刘彦佩

半闲数学集锦

Semi-Empty Collections in Mathemetics by Y.P.Liu

第六编

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第六编序

本编包含文 6.01[083]—6.15[166], 专著纵横嵌入术[135](6.16–6.27) 和专著纵横布局论—兼谈 VLSI 的布局[154](6.28—6.37).

文 6.01[083] 是在文 3.04[069]—3.05[096] 的基础上, 用与 2.09[137], 3.19[141], 以及 5.11[438] 等都不同的方法, 第一次通过寻找, 对于平面嵌入, 有向禁用构形的一个完备集, 以更有效地, 确定图的一个平面嵌入. 这里所提供的完备集只含 7 个有向构形. 虽然可以达到线性性, 但远非最简单. 而且, 文中有多处, 需要进一步澄清. 在后三者中, 都改善到只用 5 个有向禁用构形, 或者称平面嵌入障碍.

文 6.02[100]—6.05[121], 6-11[139], 6.13[145]—6.15[166], 都从我在意大利访问期间, 所构建起来框架, 逐步完备而成的.

原本打算, 确定一个图, 有每边只有一折, 平面嵌入的条件. 通过找禁用次形的完备集, 发现这些次形, 随着图的阶增大, 而增大. 这就预示此路不通. 我决定改变问题的提法.

对于任何一个整数, $k \geq 0$, 我引入至多 k 折嵌入, 简称 k-嵌入, 使得 $k = 1$ 时, 就是最接近所需要的.

如果一个 k-嵌入不是任何 l-嵌入, 其中整数 $l \leq k-1$, 这个 k-嵌入被称为称为恰 k-嵌入. 只有 0-嵌入是恰 0-嵌入, 也称为网格嵌入.

对于任何一个图类, 我们所感兴趣的是这个 k 的最小值. 首先, 证明了对于任何图, 这个 $k = 3$. 由此, 就要问对于什么样的图, $k = 2$, $k = 1$ 和 $k = 0$.

文 6.02[100]—6.06[121] 在于解决 $k = 2$ 和 $k = 1$ 这两种情形. 文 6.07[127]—6.09[132] 专门讨论网格可嵌入性的问题. 文 6.11[139], 6.13[145] 和 6.15[166] 都是讨论在计算机上的实现.

文 6.12[142], 6.14[150] 和 6.10[134], 则提供了在普遍性理论基础上的总结. 为专著[135]纵横嵌入术和专著[154] 纵横布局论中, 理论系统的形成, 提供了必要的准备.

在专著[135]纵横嵌入术中, 全书的理论主线, 无论极大极小问题, 还是总体最优问题, 都是通过确定禁用构形, 构造性地解决的. 这里的构形都不是次形. 因为组成次形的要素是边, 而这里构形的要素是半边, 特别是一个圈, 或者上圈, 甚至一个子图, 伴随所有关联半边. 它们在哲学上的意义, 与次形一样, 也是通过局部反映全局.

除 6.16—6.17, 和 6.27 是为方便阅读外, 6.18 规范化要讨论的问题, 6.19 要用到的一般嵌入理论基础, 6.20 引入纵横扩张, 纵横嵌入, 和纵横实现三个层次, 以便层层深入, 由相对容易到相对困难.

通过 6.21—6.22, 还顺便解决了, 在意大利, 困扰我和那里同事, 相当一段时日, 1-嵌入存在性的判别问题. 否定了禁用次形研究路线, 建立了禁用构形

的研究路线. 这里的构形都是特殊结构的分离三角形.

在 6.23–6.24 中, 对于全局最优性, 禁用构形不用特殊结构的圈, 就用特殊结构的上圈, 进行判别.

余下的 6.25—6.26, 则是关于这一理论的应用, 以及为便于利用这一理论, 通常需要的预处理.

在专著[154]纵横布局论中, 书的理论主线, 无论极大极小问题, 还是总体最优问题, 都是通过确定相应的方程, 或方程组, 予以实现.

除 6.28—6.29 和 6.36—6.37 为便于阅读外, 6.30—6.35 是本书的主要内容. 在专著[135]的理论结果的基础上, 这里重点讨论, 如何确定纵横嵌入的节点和边, 在平面上的位置, 以减少占用面积.

在 6.30—6.31 中, 对于要解决的问题, 给出了合适的形式, 和通过引进纵图和横图, 将问题变换为易于利用的结构.

由 6.32 提供出, 确定一般纵横嵌入的, 基本方程的基础上, 依 6.33 和 6.34 构建各种优化方程, 解在平面上的位置.

因为这种问题在实际上的复杂性, 只能利用 6.35 所建议的方法, 估计最优占用面积, 在给定的条件下, 最接近的一个界.

刘彦佩
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第六编目录

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Boolean Approach to Planar Embeddings of a Graph[1)]

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Abstract. The purpose of this paper which is a sequel of "Boolean planarity characterization of graphs"[9] is to show the following results.

(1) Both of the problems of testing the planarity of graphs and embedding a planar graph into the plane are equivalent to finding a spanning tree in another graph whose order and size are bounded by a linear function of the order and the size of the original graph, respectively.

(2) The number of topologically non-equivalent planar embeddings of a Hamiltonian planar graph G is

$$\tau(G) = 2^{c(H)-1},$$

where $c(H)$ is the number of the components of the graph H which is related to G.

1. Introduction

A graph $G=(V, E)$ here is always treated as an embedding in the 3-dimensional space R^3 with $v \in V$ being points, $e \in E$ being Jordan curves such that $\forall\ e_1, e_2 \in E$: $e_1 \cap e_2 \neq \varnothing \Longleftrightarrow e_1 \cap e_2 \in V$. Of course, it can always be done for any graph.

Let T_D be a depth first search tree on G, $\prec$ be the order relation on V in the sense that the root is the least vertex, and all the tree edges have the direction from the smaller to the greater. And, all the cotree edges are oriented to be from the greater to the smaller. For $\gamma \notin T_D$, let C_γ be the fundamental circuit generated by γ with T_D. Further, let $\mathscr{D}(G)$, or simply write $\mathscr{D}$, be defined as

$$\mathscr{D} = \{(\alpha, \beta) \mid \alpha, \beta \notin T_D \text{ and } \alpha \cap \beta = \varnothing \text{ but } C_\alpha \cap C_\beta \neq \varnothing\}.$$

For T_D on G, let f_D be a T_D-immersion, i.e., a continuous mapping f: $R^3 \to R^2$ such that

(i) $\forall\ v_1, v_2:\ v_1 \neq v_2 \Longleftrightarrow f(v_1) \neq f(v_2)$;

(ii) $\forall\ e_1 \in E, e_2 \in T_D:\ e_1 \cap e_2 \neq \varnothing \Longleftrightarrow f(e_1) \cap f(e_2) \neq \varnothing$;

(iii) $\forall\ e_1, e_2 \in E:\ e_1 \cap e_2 = \{v\} \Longleftrightarrow f(e_1) \cap f(e_2) = \{f(v)\}$.

And, let $\mathscr{F}_D$ be the set of all the T_D-immersions of G. Further, for $f_D \in \mathscr{F}_D$, let

$$\mathscr{D}_0(f_D) = \{(f_D(\alpha), f_D(\beta)) \mid f_D(\alpha) \cap f_D(\beta) = \varnothing, (\alpha, \beta) \in \mathscr{D}\};$$
$$\mathscr{D}_1(f_D) = \{(f_D(\alpha), f_D(\beta)) \mid f_D(\alpha) \cap f_D(\beta) \neq \varnothing, (\alpha, \beta) \in \mathscr{D}\}.$$

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Of course, $\mathscr{D}_1(f_D) = \mathscr{D} - \mathscr{D}_0(f_D)$.

Then, we introduce two kinds of variables x_{ts}, and $y_{t\alpha}$, which are respectively said to be tree variables and cotree variables, for edge pairs (t, s) and (t, α) at a non-univalent vertex v, probably save only for the root, of T_D where t is a tree edge going away from v, s is an edge in E of going away from v, and α is a cotree edge coming to v. Moreover we have known that for $(\alpha, \beta) \in \mathscr{D}$, in $C_\alpha \bigcup C_\beta$, there are exactly two variables, one is tree variable denoted by $x_{\langle \alpha, \beta \rangle}$, and the other, cotree variable denoted by y_γ, $\gamma = \alpha$, or β according as $h(\alpha) \succ h(\beta)$, or $h(\alpha) \prec h(\beta)$. Here, $h(\gamma)$ is the head end of γ, i.e., the smaller end for a cotree edge.

In [9], we provided a criterion for testing the planarity of a graph as follows.

Basic Criterion. *For a graph G, G is planar iff for a given T_D on G, $f_D \in \mathscr{F}_D$, the Boolean equation system*

$$I(f_D): \begin{cases} x_{\langle \alpha,\beta \rangle} \overline{y}_\gamma \vee \overline{x}_{\langle \alpha,\beta \rangle} y_\gamma = 0, & (f_D(\alpha), f_D(\beta)) \in \mathscr{D}_0(f_D); \\ \overline{x}_{\langle \alpha,\beta \rangle} \overline{y}_\gamma \vee x_{\langle \alpha,\beta \rangle} y_\gamma = 0, & (f_D(\alpha), f_D(\beta)) \in \mathscr{D}_1(f_D) \end{cases}$$

has a solution.

The purpose of this paper is to do the following four things.

In §2, we derive several criteria for testing the planarity of a graph for the further use of transforming the problem into finding a spanning tree in another graph $H_0(G, T_D)$ which is said to be an auxiliary graph of G with class 0.

In §3, we investigate the auxiliary graphs $H_1(G, T_D)$ of class 1 so that the order of $H_1(G, T_D)$ is bounded by a linear function of the order of G.

Then, in §4, we introduce the auxiliary graphs $H_2(G, T_D)$ of class 2 so that both the order and the size of $H_2(G, T_D)$ are bounded by a linear function of the order and the size of G respectively.

And, finally, in §5, the planar embeddings of a planar graph are discussed and the number of topologically non-equivalent planar embeddings of a graph is determined for the Hamiltonian case.

2. Auxiliary Graph of Class 0

From the Basic Criterion, we may see that the problem of testing the planarity of a graph has been transformed into testing if the Boolean equation system $I(f_D)$ has a solution. This is almost a trivial thing because in $I(f_D)$ each equation has exactly two Boolean variables involved.

Now, we construct a graph, denoted by

$$H_0(G, T_D) = (V_0(G, T_D), E_0(G, T_D)),$$

for T_D on G such that

$$\begin{cases} V_0(G, T_D) = \{x_{\langle \alpha,\beta \rangle}, y_\gamma \mid (\alpha, \beta) \in \mathscr{D}\}; \\ E_0(G, T_D) = \{(x_{\langle \alpha,\beta \rangle}, y_\gamma) \mid (\alpha, \beta) \in \mathscr{D}\}, \end{cases} \tag{2.1}$$

where $\gamma = \alpha$, or β as mentioned above in $I(f_D)$. $H_0(G, T_D)$ is said to be the *auxiliary graph of class* 0 for T_D on G.

Further we introduce $H_0(G,f_D;w)$ for $f_D \in \mathscr{F}_D$ such that

$$H_0 = (G,f_D;w) = H_0(G,T_D) \tag{2.2}$$

provided a weight function w on $E_0(G,T_D)$ is defined as

$$w((x_{\langle\alpha,\beta\rangle},y_\gamma)) = \begin{cases} 0, & (\alpha,\beta) \in \mathscr{D}_0(f_D); \\ 1, & (\alpha,\beta) \in \mathscr{D}_1(f_D), \end{cases} \tag{2.3}$$

for $(x_{\langle\alpha,\beta\rangle},y_\gamma) \in E_0(G,T_D)$.

For $H_0(G,f_D;w)$, if $\exists$ a kind of labeling the vertices on $V_0(G,T_D)$ by $l(v)$ = "+", or "−", $v \in V_0(G,T_d)$, such that

$$\forall (x_{\langle\alpha,\beta\rangle},y_\gamma) \in E_0(G,T_D):$$
$$l(x_{\langle\alpha,\beta\rangle}) = l(y_\gamma) \iff w((x_{\langle\alpha,\beta\rangle},y_\gamma)) = 0;$$
$$l(x_{\langle\alpha,\beta\rangle}) \neq l(y_\gamma) \iff w((x_{\langle\alpha,\beta\rangle},y_\gamma)) = 1, \tag{2.4}$$

then $H_0(G,f_D;w)$ is said to be *balanced*.

The following results are well known and easily proved.

Lemma 2.1. *The following statements are equivalent*:

(1) $I(f_D)$ *has a solution*;

(2) $\nexists$ *any fundamental circuit* $C(H_0)$ *in* $H_0(G,f_D;w)$ *with the property*:

$$w(C(H_0)) = \sum_{e\in C(H_0)} w(e) \equiv 1 \pmod 2; \tag{2.5}$$

(3) *The subset of edges* $U_1(H_0)$ *in* $H_0(G,f_D;w)$,

$$U_1(H_0) = \{e \mid w(e) = 1, e \in E_0(G,T_D)\} \tag{2.6}$$

is a cutset on $H_0(G;T_D)$ *(notice that cutsets here do not mean minimal)*.

(4) $H_0(G,f_D;w)$ *is balanced.*

Proof. (1) ⇒ (2). See Lemma 3.1 in [6].

(2) ⇒ (3). See Lemma 3.3 in [6].

(3) ⇒ (4). From (3), $U_1(H_0)$ has the form

$$U_1(H_0) = (X,Y) = \{(x,y) \mid x \in X,\ y \in Y\},$$

where $V_0(G,T_D) = X \bigcup Y$, $X \bigcap Y = \varnothing$. At this moment we may write

$$l(z) = \begin{cases} +, & z \in X; \\ -, & z \in Y. \end{cases} \tag{2.7}$$

Therefore, $H_0(G,f_D;w)$ is balanced.

(4) ⇒ (1). In this case, a solution is found for $I(f_D)$ only by substituting 1 and 0 for the labels "+" and "−" respectively, or vice versa. □

According to the Basic Criterion, we have the following criteria for the planarity.

Theorem 2.2. *G is planar iff one of the following properties holds for a given T_D on G and a T_D – immersion $f_D \in \mathscr{F}_D$:*

(1) *$H_0(G,f_D;w)$ does not have a fundamental circuit $C(H_0)$ satisfying (2.5);*
(2) *The subset of edges $U_1(H_0)$ defined by (2.6) is a cutset of $H_0(G;T_D)$;*
(3) *$H_0(G,f_D;w)$ is balanced.*

Now we may see that testing the planarity by one of above criteria shown in Theorem 2.2 in fact corresponds to finding a spanning tree on $H_0(G;T_D)$ in view of the computing complexity. However, the order and the size of $H_0(G,T_D)$ here are bounded by a quadratic function of the order and the size of G.

In what follows we investigate what the auxiliary graphs of class 0 as a whole look like. Let

$$H_0(\mathscr{G},\mathscr{T}_D)=\{H \mid \exists \ T_D \text{ on } G\in\mathscr{G},\ T_D\in\mathscr{T}_D \ \ni H=H_0(G;T_D)\}. \quad (2.8)$$

In order to characterize $H_0(\mathscr{G},\mathscr{T}_D)$, we have to introduce another kind of graphs each of which is said to be *path overlaping graph*, denoted by

$$g(\mathscr{P}(T_d))=(V(\mathscr{P}(T_d)),\ E(\mathscr{P}(T_d))); \quad (2.9)$$

$$G(\mathscr{P}(\mathscr{T}_D))=\{G(\mathscr{P}(T_d)) \mid T_d\in\mathscr{T}_d\} \quad (2.10)$$

and defined in the following way for a rooted tree $T_d\in\mathscr{T}_d$ and a set of dipaths $\mathscr{P}(T_d)$ on T_d.

For T_d is rooted, we may write $\succeq$ to be the order relation on the vertex set of T_d in the sense: $\forall\ u,v\in V(T_d)$, $v\preceq u \Longleftrightarrow \exists$ a dipath from v to u on T_d.

For a path (dipath of course) P, let

$$v=\min P:\ \forall w\in V(P),\ w\preceq v \Rightarrow w=v;$$
$$u=\max P;\ \forall w\in V(P),\ u\preceq w \Rightarrow w=u,$$

where v, u are said to be the ends of P.

And, for T_d, there is a unique minimal vertex which is the root. But, the maximal vertices generally are not unique. Let $M(T_d)$ be the set of all the maximal vertices of T_d.

Now, we may determine $V(\mathscr{P}(T_d))$ and $E(\mathscr{P}(T_d))$ in (2.9). Let

$$V_{max}=\{u \mid \exists\ P\in\mathscr{P}(T_d),\ u=\max P, u\notin M(T_d)\};$$
$$V_{min}=\{u \mid \exists\ P\in\mathscr{P}(T_d),\ v=\min P,\ v\neq 0\};$$
$$V_{inf}=\{w \mid \exists\ P_1,P_2\in\mathscr{P}(T_d),\ P_1\cap P_2\neq\varnothing, \text{ and } \inf(\max P_1,\ \max P_2)=w \succ \min(\min P_1,\ \min P_2)\}.$$

Then we have

$$V(\mathscr{P}(T_d))=V_{max}\cup V_{min}\cup V_{inf};$$
$$(u,v)\in E(\mathscr{P}(T_d)) \Longleftrightarrow \exists\ P_1,P_2\in\mathscr{P}(T_d),\ P_1\cap P_2\neq\varnothing \ni u=\max(\min P_1,\min P_2),\ \min P_1\neq\min P_2;$$

and

$$v = \inf(\max P_1, \max P_2), \max P_1 \neq \max P_2.$$

Theorem 2.3. *$H \in H_0(\mathscr{G}, \mathscr{T}_D)$, iff $H \in G(\mathscr{P}(\mathscr{T}_d))$.*

3. Reduction I

The aim of this section is to find the so called auxiliary graphs of class 1 such that their orders are bounded by a linear function of the order of the original graph. In order to do this, we have to reduce the number of variables which appear in the Boolean equation system $I(f_D)$, or find a substitute for T_D on G and $f_D \in \mathscr{T}_D$.

Now, we investigate the Boolean equation system $I(f_D)$ first. For a tree variable x_{st}, we have, without loss of generality, $s=(w,u)$, $t=(w,u) \in E(T_D)$ since otherwise we may use $\hat{G}$ as defined in the previous paper [9] to be the resultant graph of subdividing each cotree edge into two on G such that one is incident to the back end of the cotree edge for T_D and is extended to a tree edge for $\hat{T}_D$, of course, $\hat{T}_D \in \hat{\mathscr{T}}_D \equiv \mathscr{T}_D(\hat{G})$. Let $\alpha_1, \alpha_2, \cdots, \alpha_{i_v}$ be all the cotree edges incident to the branch $B_v(T_D)$ which contains all the vertices $\succeq v$ on T_D with $h(\alpha_j) \prec w$, $1 \leqslant j \leqslant i_v$, for instance $h(\alpha_1) \preceq h(\alpha_2) \preceq \cdots \preceq h(\alpha_{i_v}) \prec w$. Likewise, let $\beta_1, \beta_2, \cdots, \beta_{i_u}$ be all the cotree edges incident to the branch $B_u(T_D)$ with $h(\beta_1) \preceq h(\beta_2) \preceq \cdots \preceq h(\beta_{i_u}) \preceq w$. Because $T_D \in \mathscr{T}_D$ is allowed to choose whichever is favourable for our purpose, we here employ the L-immersion for the Basic Criterion, which was defined in [9] as such a T_D-immersion that every cotree edge is on the left hand side when one moves from the head end to the back end along T_D. Suppose $h(\beta_1)$ is the least vertex among $h(\alpha_j), h(\beta_l), 1 \leqslant j \leqslant i_v, 1 \leqslant l \leqslant i_u$, and suppose $h(\beta_1) \prec h(\alpha_1)$; otherwise, we may let α_j be α_{j-k+1}, $k = \min\{i \mid h(\alpha_i) \succ h(\beta_1)\}$, $j \geqslant k$, because no equation in $I(f_L)$ is with both y_{α_j} and x_{st} for $j < k$. Then, in $I(f_L)$, all the equations involving x_{st} are as follows:

$$\begin{cases} \overline{x}_{st}\, \overline{y}_{\alpha_j} \vee x_{st} y_{\alpha_j} = 0, & h(\beta_1) \prec h(\alpha_j), \ 1 \leqslant j \leqslant i_v; \\ x_{st} \overline{y}_{\beta_j} \vee \overline{x}_{st} y_{\beta_j} = 0, & h(\alpha_1) \prec h(\beta_1), \ 1 \leqslant j \leqslant i_u. \end{cases} \tag{3.1}$$

Further, we may arrange in a linear order all the cotree variables which appear in (3.1) in the following way: $\gamma_1 = \alpha_1$, and if γ_i is already determined then γ_{i+1} is the succeeding one appearing in the rotation at the common head end of γ_i and γ_{i+1}, whenever any; otherwise $\gamma_{i+1} = \min\{\alpha_j, \beta_j \mid h(\gamma_i) \prec h(\alpha_j), h(\beta_j)\}$. Let

$$\begin{cases} \mathscr{D}_0(f_L; x_{st}) = \{(\gamma_i, \gamma_{i+1}) \mid \gamma_i, \gamma_{i+1} \text{ appear in the same kind of equations of Eq. (3.1)}\}; \\ \mathscr{D}_1(f_L; x_{st}) = \{(\gamma_i, \gamma_{i+1}) \mid \gamma_i, \gamma_{i+1} \text{ appear in different kinds of equations of Eq. (3.1)}\}. \end{cases} \tag{3.2}$$

Then, we may find the following lemma.

Lemma 3.1. *Eq. (3.1) is equivalent to the following equation system*

$$\begin{cases} \overline{y}_{\gamma_i}\overline{y}_{\gamma_{i+1}} \vee y_{\gamma_i} y_{\gamma_{i+1}} = 0, (\gamma_i, \gamma_{i+1}) \in \mathscr{D}_1(f_L; x_{st}); \\ y_{\gamma_i}\overline{y}_{\gamma_{i+1}} \vee \overline{y}_{\gamma_i} y_{\gamma_{i+1}} = 0, (\gamma_i, \gamma_{i+1}) \in \mathscr{D}_0(f_L; x_{st}). \end{cases} \tag{3.3}$$

Proof. Let $q_j(x_{st}, y_{\gamma_j}) = 0$ be the equation, in which y_{γ_j} appears of Eq. (3.1) and $p_j(y_{\gamma_j}, y_{\gamma_{j+1}}) = 0$ be the quation, in which $y_{\gamma_j}, y_{\gamma_{j+1}}$ appear, of Eq. (3.3). And, let $I_k(x_{st}, y) = \{ q_j(x_{st}, y_{\gamma_j}) = 0 \mid j = 1, 2, \cdots, k \}$, i.e., Eq. (3.1), and $J_{k-1}(y) = \{ p_j(y_{\gamma_j}, y_{\gamma_{j+1}}) = 0 \mid j = 1, 2, \cdots, k-1 \}$, i.e., Eq. (3.3).

First, it is easily seen that $I_2(x_{st}; y)$ is equivalent to $J_1(y)$, i.e., $I_2(x_{st}, y)$ has a solution iff so does $J_1(y)$ since $\{ x_{st}; y_1, y_2 \}$ is a solution of $I_2(x_{st}; y)$ iff $\{y_1, y_2\}$ is a solution of $J_1(y)$.

Then, for the general case, on the basis of $I_k(x_{st}; y)$ being equivalent to $I_{k-1}(x_{st}; y) \bigcup \{ p_{k-1}(y_{\gamma_{k-1}}, y_{\gamma_k}) = 0 \}$, further, we may by induction find that $I_k(x_{st}; y)$ is equivalent to $J_{k-2}(y) \bigcup \{ p_{k-1}(y_{k-1}, y_k) = 0 \} = J_{k-1}(y)$. This means the lemma. □

If two successive cotree edges γ_j, γ_{j+1} for a tree variable x_{st} appear in one of the configurations: Conf. A, Conf. B, and Conf. C, then γ_j and γ_{j+1} are said to be *adjacent*. For $\gamma \notin T_D$, let $b(\gamma)$ be the back end of γ.

Conf. A. $\exists\ \alpha \notin T_D \ni$

(i) $h(\alpha) \prec h(\gamma_j) \prec h(\gamma_{j+1})$;

(ii) $\inf(b(\gamma_j), \Lambda) \prec \Lambda = \inf(b(\alpha), b(\gamma_{j+1}))$.

Conf. B. $\exists\ \alpha, \beta \notin T_D \ni$

(i) $\max(h(\alpha), h(\beta)) \prec h(\gamma_j) = h(\gamma_{j+1})$;

(ii) $\inf(b(\alpha), b(\gamma_j)) = \Lambda_1 \succ \inf(\Lambda_1, \Lambda_2) \prec \Lambda_2 = \inf(b(\beta), b(\gamma_{j+1}))$.

Conf. C. $\exists\ \alpha \notin T_D \ni$

(i) $h(\alpha) \prec h(\gamma_j) \preceq h(\gamma_{j+1})$;

(ii) $\inf(b(\alpha), \Lambda) \prec \Lambda = \inf(b(\gamma_j), b(\gamma_{j+1}))$.

Lemma 3.2. *For $\gamma_j, \gamma_{j+1} \notin T_D$ with a tree variable x_{st} we have $(\gamma_j, \gamma_{j+1}) \in \mathscr{D}_0(f_L; x_{st})$ iff γ_j and γ_{j+1} appear in Conf. C; and $(\gamma_j, \gamma_{j+1}) \in \mathscr{D}_1(f_L; x_{st})$ iff γ_j and γ_{j+1} appear in one of Conf. A and Conf. B.*

Proof. To prove the first statement. The sufficiency is easily obtained in the usual way by finding the equations. For the necessity, two cases should be discussed with $(\gamma_j, \gamma_{j+1}) \in \mathscr{D}_0(f_L; x_{st})$ and the assumptions on α_i, β_l described above.

Case 1.1. $\gamma_j = \alpha_i, \gamma_{j+1} = \alpha_{i+1}$. From Eq. (3.1), we have $h(\beta_l) \prec h(\gamma_j) \preceq h(\gamma_{j+1})$ and $\inf(b(\beta), \Lambda) \prec \Lambda = \inf(b(\gamma_j), b(\gamma_{j+1}))$ which means γ_j, γ_{j+1} appear in Conf. C.

Case 1.2 $\gamma_j = \beta_l, \gamma_{j+1} = \beta_{l+1}$. The same as that in Case 1.1 can be used except only for β instead of α.

To prove the last statement. Similarly, the sufficiency is obvious. For the necessity, we discuss the following two cases.

Case 2.1. $\gamma_j = \alpha_i, \gamma_{j+1} = \beta_l$. From Eq. (3.1), we have $h(\beta_l) \prec h(\alpha_1) \preceq h(\alpha_i) \preceq h(\beta_l)$. If $h(\alpha_1) = h(\alpha_i)$, then $h(\alpha_i) \prec h(\beta_l)$. That means γ_j, γ_{j+1} appear in Conf. A. Otherwise, γ_j and γ_{j+1} appear in Conf. A or Conf. B according as $h(\alpha_i) \prec h(\beta_l)$ or $h(\alpha_i) = h(\beta_l)$.

Case 2.2. $\gamma_j=\beta_l$, $\gamma_{j+1}=\alpha_i$. The same as that in Case 2.1 only by exchanging α and β. □

This lemma enables us to chech up on the adjacency of two cotree edges without using the tree variables. Let $\mathscr{A}$ be the set of all the pairs of cotree edges which are adjacent. And, let

$$\begin{cases}\mathscr{A}_0=\{(\alpha,\beta)\in\mathscr{A}\mid\alpha,\beta \text{ appear in Conf. C}\};\\ \mathscr{A}_1=\{(\alpha,\beta)\in\mathscr{A}\mid\alpha,\beta \text{ appear in Conf. A or Conf. B}\}.\end{cases}\tag{3.4}$$

Now, we have to mention that because of the uniquness of f_L for a given T_D on G and $f_D\in\mathscr{F}_D$, all of $\mathscr{A}$, $\mathscr{A}_0$ and $\mathscr{A}_1$ do not depend on the immersions.

Lemma 3.3. *Equation system $I(f_D)$ for T_D on G and $f_D\in\mathscr{F}_D$ is equivalent to the following Boolean equation system:*

$$\begin{cases}y_\alpha\overline{y}_\beta\vee\overline{y}_\alpha y_\beta=0, & (\alpha,\beta)\in\mathscr{A}_0;\\ \overline{y}_\alpha\overline{y}_\beta\vee y_\alpha y_\beta=0, & (\alpha,\beta)\in\mathscr{A}_1.\end{cases}\tag{3.5}$$

Proof. Let $I(f_D)$ represent the set of all the equations appearing in the Basic Criterion. And let $I(f_D;x)$ be the set of all the equations in Eq. (3.1) for the tree variable $x\in X$, the set of all the tree variables. Then, $I(f_D)=\sum_{x\in X}I(f_D;x)$. On the other hand, let Y be the set of all the cotree variables, and let $J(f_D;x)$ be the set of all the equations appearing in Eq. (3.3) for a given $x\in X$. Then, the set of all equations in the form (3.3) for all tree variables is $J(f_D)=\sum_{x\in X}J(f_D;x)$. From Lemma 3.1, we have

$$I(f_D)\Leftrightarrow J(f_D).$$

Further, let $K(T_D)$ be the set of all equations in Eq. (3.5). Then, from Lemma 3.2, we have

$$J(f_D)\Leftrightarrow K(T_D).$$

This is the lemma. □

Criterion 3.4. *G is planar iff for a T_D on G, Eq.* (3.5) has a solution.

Proof. From Lemma 3.3, a direct result of the Basic Criterion. □

This criterion allows us to go back to a graph again, which is denoted by $H_1(G,T_D)=(V(H_1),E(H_1))$ and is said to be an *auxiliary graph of class* 1 *of* G, where

$$V(H_1)=\{\gamma\mid y_\gamma\in Y\};\quad E(H_1)=\mathscr{A}.$$

And, we may also introduce a weight function w on $E(H_1)$ as follows:

$$w((\alpha,\beta))=\begin{cases}0, & (\alpha,\beta)\in\mathscr{A}_0;\\ 1, & (\alpha,\beta)\in\mathscr{A}_1.\end{cases}\tag{3.6}$$

Then, in the same way as that of proving Theorem 2.2, we may obtain the following theorem.

Theorem 3.5. *The following statements are equivalent:*

(1) G *is planar*;

(2) $H_1(G;T_D)$ *does not have a fundamental circuit* $C(H_1)$ *such that*
$$w(C(H_1))=\sum_{e\in C(H_1)} w(e)\equiv 1 \pmod 2;$$

(3) $\mathscr{A}_1$ *is a cutset of* $H_1(G;T_D)$;

(4) $H_1(G;T_D)$ *is balanced.*

Theorem 3.6. *The order of* $H_1(G;T_D)$ *is bounded by a linear function of the order of a planar graph* G.

Proof. Let n be the order of G, i.e., $n=|V|$. And, we have known that the size of a planar graph is at most $3n-6$, and the order of $H_1(G;T_D)$ is less than the number $(3n-6)-n+1$ of cotree edges in G. Therefore, the order of $H_1(G;T_D)$ is less than $2n-5$. □

4. Reduction II

In the previous paper, we found six obstacles which were denoted by Conf. I—VI [9]. However, we may see that there are in fact seven configurations by dividing Conf. II into two. After eliminating repetitions between them, the seven configurations are listed as follows.

Conf. 1 $\exists\ \alpha,\beta$, and $\gamma,\delta\notin T_D \ni$
$\max(h(\gamma),h(\delta))\prec h(\alpha)\prec h(\beta)\prec C\prec B\prec A$,
where $C=\inf(b(\gamma),B)$, $B=\inf(b(\alpha),A)$, and $A=\inf(b(\beta),b(\delta))$.

Conf. 2 $\exists\ \alpha,\beta,\eta$ and $\gamma,\delta\notin T_D\ni$
(i) $B=\inf(A,\inf(b(\eta),b(\alpha)))\prec\inf(b(\gamma),b(\delta))=A$;
(ii) $h(\eta)=h(\beta)\prec h(\delta)\prec h(\alpha)\prec\min(C,h(\gamma))\preceq\max(C,h(\gamma))\prec B$,
where $C=\inf(b(\beta),b(\alpha))$.
(iii) $B=\inf(b(\eta),b(\alpha))$.

Conf. 3 $\exists\ \alpha,\beta,\eta$ and $\gamma,\delta\notin T_D\ni$
(i) $h(\beta)=h(\eta)\prec h(\delta)\preceq h(\alpha)\prec D\preceq h(\gamma)\prec C$;
(ii) $D=\inf(b(\beta),C)\preceq\max(h(\gamma),D)\prec C\prec B\prec A$.
where $A=\inf(b(\gamma),b(\delta))$, $B=\inf(b(\eta),A)$, $C=\inf(b(\alpha),B)$.

Conf. 4 $\exists\ \alpha,\beta,\gamma$ and $\xi,\eta,\zeta\notin T_D\ni$
(i) $\max(h(\xi),h(\eta),h(\zeta))\prec\min(h(\alpha),h(\beta))$;
(ii) $\max(h(\alpha),h(\beta))\prec\inf(b(\eta),h(\gamma))\preceq h(\gamma)$;
(iii) $h(\gamma)\prec\inf(b(\xi),b(\beta),A)\prec\inf(b(\gamma),b(\beta),b(\zeta))=A$.

Conf. 5 $\exists\ \alpha,\beta,\gamma$ and $\delta,\eta\notin T_D\ni$
(i) $\max(h(\beta),h(\eta))\prec h(\gamma)\prec\min(h(\delta),h(\alpha))$;
(ii) $\max(h(\delta),h(\alpha))\prec\inf(A,B)\prec\inf(b(\delta),b(\eta))=A$;
(iii) $B=\inf(b(\gamma),C)\prec\inf(b(\alpha),b(\beta))=C$.

Conf. 6 $\exists\ \alpha,\beta,\gamma,\delta$ and $\xi\notin T_D\ni$
(i) $h(\delta)=h(\beta)\prec h(\xi)\prec\min(h(\alpha),h(\gamma))$;
(ii) $\max(h(\alpha),h(\gamma))\prec A=\inf(b(\xi),B)\prec B=\inf(C,D)$;
(iii) $B\prec C=\inf(b(\delta),b(\gamma))$, $B\prec D=\inf(b(a),b(\beta))$.

Conf. 7 $\exists$ α, β, γ and $\xi, \eta, \zeta \notin T_D \ni$

(i) $\max(h(\zeta), h(\eta), h(\xi)) \prec h(\alpha) = h(\beta) = h(\gamma)$;

(ii) $h(\gamma) \prec D = \inf(A, B, C)$;

(iii) $D \prec \inf(b(\beta), b(\eta)) = A$, $D \prec \inf(b(\alpha), b(\xi)) = B$, $D \prec \inf(b(\zeta), b(\gamma)) = C$.

Criterion 4.1 *G is planar iff none of the configurations: Conf. 1—7 as shown above appears in G as a subgraph.*

Proof. A direct result of Lemma 4.8 in [9] and the Basic Criterion. □

According to the criterion we may now introduce another kind of auxiliary graphs for T_D on G, which is said to be the *auxiliary graphs of class* 2, denoted by $H_2(G;T_D)$, in order to reduce the sizes of those of class 1.

The vertex set of $H_2(G; T_D)$ is the same as that of $H_1(G; T_D)$. And, the edge set $E(H_2)$ of $H_2(G; T_D)$ has to be partitioned into seven parts corresponding to Conf. 1—7 and represented by $E^{(i)}(H_2)$, $i=1, \cdots 7$. Thus, we have

$$H_2(G; T_D) = \bigcup_{i=1}^{7} H_2^{(i)}(G; T_D); \quad H_2^{(i)}(G; T_D) = (V(H_1), E^{(i)}(H_2)). \tag{4.1}$$

In what follows, we determine $E^{(i)}(H_2)$, therefore $H^{(i)}(G; T_D)$ for $i = 1, 2, \cdots, 7$. For the sak of brevity, we employ $\langle \alpha, \beta \rangle$, α, $\beta \notin T_D$, instead of $\inf(b(\alpha), b(\beta))$.

Proc. 1. To find $E^{(1)}(H_2)$.

1. For $\alpha \notin T_D$, if $\nexists \beta \notin T_D \ni \beta \in P(\alpha)$:
$$P(\alpha) = \{\gamma \notin T_D \mid h(\alpha) \prec h(\gamma) \prec \langle \alpha, \gamma \rangle \prec b(\gamma)\}, \tag{4.2}$$
then choose a new $\alpha \notin T_D$ and go back to 1; otherwise choose $\beta \notin T_D \ni$ $h(\beta) = \min\{h(\gamma) \mid \gamma \in P(\alpha)\}$.
2. If $\exists\ \gamma \notin T_D \ni h(\beta) \prec \langle \gamma, \langle \alpha, \beta \rangle\rangle \prec \langle \alpha, \beta \rangle$ and $h(\gamma) \prec h(\alpha)$, then let $(\alpha, \beta) \in E^{(1)}(H_2)$ with weight 0.
3. If $\nexists\ \delta \notin T_D \ni h(\delta) \prec h(\alpha)$, and $\langle \alpha, \beta \rangle \prec b(\delta)$, then choose a new $\alpha \notin T_D$ and go back to 1; otherwise, let $(\alpha, \beta) \in E^{(1)}(H_2)$ with weight 1. Then choose a new $\alpha \notin T_D$ and go back to 1.
4. Until all the cotree edges have been chosen.

Lemma 4.2. *If G has a subgraph isomorphic to Conf.* 1, *then for the* T_D *on* G, $H_2^{(1)}(G; T_D)$ *has a circuit C such that* $w(C) \equiv 1 \pmod 2$.

Proof. If G has a subgraph isomorphic to Conf. 1, then there is a cotree edge corresponding to α in the subgraph. Hence, for the edge as α, Proc. 1 provides a circuit C of length 2 with one edge of weight 0 and the other of weight 1. A circuit C such that $w(C) \equiv 1 \pmod 2$ is found. □

Proc. 2. To find $E^{(2)}(H_2)$.

1. For $\eta \notin T_D$, if $\neg\ \alpha \notin T_D$: $\exists\ \beta \notin T_D \ni h(\beta) = h(\eta) \prec h(\alpha) \prec \langle \beta, \alpha \rangle \prec \langle \alpha, \eta \rangle$, then choose a new $\eta \notin T_D$ and go back to 1; otherwise, choose $\beta \notin T_D$:

$\langle \beta,\eta \rangle = \max\{\langle \gamma,\eta \rangle \mid \exists \ \alpha \notin T_D \ni h(\gamma)=h(\eta) \prec h(\alpha) \prec \langle \gamma,\alpha \rangle \prec \langle \alpha,\eta \rangle\}$. (4.3)

2. Let $P(\beta)=\{\zeta \notin T_D \mid h(\beta) \prec h(\zeta) \prec \langle \zeta,\beta \rangle \prec \langle \zeta,\eta \rangle\}$. If

$$\neg \alpha \in P(\beta): \exists \ \delta,\gamma \notin T_D \ni h(\delta) \prec h(\alpha) \prec h(\gamma) \prec \langle \alpha,\eta \rangle = \langle\langle \delta,\gamma \rangle, \alpha \rangle \prec \langle \delta,\gamma \rangle, \quad (4.4)$$

then choose a new $\eta \notin T_D$ and go back to 1.

3. Let $(\alpha,\delta) \in E^{(2)}(H_,)$ with weight 0, $(\delta,\gamma) \in E^{(2)}(H_2)$ with weight 0, and $(\alpha,\gamma) \in E^{(2)}(H_2)$ with weight 1. Choose a new $\eta \notin T_D$ and go back to 1.

4. Until all the cotree edges have been chosen.

Lemma 4.3. *If G has a subgraph isomorphic to Conf. 2, then for the T_D on G, $H^{(2)}(G;T_D)$ has a circuit C such that $w(C) \equiv 1 \pmod 2$.*

Proof. Because if G has a subgraph which isomorphic to Conf. 2, then $\exists \ \eta \notin T_D$ as indicated in Proc. 2. Therefore, a circuit C of length 3 with one edge of wieght 1, i.e., $w(C) \equiv 1 \pmod 2$ occurs in $H^{(2)}(G;T_D)$. □

Proc. 3. To find $E^{(3)}(H_2)$.

1. For $\eta \notin T_D$, if

$$\neg \alpha \notin T_D: \exists \ \beta \notin T_D \ni h(\beta)=h(\eta) \prec h(\alpha) \prec \langle \beta,\alpha \rangle \prec \langle \alpha,\eta \rangle, \quad (4.5)$$

then choose a new $\eta \notin T_D$ and go back to 1; otherwise, choose $\beta \notin T_D$: $\langle \beta,\eta \rangle = \max\{\langle \gamma,\eta \rangle \mid \gamma \notin T_D: \exists \ \alpha \notin T_D \ni h(\gamma)=h(\eta) \prec h(\alpha) \prec \langle \gamma,\alpha \rangle \prec \langle \alpha,\eta \rangle\}$.

2. Let $P(\beta)=\{\gamma \notin T_D \mid h(\beta) \prec h(\gamma) \prec \langle \gamma,\beta \rangle \prec \langle \gamma,\eta \rangle\}$. If $\neg \alpha \in P(\beta)$: $\exists \ \delta,\gamma \notin T_D \ni h(\beta)=h(\eta) \prec h(\delta) \preceq h(\alpha) \prec \langle \beta,\eta \rangle \preceq h(\gamma) \prec \langle \alpha,\eta \rangle \prec \langle \eta, \langle \delta,\gamma \rangle \rangle \prec \langle \delta,\gamma \rangle$, then choose a new $\eta \notin T_n$ and go back to 1.

3. Let (δ,γ), $(\delta,\alpha) \in E^{(3)}(H_2)$ with weight 0, $(\alpha,\gamma) \in E^{(3)}(H_2)$ with weight 1. Choose a new $\eta \notin T_D$ and go back to 1.

4. Until all the cotree edges have been chosen.

Lemma 4.4. *If G has a subgraph isomorphic to Conf. 3, then for the T_D on G, $H^{(3)}(G;T_D)$ has a circuit C such that $w(C) \equiv 1 \pmod 2$.*

Proof. Similar to the proof of Lemma 4.3, if G has a subgraph which is isomorphic to Conf. 3, then $H^{(3)}(G;T_D)$ has a circuit C of length 3, in which only one edge has weight 1. □

Proc. 4. To find $E^{(4)}(H_2)$.

1. For $\zeta \notin T_D$, if

$$\neg \gamma,\eta \notin T_D: \max(h(\zeta),h(\eta)) \prec \langle \eta,\zeta \rangle \prec h(\gamma) \prec \langle \gamma,\zeta \rangle, \quad (4.6)$$

then choose a new $\zeta \notin T_D$ and go back to 1.

2. Let $P(\zeta)=\{\gamma,\eta \notin T_D \mid \gamma$ and η satisfy (4.6)$\}$.
If $\neg \alpha,\beta,\xi \notin T_D$: $\max(h(\xi),h(\eta),h(\zeta)) \prec \min(h(\alpha),h(\beta)) \preceq \max(h(\alpha),h(\beta)) \prec \langle \eta,\zeta \rangle \prec h(\gamma) \prec \min(\langle \alpha,\zeta \rangle, \langle \xi,\zeta \rangle)$