L(p, z, x) = L(p_1, \cdots, p_n, z, x_1, \cdots, x_n)$ the Lagrangian. Define the energy functional $I:\ C_0^1(\Omega) \to \mathbb{R}$ by

$$I[v] = \int_\Omega L(\mathrm{D}v(x), v(x), x)\,\mathrm{d}x. \tag{1.3.1}$$

Suppose some $u \in C_0^1(\Omega)$ happens to be a minimizer of $I[\cdot]$. We explicitly compute the first variation $\delta I(u)$ of $I[\cdot]$ at u. For any function $v \in C_0^\infty(\Omega)$,