序

《电动力学》课程是物理学专业重要的理论必修课，是物理学的四大力学之一，其与量子力学统属物理专业最重要的理论基础课程；也是目前高等学校理、工科和师范院校本科物理类专业的一门重要基础理论课。由于该课程涉及的数学知识较多，例如，复变函数、格林函数、特殊函数、数理方程、矩阵、矢量分析和张量分析。这给学习《电动力学》的学生带来了一定的难度，为给学习《电动力学》的学生提供一定的帮助，作者将自己80年代初以来收集到的270题进行了较为详细解答和整理，历经数年才得以完成此书.其中包含了郭硕鸿先生著《电动力学》第二版几乎所有习题的解答，还有不少习题是研究生的入学考试试题。

由于作者多年来一直使用郭硕鸿先生著《电动力学》第二版作为参考教材，所以，本书中将该参考教材称为"教材"。出版本书旨在对学习该门课程的学生有所帮助，且对他们考研究生也有所作用，同时也希望为初讲授《电动力学》这门课程的年轻教师提供一定的教学参考。本书后面附有数学附录和部分基本物理常数，列出了所需的数学知识和计算所需的一些基本物理常数值。

由于作者学识水平有限，错误在所难免，敬请读者批评指正。

作　者

2007 年 10 月 2 日

目　录

第一章 电磁现象的普遍规律

1.1. 证明下列恒等式

(1) $(\vec{a}\cdot\vec{b})^2+(\vec{a}\times\vec{b})^2=a^2b^2$;

(2) $(\vec{a}+\vec{b})\cdot[(\vec{a}+\vec{c})\times\vec{b}]=-\vec{a}\cdot(\vec{b}\times\vec{c})$;

(3) $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$;

(4) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{c}[\vec{d}\cdot(\vec{a}\times\vec{b})]-\vec{d}[\vec{c}\cdot(\vec{a}\times\vec{b})]$
$=\vec{b}[\vec{a}\cdot(\vec{c}\times\vec{d})]-\vec{a}[\vec{b}\cdot(\vec{c}\times\vec{d})]$.

【证明】Ⅰ. 方程的左边为

$$(\vec{a}\cdot\vec{b})^2+(\vec{a}\times\vec{b})^2=(ab\cos\theta)^2+(ab\sin\theta)^2=a^2b^2(\cos^2\theta+\sin^2\theta)=a^2b^2$$

方程左边等于方程右边,证毕.

Ⅱ. 方程的左边为

$$\begin{aligned}&(\vec{a}+\vec{b})\cdot[(\vec{a}+\vec{c})\times\vec{b}]\\&=(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{b})+(\vec{a}+\vec{b})\cdot(\vec{c}\times\vec{b})\\&=\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{a}\times\vec{b})+a\cdot(\vec{c}\times\vec{b})+\vec{b}\cdot(\vec{c}\times\vec{b}),\end{aligned}$$

式中

$$\vec{a}\cdot(\vec{a}\times\vec{b})=0,\vec{b}\cdot(\vec{a}\times\vec{b})=0,\vec{b}\cdot(\vec{c}\times\vec{b})=0,$$

所以

$$(\vec{a}+\vec{b})\cdot[(\vec{a}+\vec{c})\times\vec{b}]=\vec{a}\cdot(\vec{c}\times\vec{b})=-\vec{a}\cdot(\vec{b}\times\vec{c}).$$

方程左边等于方程右边,证毕.

Ⅲ. 方程的左边为

$$\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})\\&=[\vec{b}(\vec{a}\cdot\vec{c})+\vec{b}(\vec{c}\cdot\vec{a})-\vec{c}(\vec{a}\cdot\vec{b})-\vec{c}(\vec{b}\cdot\vec{a})]\\&\quad+[\vec{c}(\vec{b}\cdot\vec{a})+\vec{c}(\vec{a}\cdot\vec{b})-\vec{a}(\vec{b}\cdot\vec{c})-\vec{a}(\vec{c}\cdot\vec{b})]\\&\quad+[\vec{a}\cdot(\vec{c}\cdot\vec{b})+\vec{a}(\vec{b}\cdot\vec{c})-\vec{b}(\vec{c}\cdot\vec{a})-\vec{b}(\vec{a}\cdot\vec{c})]=0,\end{aligned}$$

方程左边等于方程右边,证毕.

Ⅳ. 把第一个括弧看成一矢量,根据矢量代数公式[数学附录(1.17)],方程的左边为

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{c}[\vec{d}\cdot(\vec{a}\times\vec{b})]-\vec{d}[\vec{c}\cdot(\vec{a}\times\vec{b})],$$

把第二个括弧看成一矢量,同理有

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{b}[\vec{a}\cdot(\vec{c}\times\vec{d})]-\vec{a}[\vec{b}\cdot(\vec{c}\times\vec{d})],$$

方程左边等于方程右边,证毕.

1.2. 根据算符∇的微分性与矢量性证明下列公式:

(1) $\nabla(\vec{A}\cdot\vec{B})=\vec{B}\times(\nabla\times\vec{A})+(\vec{B}\cdot\nabla)\vec{A}+\vec{A}\times(\nabla\times\vec{B})+(\vec{A}\cdot\nabla)\vec{B}$;

(2) $\vec{A}\times(\nabla\times\vec{A})=\dfrac{1}{2}\nabla A^2-(\vec{A}\cdot\nabla)\vec{A}$.

【证明】Ⅰ. 方程的左边为

$$\nabla(\vec{A}\cdot\vec{B}) = \nabla_A(\vec{A}\cdot\vec{B}) + \nabla_B(\vec{A}\cdot\vec{B}) \tag{1}$$

根据公式 $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{c}\cdot\vec{a}) - \vec{c}(\vec{a}\cdot\vec{b})$,

即

$$\vec{b}(\vec{c}\cdot\vec{a}) = \vec{a}\times(\vec{b}\times\vec{c}) + \vec{c}(\vec{a}\cdot\vec{b}),\text{注意 } \nabla_A\to\vec{b}, \vec{A}\to\vec{c}, \vec{B}\to\vec{a},$$

则有

$$\nabla_A(\vec{A}\cdot\vec{B}) = \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}. \tag{2}$$

同理,

$$\nabla_B(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B}. \tag{3}$$

将(2)式和(3)式代入(1)式得

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\times\nabla)\vec{B}. \tag{4}$$

方程的左边等于方程右边,证毕.

Ⅱ. 在(4)式中令 $\vec{A}=\vec{B}$,则有

$$\begin{aligned}\nabla A^2 &= \vec{A}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{A}\\ &= 2\vec{A}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{A},\end{aligned} \tag{5}$$

即

$$\vec{A}\times(\nabla\times\vec{A}) = \frac{1}{2}\nabla A^2 - (\vec{A}\cdot\nabla)\vec{A}. \tag{6}$$

证毕.

1.3. 设 u 是空间坐标的函数,证明:

$$(1)\ \nabla f(u) = \frac{df}{du}\nabla u;\quad (2)\ \nabla\cdot\vec{A}(u) = \nabla u\cdot\frac{d\vec{A}}{du};\quad (3)\ \nabla\times\vec{A}(u) = \nabla u\times\frac{d\vec{A}}{du}.$$

【证明】Ⅰ. 方程的左边为

$$\begin{aligned}\nabla f(u) &= \frac{\partial f(u)}{\partial x}\vec{e}_x + \frac{\partial f(u)}{\partial y}\vec{e}_y + \frac{\partial f(u)}{\partial z}\vec{e}_z = \frac{df}{du}\frac{du}{dx}\vec{e}_x + \frac{df}{du}\frac{du}{dy}\vec{e}_y + \frac{df}{du}\frac{du}{dz}\vec{e}_z\\ &= \frac{df}{du}\left(\frac{du}{dx}\vec{e}_x + \frac{du}{dy}\vec{e}_y + \frac{du}{dz}\vec{e}_z\right) = \frac{df}{du}\nabla u,\end{aligned}$$

方程的左边等于方程右边,证毕.

Ⅱ. 方程的左边为

$$\begin{aligned}\nabla\cdot\vec{A}(u) &= \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial u}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z\right)\cdot[A_x(u)\vec{e}_x + A_y(u)\vec{e}_y + A_z(u)\vec{e}_z]\\ &= \frac{\partial A_x(u)}{\partial x} + \frac{\partial A_y(u)}{\partial y} + \frac{\partial A_z(u)}{\partial z} = \frac{dA_x(u)}{du}\frac{du}{dx} + \frac{dA_y(u)}{du}\frac{du}{dy} + \frac{dA_z(u)}{du}\frac{du}{dz}\\ &= \nabla u\cdot\frac{d\vec{A}(u)}{du},\end{aligned}$$

方程的左边等于方程右边,证毕.

Ⅲ. 方程的左边为

$$\nabla\times\vec{A}(u)=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x(u) & A_y(u) & A_z(u)\end{vmatrix}$$

$$=\left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]\vec{e}_x+\left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right]\vec{e}_y+\left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right]\vec{e}_z$$

$$=\left[\frac{dA_z}{du}\frac{du}{dy}-\frac{dA_y}{du}\frac{du}{dz}\right]\vec{e}_x+\left[\frac{dA_x}{du}\frac{du}{dz}-\frac{dA_z}{du}\frac{du}{dx}\right]\vec{e}_y+\left[\frac{dA_y}{du}\frac{du}{dx}-\frac{dA_x}{du}\frac{du}{dy}\right]\vec{e}_z.$$

方程的右边为

$$\nabla u\times\frac{d\vec{A}}{du}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz}\\ \frac{dA_x}{du} & \frac{dA_y}{du} & \frac{dA_z}{du}\end{vmatrix}$$

$$=\left[\frac{dA_z}{du}\frac{du}{dy}-\frac{dA_y}{du}\frac{du}{dz}\right]\vec{e}_x+\left[\frac{dA_x}{du}\frac{du}{dz}-\frac{dA_z}{du}\frac{du}{dx}\right]\vec{e}_y+\left[\frac{dA_y}{du}\frac{du}{dx}-\frac{dA_x}{du}\frac{du}{dy}\right]\vec{e}_z.$$

方程的左边等于方程右边，证毕.

1.4. 设 $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ 为源点 $\vec{x}'$ 到场点 $\vec{x}$ 的距离，$\vec{r}$ 的方向规定为源点指向场点.

(1) 证明下列结果，并体会对源变数求微商（$\nabla'=\vec{e}_x\frac{\partial}{\partial x'}+\vec{e}_y\frac{\partial}{\partial y'}+\vec{e}_z\frac{\partial}{\partial z'}$）与对场变数求微商（$\nabla=\vec{e}_x\frac{\partial}{\partial x}+\vec{e}_y\frac{\partial}{\partial y}+\vec{e}_z\frac{\partial}{\partial z}$）的关系：

$$\nabla r=-\nabla' r=\frac{\vec{r}}{r},\ \nabla\frac{1}{r}=-\nabla'\frac{1}{r}=-\frac{\vec{r}}{r^3},\ \nabla\times\frac{\vec{r}}{r^3}=0,\ \nabla\cdot\frac{\vec{r}}{r^3}=-\nabla'\cdot\frac{\vec{r}}{r^3}=0,\quad(r\neq0)$$

（最后一式在 $r=0$ 点不成立）.

(2) 求 ∇r, $\nabla\cdot\vec{r}$, $\nabla\times\vec{r}$, $(\vec{a}\cdot\nabla)\vec{r}$, $\nabla(\vec{a}\cdot\vec{r})$, $\nabla\cdot[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]$ 及 $\nabla\times[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]$，其中 $\vec{a}$、$\vec{k}$ 及 $\vec{E}_0$ 均为常矢量.

【解】 Ⅰ. 证明如下：

(1) 设 $\vec{r}=(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z$,

则

$$r=[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}},$$

所以有

$$\nabla r=\frac{\partial r}{\partial x}\vec{e}_x+\frac{\partial r}{\partial y}\vec{e}_y+\frac{\partial r}{\partial z}\vec{e}_z=\frac{(x-x')}{r}\vec{e}_x+\frac{(y-y')}{r}\vec{e}_y+\frac{(z-z')}{r}\vec{e}_z=\frac{\vec{r}}{r}.$$

而

$$\nabla' r=\frac{\partial r}{\partial x'}\vec{e}_x+\frac{\partial r}{\partial y'}\vec{e}_y+\frac{\partial r}{\partial z'}\vec{e}_z=-\frac{(x-x')}{r}\vec{e}_x-\frac{(y-y')}{r}\vec{e}_y-\frac{(z-z')}{r}\vec{e}_z=-\frac{\vec{r}}{r},$$

所以有

$$\nabla r=-\nabla' r,$$

(2)同理，

$$\nabla\frac{1}{r}=-\frac{1}{2}\frac{1}{r^3}[2(x-x')\vec{e}_x+2(y-y')\vec{e}_y+2(z-z')\vec{e}_z]=-\frac{\vec{r}}{r^3},$$

而

$$\nabla'\frac{1}{r}=-\frac{1}{2}\frac{1}{r^3}[-2(x-x')\vec{e}_x-2(y-y')\vec{e}_y-2(z-z')\vec{e}_z]=\frac{\vec{r}}{r^3},$$

所以有

$$\nabla\frac{1}{r}=-\nabla'\frac{1}{r}=-\frac{\vec{r}}{r^3}.$$

(3)同理，

$$\nabla\times\frac{\vec{r}}{r^3}=\left(\nabla\frac{1}{r^3}\right)\times\vec{r}+\frac{1}{r^3}\nabla\times\vec{r}=-\frac{3\vec{r}}{r^5}\times\vec{r}+\frac{1}{r^3}\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (x-x') & (y-y') & (z-z')\end{vmatrix}=0+0=0,$$

而

$$\nabla'\times\frac{\vec{r}}{r^3}=\left(\nabla'\frac{1}{r^3}\right)\times\vec{r}+\frac{1}{r^3}\nabla'\times\vec{r}=\frac{3\vec{r}}{r^5}\times\vec{r}+\frac{1}{r^3}\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'}\\ (x-x') & (y-y') & (z-z')\end{vmatrix}=0+0=0,$$

所以有

$$\nabla\times\frac{\vec{r}}{r^3}=-\nabla'\times\frac{\vec{r}}{r^3}=0.$$

(4)同理，

$$\nabla\cdot\frac{\vec{r}}{r^3}=\left(\nabla\frac{1}{r^3}\right)\cdot\vec{r}+\frac{1}{r^3}\nabla\cdot\vec{r}=-\frac{3\vec{r}}{r^5}\cdot\vec{r}+\frac{3}{r^3}=0,$$

而

$$\nabla'\cdot\frac{\vec{r}}{r^3}=\left(\nabla'\frac{1}{r^3}\right)\cdot\vec{r}+\frac{1}{r^3}\nabla'\cdot\vec{r}=\frac{3\vec{r}}{r^5}\cdot\vec{r}-\frac{3}{r^3}=0,$$

所以有

$$\nabla\cdot\frac{\vec{r}}{r^3}=-\nabla\cdot\frac{\vec{r}}{r^3}=0.\quad(r\neq0)$$

证毕.从以上证明看出，矢量微分算符 ∇是对场变数求微分，即对场点坐标(z,y,z)进行微分作用，而∇'是对源变数求微分，即对源点坐标(x',y',z')进行微分作用，两种作用相差一负号.

Ⅱ.计算如下：

(1)设 $\vec{r}=(x-x_0)\vec{e}_x+(y-y_0)\vec{e}_y+(z-z_0)\vec{e}_z$,

则

$$r=[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{\frac{1}{2}},$$

所以有

$$\nabla r=\frac{\partial r}{\partial x}\vec{e}_x+\frac{\partial r}{\partial y}\vec{e}_y+\frac{\partial r}{\partial z}\vec{e}_z=\frac{(x-x_0)}{r}\vec{e}_x+\frac{(y-y_0)}{r}\vec{e}_y+\frac{(z-z_0)}{r}\vec{e}_z=\frac{\vec{r}}{r}.$$

(2)同理,

$$\nabla\cdot\vec{r}=\frac{\partial}{\partial x}(x-x_0)+\frac{\partial}{\partial y}(y-y_0)+\frac{\partial}{\partial z}(z-z_0)=1+1+1=3.$$

(3)同理,

$$\nabla\times\vec{r}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ (x-x_0) & (y-y_0) & (z-z_0)\end{vmatrix}$$

$$=\left[\frac{\partial(z-z_0)}{\partial y}-\frac{\partial(y-y_0)}{\partial z}\right]\vec{e}_x+\left[\frac{\partial(x-x_0)}{\partial z}-\frac{\partial(z-z_0)}{\partial x}\right]\vec{e}_y$$

$$+\left[\frac{\partial(y-y_0)}{\partial x}-\frac{\partial(z-z_0)}{\partial y}\right]\vec{e}_z=0.$$

(4) $(\vec{a}\cdot\nabla)\vec{r}=(a_x\nabla_x+a_y\nabla_y+a_z\nabla_z)(x\vec{e}_x+y\vec{e}_y+z\vec{e}_z)=a_x\vec{e}_x+a_y\vec{e}_y+a_z\vec{e}_z=\vec{a}$,

或

$$(\vec{a}\cdot\nabla)\vec{r}=\vec{a}\cdot\nabla\vec{r}=\vec{a}\cdot\vec{\vec{I}}=\vec{a}.$$

(5) $\nabla(\vec{a}\cdot\vec{r})=(\nabla_x+\nabla_y+\nabla_z)(a_xx+a_yy+a_zz)$

$$=a_x\vec{e}_x+a_y\vec{e}_y+a_z\vec{e}_z=\vec{a}.$$

(6) $\nabla\cdot[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]=(\nabla_x+\nabla_y+\nabla_z)\cdot[(E_{0x}\vec{e}_x+E_{0y}\vec{e}_y+E_{0z}\vec{e}_z)\sin(k_xx+k_yy+k_zz)]$

$$=(\nabla_xE_{0x}+\nabla_yE_{0y}+\nabla_zE_{0z})\sin(k_xx+k_yy+k_zz)$$

$$=(E_{0x}k_x+E_{0y}k_y+E_{0z}k_z)\cos(k_xx+k_yy+k_zz)$$

$$=(\vec{E}_0\cdot\vec{k})\cos(\vec{k}\cdot\vec{r}),$$

或

$$\nabla\cdot[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]=(\nabla\cdot\vec{E}_0)\sin(\vec{k}\cdot\vec{r})+[\nabla\sin(\vec{k}\cdot\vec{r})]\cdot\vec{E}_0$$

$$=0+\cos(\vec{k}\cdot\vec{r})\ \nabla(\vec{k}\cdot\vec{r})\cdot\vec{E}_0$$

$$=\cos(\vec{k}\cdot\vec{r})\ \nabla(xk_x+yk_y+zk_z)\cdot\vec{E}_0$$

$$=\cos(\vec{k}\cdot\vec{r})(k_x\vec{e}_x+k_y\vec{e}_y+k_z\vec{e}_z)\cdot\vec{E}_0$$

$$=(\vec{E}_0\cdot\vec{k})\cos(\vec{k}\cdot\vec{r}).$$

(7) $\nabla\times[\vec{E}_0\sin(\vec{k}\cdot\vec{r})]=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \nabla_x & \nabla_y & \nabla_z\\ E_{0x}\sin(\vec{k}\cdot\vec{r}) & E_{0y}\sin(\vec{k}\cdot\vec{r}) & E_{0z}\sin(\vec{k}\cdot\vec{r})\end{vmatrix}$

$$=[(\nabla_yE_{0z}-\nabla_zE_{0y})\sin(\vec{k}\cdot\vec{r})]\vec{e}_x+[(\nabla_zE_{0x}-\nabla_xE_{0z})\sin(\vec{k}\cdot\vec{r})]\vec{e}_y$$

$$+[(\nabla_xE_{0y}-\nabla_yE_{0x})\sin(\vec{k}\cdot\vec{r})]\vec{e}_z$$

$$=(E_{0z}k_y-E_{0y}k_z)\cos(\vec{k}\cdot\vec{r})\vec{e}_x+(E_{0x}k_z-E_{0z}k_x)\cos(\vec{k}\cdot\vec{r})\vec{e}_x$$

$$+(E_{0y}k_x-E_{0x}k_y)\cos(\vec{k}\cdot\vec{r})\vec{e}_x$$

$$=-(\vec{E}_0\times\vec{k})\cos(\vec{k}\cdot\vec{r})$$

$$=\vec{k}\times\vec{E}_0\cos(\vec{k}\cdot\vec{r}).$$

1.5. **计算下列各式：**

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right),\ \nabla\times(r^n\vec{r}),\ \nabla\cdot(r^n\vec{r}).$$

【解】(1)设 $\vec{r}=(x-x_0)\vec{e}_x+(y-y_0)\vec{e}_y+(z-z_0)\vec{e}_z$,则有

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right)=\nabla_{\vec{r}}\cdot\left(\frac{\vec{r}}{r}\right)+\nabla_r\cdot\left(\frac{\vec{r}}{r}\right)=\frac{1}{r}\nabla\cdot\vec{r}+\left(\nabla\frac{1}{r}\right)\cdot\vec{r}$$

$$=3\frac{1}{r}+\left(-\frac{\vec{r}}{r^3}\cdot\vec{r}\right)=3\frac{1}{r}-\frac{1}{r}=2\frac{1}{r}.$$

(2)同理,设 $\vec{r}=(x-x_0)\vec{e}_x+(y-y_0)\vec{e}_y+(z-z_0)\vec{e}_z$,

而

$$r^n=[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{\frac{n}{2}},$$

则有

$$\nabla\times(r^n\vec{r})=\nabla_{\vec{r}}\times(r^n\vec{r})+\nabla_{r^n}\times(r^n\vec{r})=r^n\nabla_{\vec{r}}\times\vec{r}+(\nabla_{r^n}r^n)\times\vec{r}=(\nabla_{r^n}r^n)\times\vec{r}$$

$$=\frac{n}{2}[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{\frac{n}{2}-1}2[(x-x_0)\vec{e}_x+(y-y_0)\vec{e}_y+(z-z_0)\vec{e}_z]\times\vec{r}$$

$$=nr^n\frac{\vec{r}}{r^2}\times\vec{r}=0.$$

(3)同理,设 $\vec{r}=(x-x_0)\vec{e}_x+(y-y_0)\vec{e}_y+(z-z_0)\vec{e}_z$,

而

$$r^n=[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{\frac{n}{2}},$$

则有

$$\nabla\cdot(r^n\vec{r})=\nabla_{\vec{r}}\cdot(r^n\vec{r})+\nabla_{r^n}\cdot(r^n\vec{r})=r^n(\nabla_{\vec{r}}\cdot\vec{r})+(\nabla_{r^n}r^n)\cdot\vec{r}$$

$$=3r^n+nr^n=(3+n)r^n.$$

1.6. **证明下式：**

$$\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)=-\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right).$$

【证明】方程左边为

$$\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)=\nabla_{r^3}\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)+\nabla_{\vec{a}}\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)+\nabla_{\vec{r}}\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)$$

$$=\nabla_{r^3}\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)+\nabla_{\vec{r}}\left(\frac{\vec{a}\cdot\vec{r}}{r^3}\right)$$

$$=\left(\nabla_r\frac{1}{r^3}\right)(\vec{a}\cdot\vec{r})+\left(\frac{1}{r^3}\nabla_{\vec{r}}\right)(\vec{a}\cdot\vec{r})$$

$$=\vec{a}\times\left(\nabla_{r^3}\frac{1}{r^3}\times\vec{r}\right)+\left(\vec{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\vec{r}+\vec{a}\times\left(\frac{1}{r^3}\nabla_{\vec{r}}\times\vec{r}\right)+\left(\vec{a}\cdot\frac{1}{r^3}\nabla_{\vec{r}}\right)\vec{r}$$

$$=0+\left(\vec{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\vec{r}+0+\left(\vec{a}\cdot\frac{1}{r^3}\nabla_{\vec{r}}\right)\vec{r}$$

$$=\left(\vec{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\vec{r}+\left(\vec{a}\cdot\frac{1}{r^3}\nabla_{\vec{r}}\right)\vec{r}.$$

方程右边为

$$-\nabla\times\left(\frac{\vec a\times\vec r}{r^3}\right)=-\nabla_{r^3}\times\left(\frac{\vec a\times\vec r}{r^3}\right)-\nabla_{\vec r}\times\left(\frac{\vec a\times\vec r}{r^3}\right)$$

$$=-\nabla_{r^3}\frac{1}{r^3}\times(\vec a\times\vec r)-\frac{1}{r^3}\nabla_{\vec r}\times(\vec a\times\vec r)$$

$$=-\left[\left(\nabla_{r^3}\frac{1}{r^3}\cdot\vec r\right)\vec a-\left(\nabla_{r^3}\frac{1}{r^3}\cdot\vec a\right)\vec r\right]-\left[\left(\frac{1}{r^3}\nabla_{\vec r}\cdot\vec r\right)\vec a-\left(\frac{1}{r^3}\nabla_{\vec r}\cdot a\right)\vec r\right]$$

$$=-\left(-3\frac{\vec r}{r^5}\cdot\vec r\right)\vec a+\left(\vec a\cdot\ \nabla_{r^3}\frac{1}{r^3}\right)\vec r-3\frac{1}{r^3}\vec a+\left(\vec a\cdot\frac{1}{r^3}\nabla_{\vec r}\right)\vec r$$

$$=\left(\vec a\cdot\ \nabla_{r^3}\frac{1}{r^3}\right)\vec r+\left(\vec a\cdot\frac{1}{r^3}\nabla_{\vec r}\right)\vec r.$$

方程左边等于右边,证毕.

1.7. 设 $\vec c$ 为恒矢量,$\vec r$ 为位置矢量,计算下列各式:

(1) $\nabla\times(\vec c\times\vec r)$;(2) ∇r^2;(3) $\nabla^2\left(\frac{1}{r}\right)(r\neq0)$.

【解】(1)将 $\vec c$ 和 $\vec r$ 写为分量式

$$\vec c=c_x\vec e_x+c_y\vec e_y+c_z\vec e_z,$$
$$\vec r=(x-x_0)\vec e_x+(y-y_0)\vec e y+(z-z_0)\vec e_z,$$

所以

$$\vec c\times\vec r=\begin{vmatrix}\vec e_x & \vec e_y & \vec e_z\\ c_x & c_y & c_z\\ x & y & z\end{vmatrix}$$

$$=(c_yz-c_zy)\vec e_x+(c_zx-c_xz)\vec e_y+(c_xy-c_yx)\vec e_z.$$

$$\nabla\times(\vec c\times\vec r)=\begin{vmatrix}\vec e_x & \vec e_y & \vec e_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (c_yz-c_zy) & (c_zx-c_xz) & (c_xy-c_yx)\end{vmatrix}$$

$$=2(c_x\vec e_x+c_y\vec e_y+c_z\vec e_z)=2\vec c.$$

(2)同理,

$$\nabla r^2=\frac{d}{dr}r^2=2r\vec e_r=2\vec r.$$

(3)同理,

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\ \nabla\frac{1}{r}=\nabla\left(-\frac{\vec r}{r^3}\right)=-\frac{1}{r^3}\nabla\cdot\vec r-\nabla\left(\frac{1}{r^3}\right)\cdot\vec r=-\frac{3}{r}+3\frac{\vec r}{r^r}\cdot\vec r=0.$$

1.8. 直接由 $\vec E(\vec x)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec x')\vec r}{r^3}dV'$ 证明:

(1) $\nabla\cdot\vec E(\vec x)=\frac{1}{\varepsilon_0}\rho(\vec x)$;