刘彦佩

# 半闲数学集锦

# Semi-Empty Collections in Mathematics by Y.P.Liu

## 第十九编

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# 第十九编序

本编的前部为文 19,01[467]—19.10[478], 和后部为专著[486] *Elements of Algebraic Graphs*(19.11—19.36).

在前部中, 文 19.01[467], 19.04[470] 和 19.05[471], 为下面的专著, 提供了一些基本素材.

因为循环图和轮图, 还未得到嵌入的亏格分布, 只能用联树法, 确定在小亏格曲面上的嵌入数, 这就是文 19.02[468] 和文 19.03[469] 所要完成的任务.

文 19.06[472] 和文 19.08[475], 都是关联, 图的上可嵌入性.

文 19.07[473], 属于图上多面形理论, 的一部分, 因为尚未得到应有的注意, 又重新整理的. 其他的, 即文 19.09[477] 和文 19.10[478], 都是有关地图计数的.

在后部中, 专著 [486](19.11—19.36), 着意从图的新代数表示出发, 研究各种组合构形, 例如曲面, 地图, 图在曲面上的嵌入, 多面形, 以及图本身等, 在由各种代数运算, 形成的等价关系下, 的分类与计数.

这就是从调查不对称对象, 本身的局部对称性着手. 在书中, 只是以图和地图为对象, 用它们的局部结构对称性, 揭示它们的内在的全局性质.

款 19.13 提供基本概念形成的渊由. 通过单面形研究多面形, 曲面, 图在曲面上的嵌入, 地图, 以及图等组合构形. 而不是反之.

实际上, 建立了在单面形上的一种新的代数. 这一路线是, 通过作者本人, 所创建的联树模型实现的.

联树模型, 比至今一直在国际上被人们利用的 Heftter- Edmonds 嵌入模型, 优越之处在于, 能十分有效地, 构造出这些构形. 也简化了 Klein-Tutte 地图模型.

一个不易被人们理解之处在于, 联树以某个树结构为出发点. 但从理论上已经证明, 用联树处理这些构形, 不依赖此树形结构的选择.

款 19.14—19.24, 就是在完成这一任务的同时, 确定出各种等价类的完全不变量(理论上, 全是多项式型的不变量).

款 19.25, 则在前面建立的理论指导下, 将 Tutte, Harary, 以及作者本人, 曾用高等方法, 解决的有关树的计数问题, 变为一种, 纯组合分类, 的初等问题. 以及由此的延伸.

款 19.26 和款 19.27, 讨论上述构形与泛函方程, 以及与图本身的关系.

款 19.28 在这种理论的基础上, 提出了确定一般(不一定带有对称性, 即自同构群非平凡!)图的, 各种亏格的理论. 第一次脱离了, 只考虑具有, 好对称性的图(因为只有这种图, 才能用电流图与电压图, 或者说拓扑学中, 覆盖空间的方法等).

款 19.29 是图论中的一个新专题, 也是由联树法引起的. 这就是考虑是否能, 用一个图的亏格多项式, 刻划这个图.

本章提供了一个准则, 以构造出具有相同亏格多项式, 而不同构图的无穷类, 推进了对于图同构的研究.

款 19.30, 对于一直无从着手, 几乎快要被人们忘却的, 图的曲面(非平面情形!)可嵌入性问题. 在联树原理下, 给出了四个具有独立理论意义的表征. 目前, 除在国外出现一篇新出版的文章外, 都只是讨论小亏格的曲面(如, 射影平面, 环面), 而且十分复杂.

这里的四个表征都是对于一般的曲面, 即亏格不受限制. 值得一提的, 其中一个表征, 在亏格为 0 的特殊情形(及平面情形), 却一举同时导出 Lefschetz(Solomon,1884-1972, 代数拓扑奠基人, 美国国家科学勋章获得者), MacLane(Saunders, 1909-2005, 美国国家科学院院士), Whitney(Hessler, 1907-1989, 美国国家科学院院士), 从三个不同的方向, 具有独立理论意义, 图的三个平面表征. 它们同发表在, 上世纪 30 年代. 而上面提到的那篇新文章, 只是将 MacLane 平面表征, 推广到曲面上.

前几年, 在国外, 也有一篇讨论 Whitney 平面表征, 推广到射影平面上的文章(Abrums 和 Slilaty "An algebraic characterization of projective-planar graphs", Jounal of Graph Theory, 42(2003), 320-331. MR2003k: 05041). 不过, 这两篇文章, 都比本章所提供的, 要复杂得多.

在每一款最后一节的注记, 都提供 10 余道问题, 共 200 余道, 皆本书作者从研究中得到, 或体会到, 虽然都程度不同是很困难的, 但具有近期可接近性.

在款 19.31—19.33 中, 除澄清一些理论概念外, 还提供了基于联树法的计算机实现.

刘彦佩<br>2015 年 9 月<br>於北京上园村

# 第十九编目录

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# Surface Embeddability of Graphs via Tree-travels

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**Abstract:** This paper provides a characterization for surface embeddability of a graph with any given orientable and nonorientable genus not zero via a method discovered by the author thirty years ago.

**Key Words**: Surface, graph,Smarandache $\lambda^S$-drawing, embeddability, tree-travel.

**AMS(2010)**: 05C15, 05C25

## §1. Introduction

A drawing of a graph $G$ on a surface $S$ is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache $\lambda^S$-drawing* of G on $S$ is a drawing of G on $S$ with minimal intersections $\lambda^S$. Particularly, a Smarandache 0-drawing of $G$ on $S$, if existing, is called an embedding of $G$ on $S$. Along the Kurotowski research line for determining the embeddability of a graph on a surface of genus not zero, the number of forbidden minors is greater than a hundred even for the projective plane, a nonorientable surface of genus 1 in [1].

However, this paper extends the results in [3] which is on the basis of the method established in [3-4] by the author himself for dealing with the problem on the maximum genus of a graph in 1979. Although the principle idea looks like from the joint trees, a main difference of a tree used here is not corresponding to an embedding of the graph considered.

Given a graph $G = (V, E)$, let $T$ be a spanning tree of $G$. If each cotree edge is added to $T$ as an articulate edge, what obtained is called a *protracted tree* of $G$, denoted by $\check{T}$. An protracted tree $\check{T}$ is oriented via an orientation of $T$ or its fundamental circuits. In order to guarantee the well-definedness of the orientation for given rotation at all vertices on $G$ and a selected vertex of $T$, the direction of a cotree edge is always chosen in coincidence with its direction firstly appeared along the the face boundary of $\check{T}$. For convenience, vertices on the boundary are marked by the ordinary natural numbers as the root vertex, the starting vertex, by 0. Of course, the boundary is a travel on $G$, called a *tree-travel*.

In Fig.1, (a) A spanning tree $T$ of $K_5$(i.e., the complete graph of order 5), as shown by bold lines; (b) the protracted tree $\check{T}$ of $T$.

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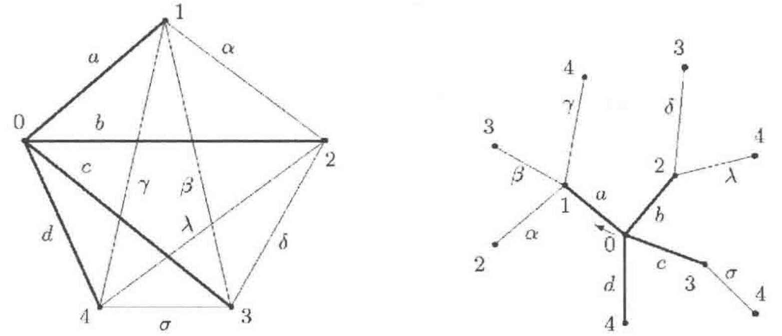



(a) $T$ (b) $\check{T}$

**Fig.1**

## §2. Tree-Travels

Let $C = C(V;\epsilon)$ be the tree travel obtained from the boundary of $\check{T}$ with 0 as the starting vertex. Apparently, the travel as a edge sequence $C = C(\epsilon)$ provides a double covering of $G = (V, E)$, denoted by

$$C(V;\epsilon) = 0P_{0,i_1}v_{i_1}P_{i_1,i_2}v_{i_2}P_{i_2,i'_1}v_{i'_1}P_{i'_1,i'_2}v_{i'_2}P_{i'_2,2\epsilon}0 \tag{1}$$

where $\epsilon = |E|$.

For a vertex-edge sequence $Q$ as a tree-travel, denote by $[Q]_{\mathrm{eg}}$ the edge sequence induced from $Q$ missing vertices, then $C_{\mathrm{eg}} = [C(V;\epsilon)]_{\mathrm{eg}}$ is a polyhegon(*i.e.*, a polyhedron with only one face).

**Example 1** From $\check{T}$ in Fig.1(b), obtain the thee-travel

$$C(V;\epsilon) = 0P_{0,8}0P_{8,14}0P_{14,18}0P_{18,20}0$$

where $v_0 = v_8 = v_{14} = v_{18} = v_{20} = 0$ and

$$P_{0,8} = a1\alpha2\alpha^{-1}1\beta3\beta^{-1}1\gamma4\gamma^{-1}1a^{-1};$$
$$P_{8,14} = b2\delta3\delta^{-1}2\lambda4\lambda^{-1}2b^{-1};$$
$$P_{14,18} = c3\sigma4\sigma^{-1}3c^{-1};$$
$$P_{18,20} = d4d^{-1}.$$

For natural number $i$, if $av_ia^{-1}$ is a segment in $C$, then $a$ is called a *reflective edge* and then $v_i$, the *reflective vertex* of $a$.

Because of nothing important for articulate vertices(1-valent vertices) and 2-valent vertices in an embedding, we are allowed to restrict ourselves only discussing graphs with neither 1-valent nor 2-valent vertices without loss of generality. From vertices of all greater than or equal to 3, we are allowed only to consider all reflective edges as on the cotree.

If $v_{i_1}$ and $v_{i_2}$ are both reflective vertices in (1), their reflective edges are adjacent in $G$ and $v_{i'_1} = v_{i_1}$ and $v_{i'_2} = v_{i_2}$, $[P_{v_{i_1,i_2}}]_{eg} \cap [P_{v_{i'_1,i'_2}}]_{eg} = \emptyset$, but neither $v_{i'_1}$ nor $v_{i'_2}$ is a reflective vertex, then the transformation from $C$ to

$$\triangle_{v_{i_1},v_{i_2}} C(V;e) = 0P_{0,i_1} v_{i_1} P_{i'_1,i'_2} v_{i_2} P_{i_2,i'_1} v_{i'_1} P_{i_1,i_2} v_{i'_2} P_{i'_2,0}0. \tag{2}$$

is called an operation of *interchange segments* for $\{v_{i_1}, v_{i_2}\}$.

**Example 2** In $C = C(V;e))$ of Example 1, $v_2 = 2$ and $v_4 = 3$ are two reflective vertices, their reflective edges $\alpha$ and $\beta$, $v_9 = 2$ and $v_15 = 3$. For interchange segments once on $C$, we have

$$\triangle_{2,3} C = 0P_{0,2}2P_{9,15}3P_{4,9}2P_{2,4}3P_{15,20}0 \ (= C_1).$$

where

$$\begin{aligned}
&P_{0,2} = a1\alpha \ (= P_{1;0,2});\\
&P_{9,15} = \delta 3\delta^{-1}2\lambda 4\lambda^{-1}2b^{-1}0c3 \ (= P_{1;2,8});\\
&P_{4,9} = \beta^{-1}1\gamma 4\gamma^{-1}1a^{-1}0b2 \ (= P_{1;8,13});\\
&P_{2,4} = \alpha^{-1}1\beta \ (= P_{1;13,15});\\
&P_{15,20} = \sigma 4\sigma^{-1}3c^{-1}0d4d^{-1} \ (= P_{1;15,20}).
\end{aligned}$$

**Lemma 1** Polyhegon $\triangle_{v_i,v_j} C_{eg}$ is orientable if, and only if, $C_{eg}$ is orientable and the genus of $\triangle_{v_{i_1},v_{i_2}} C_{eg}$ is exactly 1 greater than that of $C_{eg}$.

*Proof* Because of the invariant of orientability for $\triangle$-operation on a polyhegon, the first statement is true.

In order to prove the second statement, assume cotree edges $\alpha$ and $\beta$ are reflective edges at vertices, respectively,$v_{i_1}$ and $v_{i_2}$. Because of

$$C_{eg} = A\alpha\alpha^{-1}B\beta\beta^{-1}CDE$$

where

$$\begin{aligned}
&A\alpha = [P_{0,i_1}]_{eg};\ \alpha^{-1}B\beta = [P_{i_1,i_2}]_{eg};\\
&\beta^{-1}C = [P_{i_2,i'_1}]_{eg};\ D = [P_{i'_1,i'_2}]_{eg};\\
&E = [P_{i'_2,i_r}]_{eg},
\end{aligned}$$

we have

$$\begin{aligned}
\triangle_{v_{i_1},v_{i_2}} C_{eg} &= A\alpha D\beta^{-1}C\alpha^{-1}B\beta E\\
&\sim_{top} ABCDE\alpha\beta\alpha^{-1}\beta^{-1}, \text{ (Theorem 3.3.3 in [5])}\\
&= C_{eg}\alpha\beta\alpha^{-1}\beta^{-1} \text{ (Transform 1, in §3.1 of [5])}.
\end{aligned}$$

Therefore, the second statement is true. □

If interchange segments can be done on $C$ successively for $k$ times, then $C$ is called a *k-tree travel*. Since one reflective edge is reduced for each interchange of segments on $C$ and $C$ has at most $m = \lfloor \beta/2 \rfloor$ reflective edges, we have $0 \leqslant k \leqslant m$ where $\beta = \beta(G)$ is the Betti number(or corank) of $G$. When $k = m$, $C$ is also called *normal*.

For a $k$-tree travel $C_k(V; e, e^{-1})$ of $G$, graph $G_k$ is defined as

$$G_k = T \bigcup [E_{\text{ref}} \bigcap E_{\bar{T}} - \sum_{j=1}^{k} \{e_j, e'_j\}] \tag{3}$$

where $T$ is a spanning tree, $[X]$ represents the edge induced subgraph by edge subset $X$, and $e \in E_{\text{ref}}$, $e \in E_T$, $\{e_j, e'_j\}$ are, respectively, reflective edge, cotree edge, pair of reflective edges for interchange segments.

**Example 3** On $C_1$ in Example 2, $v_{1;3} = 3$ and $v_{1;5} = 4$ are two reflective vertices, $v_{1;8} = 3$ and $v_{1;10} = 4$. By doing interchange segments on $C_1$, obtain

$$\triangle_{3,4} C_1 = 0P_{1;0,10}3P_{1;17,19}4P_{1;12,15}3P_{1;10.12}4P_{1;19,20}0\ (= C_2)$$

where

$$\begin{aligned} &P_{1;0,10} = a1\alpha 2b^{-1}0c3\beta^{-1}1\gamma 4\gamma^{-1}1a^{-1}0b2\delta(= P_{2;0,10});\\ &P_{1;17,19} = c^{-1}0d(= P_{2;10,12});\\ &P_{1;12.17} = \alpha^{-1}2\alpha^{-1}1\beta 3\sigma 4\sigma^{-1}(= P_{2;12,17});\\ &P_{1;10,12} = \delta^{-1}2\lambda(= P_{2;17,19});\\ &P_{1;19,20} = d^{-1}(= P_{2;19,20}). \end{aligned}$$

Because of $[P_{2;6,16}]_{\text{eg}} \cap [P_{2;12,19}]_{\text{eg}} \neq \emptyset$ for $v_{2;12} = 4$ and $v_{2;19} = 4$, only $v_{2;6} = 4$ and $v_{2;16} = 4$ with their reflective edges $\gamma$ and $\sigma$ are allowed for doing interchange segments on $C_2$. The protracted tree $\check{T}$ in Fig.1(b) provides a 2-tree travel $C$, and then a 1-tree travel as well.

However, if interchange segments are done for pairs of cotree edges as $\{\beta, \gamma\}$, $\{\delta, \lambda\}$ and $\{\alpha, \sigma\}$ in this order, it is known that $C$ is also a 3-tree travel.

On $C$ of Example 1, the reflective vertices of cotree edges $\beta$ and $\gamma$ are, respectively, $v_4 = 3$ and $v_6 = 4$, choose $4' = 15$ and $6' = 19$, we have

$$\triangle_{4,6} C = 0P_{1;0,4}3P_{1;4,8}4P_{1;8,17}3P_{1;17,19}4P_{1;19,20}0(= C_1)$$

where

$$\begin{aligned} &P_{1;0,4} = P_{0,4};\ P_{1;4,8} = P_{15,19};\ P_{1;8,17} = P_{6,15};\\ &P_{1;17,19} = P_{4,6};\ P_{1;19,20} = P_{19,20}. \end{aligned}$$

On $C_1$, subindices of the reflective vertices for reflective edges $\delta$ and $\lambda$ are 5 and 8, choose $5' = 17$ and $8' = 19$, find

$$\triangle_{5,8} C_1 = 0P_{2;0,5}3P_{2;5,7}4P_{2;7,16}3P_{2;16,19}4P_{2;19,20}0(= C_2)$$

where

$$P_{2;0,12} = P_{1;0,12};\ P_{2;12,14} = P_{1;17,19};\ P_{2;14,17} = P_{1;14,17};$$
$$P_{2;17,19} = P_{1;12,14};\ P_{2;19,20} = P_{1;19,20}.$$

On $C_2$, subindices of the reflective vertices for reflective edges $\alpha$ and $\sigma$ are 2 and 5, choose $2' = 18$ and $5' = 19$, find

$$\triangle_{5,8} C_2 = 0P_{3;0,2}3P_{3;2,3}4P_{3;3,16}3P_{3;16,19}4P_{3;19,20}0 (= C_3)$$

where

$$P_{3;0,2} = P_{2;0,2};\ P_{2;2,3} = P_{2;18,19};\ P_{3;3,16} = P_{2;5,18};$$
$$P_{3;16,19} = P_{2;2,5};\ P_{3;19,20} = P_{2;19,20}.$$

Because of $\beta(K_5) = 6$, $m = 3 = \lfloor \beta/2 \rfloor$. Thus, the tree-travel $C$ is normal.

This example tells us the problem of determining the maximum orientable genus of a graph can be transformed into that of determining a $k$-tree travel of a graph with $k$ maximum as shown in [4].

**Lemma 2** Among all $k$-tree travel of a graph $G$, the maximum of $k$ is the maximum orientable genus $\gamma_{\max}(G)$ of $G$.

*Proof* In order to prove this lemma, the following two facts have to be known(both of them can be done via the finite recursion principle in §1.3 of [5]!).

**Fact 1** In a connected graph $G$ considered, there exists a spanning tree such that any pair of cotree edges whose fundamental circuits with vertex in common are adjacent in $G$.

**Fact 2** For a spanning tree $T$ with Fact 1, there exists an orientation such that on the protracted tree $\check{T}$, no two articulate subvertices(articulate vertices of $T$) with odd out-degree of cotree have a path in the cotree.

Because of that if two cotree edges for a tree are with their fundamental circuits without vertex in common then they for any other tree are with their fundamental circuits without vertex in common as well, Fact 1 enables us to find a spanning tree with number of pairs of adjacent cotree edges as much as possible and Fact 2 enables us to find an orientation such that the number of times for dong interchange segments successively as much as possible. From Lemma 1, the lemma can be done. □

## §3. Tree-Travel Theorems

The purpose of what follows is for characterizing the embeddability of a graph on a surface of genus not necessary to be zero via $k$-tree travels.

**Theorem 1** *A graph $G$ can be embedded into an orientable surface of genus $k$ if, and only if, there exists a $k$-tree travel $C_k(V;e)$ such that $G_k$ is planar.*

*Proof* Necessity. Let $\mu(G)$ be an embedding of $G$ on an orientable surface of genus $k$. From Lemma 2, $\mu(G)$ has a spanning tree $T$ with its edge subsets $E_0$, $|E_0| = \beta(G) - 2k$, such that $\hat{G} = G - E_0$ is with exactly one face. By successively doing the inverse of interchange segments for $k$ times, a $k$-tree travel is obtained on $\hat{G}$. Let $K$ be consisted of the $k$ pairs of cotree edge subsets. Thus, from Operation 2 in §3.3 of [5], $G_k = G - K = \hat{G} - K + E_0$ is planar.

Sufficiency. Because of $G$ with a $k$-tree travel $C_k(V;e)$, Let $K$ be consisted of the $k$ pairs of cotree edge subsets in successively doing interchange segments for $k$ times. Since $G_k = G - K$ is planar, By successively doing the inverse of interchange segments for $k$ times on $C_k(V;e)$ in its planar embedding, an embedding of $G$ on an orientable surface of genus $k$ is obtained. □

**Example 4** In Example 1, for $G = K_5$, $C$ is a 1-tree travel for the pair of cotree edges $\alpha$ and $\beta$. And, $G_1 = K_5 - \{\alpha, \beta\}$ is planar. Its planar embedding is

$$[4\sigma^{-1}3c^{-1}0d4]_{\text{eg}} = (\sigma^{-1}c^{-1}d);$$
$$[4d^{-1}0a1\gamma4]_{\text{eg}} = (d^{-1}a\gamma);$$
$$[3\sigma4\lambda^{-1}2\delta3]_{\text{eg}} = (\sigma\lambda^{-1}\delta);\ [0c3\delta^{-1}2b^{-1}0]_{\text{eg}} = (c\delta^{-1}b^{-1});$$
$$[2\lambda4\gamma^{-1}1a^{-1}0b2]_{\text{eg}} = (\lambda\gamma^{-1}a^{-1}b).$$

By recovering $\{\alpha, \beta\}$ to $G$ and then doing interchange segments once on $C$, obtain $C_1$. From $C_1$ on the basis of a planar embedding of $G_1$, an embedding of $G$ on an orientable surface of genus 1(the torus) is produced as

$$[4\sigma^{-1}3c^{-1}0d4]_{\text{eg}} = (\sigma^{-1}c^{-1}d);\ [4d^{-1}0a1\gamma4]_{\text{eg}} = (d^{-1}a\gamma);$$
$$[3\sigma4\lambda^{-1}2\delta3\beta^{-1}1a^{-1}0b2\alpha^{-1}1\beta3]_{\text{eg}} = (\sigma\lambda^{-1}\delta\beta^{-1}a^{-1}b2\alpha^{-1}\beta);$$
$$[0c3\delta^{-1}2b^{-1}0]_{\text{eg}} = (c\delta^{-1}b^{-1});\ [2\lambda4\gamma^{-1}1\alpha2]_{\text{eg}} = (\lambda\gamma^{-1}\alpha).$$

Similarly, we further discuss on nonorientable case. Let $G = (V, E)$, $T$ a spanning tree, and

$$C(V;e) = 0P_{0,i}v_iP_{i,j}v_jP_{j,2\epsilon}0 \tag{4}$$

is the travel obtained from 0 along the boundary of protracted tree $\check{T}$. If $v_i$ is a reflective vertex and $v_j = v_i$, then

$$\widetilde{\triangle}_\xi C(V;\epsilon) = 0P_{0,i}v_iP_{i,j}^{-1}v_jP_{j,2\epsilon}0 \tag{5}$$

is called what is obtained by doing a *reverse segment* for the reflective vertex $v_i$ on $C(V;e)$.

If reverse segment can be done for successively $k$ times on $C$, then $C$ is called a $\tilde{k}$*-tree travel*. Because of one reflective edge reduced for each reverse segment and at most $\beta$ reflective edges on $C$, we have $0 \leqslant k \leqslant \beta$ where $\beta = \beta(G)$ is the Betti number of $G$(or corank). When $k = \beta$, $C$(or $G$) is called *twist normal*.

**Lemma 3** A connected graph is twist normal if, and only if, the graph is not a tree.

*Proof* Because of trees no cotree edge themselves, the reverse segment can not be done, this leads to the necessity. Conversely, because of a graph not a tree, the graph has to be with a circuit, a tree-travel has at least one reflective edge. Because of no effect to other reflective edges after doing reverse segment once for a reflective edge, reverse segment can always be done for successively $\beta = \beta(G)$ times, and hence this tree-travel is twist normal. Therefore, sufficiency holds. □

**Lemma 4** Let $C$ be obtained by doing reverse segment at least once on a tree-travel of a graph. Then the polyhegon $[\triangle_i C]_{eg}$ is nonorientable and its genus

$$\widetilde{g}([\triangle_\xi C]_{eg}) = \begin{cases} 2g(C)+1, & \text{when } C \text{ orientable;} \\ \widetilde{g}(C)+1, & \text{when } C \text{ nonorientable.} \end{cases} \tag{6}$$

*Proof* Although a tree-travel is orientable with genus 0 itself, after the first time of doing the reverse segment on what are obtained the nonorientability is always kept unchanged. This leads to the first conclusion. Assume $C_{eg}$ is orientable with genus $g(C)$(in fact, only $g(C) = 0$ will be used!). Because of

$$[\triangle_i C]_{eg} = A\xi B^{-1}\xi C$$

where $[P_{0,i}]_{eg} = A\xi$, $[P_{i,j}]_{eg} = \xi^{-1}B$ and $[P_{j,\epsilon}]_{eg} = C$, From (3.1.2) in [5]

$$[\triangle_i C]_{eg} \sim_{top} ABC\xi\xi.$$

Noticing that from Operation 0 in §3.3 of [5], $C_{rseg} \sim_{top} ABC$, Lemma 3.1.1 in [5] leads to

$$\widetilde{g}([\triangle_\xi C]_{eg}) = 2g([C]_{eg}) + 1 = 2g(C) + 1.$$

Assume $C_{eg}$ is nonorientable with genus $g(C)$. Because of

$$C_{eg} = A\xi\xi^{-1}BC \sim_{top} ABC,$$

$\widetilde{g}([\triangle_\xi C]_{eg}) = \widetilde{g}(C) + 1$. Thus, this implies the second conclusion. □

As a matter of fact, only reverse segment is enough on a tree-travel for determining the nonorientable maximum genus of a graph.

**Lemma 5** Any connected graph, except only for trees, has its Betti number as the nonorientable maximum genus.

*Proof* From Lemmas 3-4, the conclusion can soon be done. □

For a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$ on $G$, the graph $G_{\widetilde{k}}$ is defined as

$$G_{\widetilde{k}} = T\bigcup[E_{ref} - \sum_{j=1}^{k}\{e_j\}] \tag{7}$$

where $T$ is a spanning tree, $[X]$ the induced graph of edge subset $X$, and $e \in E_{ref}$ and $\{e_j, e_j'\}$, respectively, a reflective edge and that used for reverse segment.

**Theorem 2** *A graph $G$ can be embedded into a nonorientable surface of genus $k$ if, and only if, $G$ has a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$ such that $G_{\widetilde{k}}$ is planar.*

*Proof* From Lemma 3, for $k$, $1 \leqslant k \leqslant \beta(G)$, any connected graph $G$ but tree has a $\widetilde{k}$-tree travel.

Necessity. Because of $G$ embeddable on a nonorientable surface $S_{\widetilde{k}}$ of genus $k$, let $\widetilde{\mu}(G)$ be an embedding of $G$ on $S_{\widetilde{k}}$. From Lemma 5, $\widetilde{\mu}(G)$ has a spanning tree $T$ with cotree edge set $E_0$, $|E_0| = \beta(G) - k$, such that $\widetilde{G} = G - E_0$ has exactly one face. By doing the inverse of reverse segment for $k$ times, a $\widetilde{k}$-tree travel of $\widetilde{G}$ is obtained. Let $K$ be a set consisted of the $k$ cotree edges. From Operation 2 in §3.3 of [5], $G_{\widetilde{k}} = G - K = \widetilde{G} - K + E_0$ is planar.

Sufficiency. Because of $G$ with a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$, let $K$ be the set of $k$ cotree edges used for successively dong reverse segment. Since $G_{\widetilde{k}} = G - K$ is planar, by successively doing reverse segment for $k$ times on $C_{\widetilde{k}}(V;e)$ in a planar embedding of $G_{\widetilde{k}}$, an embedding of $G$ on a nonorientable surface $S_{\widetilde{k}}$ of genus $k$ is then extracted. □

**Example 5** On $K_{3,3}$, take a spanning tree $T$, as shown in Fig.2(a) by bold lines. In (b), given a protracted tree $\check{T}$ of $T$. From $\check{T}$, get a tree-travel

$$C = 0P_{0,11}2P_{11,15}2P_{15,0}0 \ (= C_0)$$

where $v_0 = v_{18}$ and

$$P_{0,11} = c4\delta5\delta^{-1}4\gamma3\gamma^{-1}4c^{-1}0d2e3\beta1\beta^{-1}3e^{-1};$$
$$P_{11,15} = d^{-1}0a1b5\alpha;$$
$$P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}.$$

Because of $v_{15} = 2$ as the reflective vertex of cotree edge $\alpha$ and $v_{11} = v_{15}$,

$$\triangle_3 C_0 = 0P_{1;0,11}2P_{1;11,15}2P_{1;15,0}0 \ (= C_1)$$

where

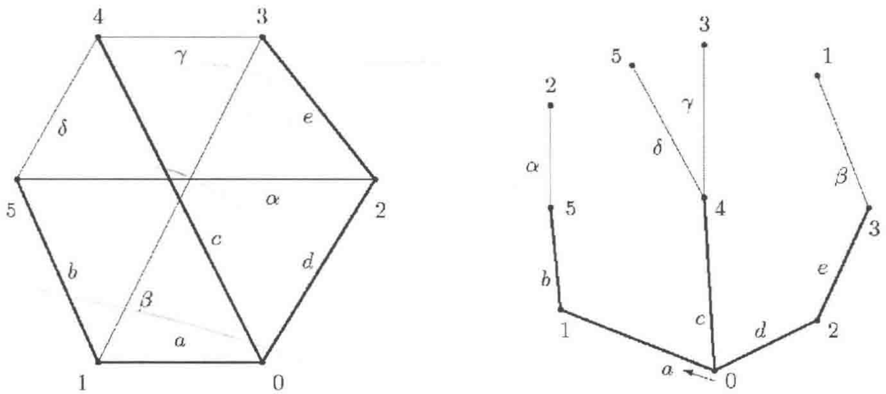
$$P_{1;0,11} = P_{0,11} = c4\delta5\delta^{-1}4\gamma3\gamma^{-1}4c^{-1}0d2e3\beta1\beta^{-1}3e^{-1};$$
$$P_{1;11,15} = P_{11,15}^{-1} = \alpha^{-1}5b^{-1}1a^{-1}0d;$$
$$P_{1;15,0} = P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}.$$

Since $G_{\widetilde{1}} = K_{3,3} - \alpha$ is planar, from $C_0$ we have its planar embedding

$$f_1 = [5P_{16,0}0P_{0,2}0]_{\text{eg}} = (b^{-1}a^{-1}c\delta);$$
$$f_2 = [3P_{4,8}3]_{\text{eg}} = (\gamma^{-1}c^{-1}de);$$
$$f_3 = [1P_{13,14}5P_{2,4}3P_{8,9}1]_{\text{eg}} = (\delta^{-1}\gamma\beta b);$$
$$f_4 = [1P_{9,13}1]_{\text{eg}} = (\beta^{-1}e^{-1}d^{-1}a).$$

By doing reverse segment on $C_0$, get $C_1$. On this basis, an embedding of $K_{3,3}$ on the projective plane(*i.e.*, nonorientable surface $S_{\widetilde{1}}$ of genus 1) is obtained as

$$\begin{cases} \widetilde{f}_1 = [5P_{1;16,0}0P_{1;0,2}0]_{\text{eg}} = f_1 = (b^{-1}a^{-1}c\delta); \\ \widetilde{f}_2 = [3P_{1;4,8}3]_{\text{eg}} = f_2 = (\gamma^{-1}c^{-1}de); \\ \widetilde{f}_3 = [1P_{1;9,11}2P_{1;11,13}1]_{\text{eg}} = be^{-1}e^{-1}\alpha^{-1}b^{-1}); \\ \widetilde{f}_4 = [0P_{1;14,15}2P_{1;15,16}5P_{1;2,4}3P_{1;8,9}1P_{1;13,14}0]_{\text{eg}} \\ \quad = (d\alpha^{-1}\delta^{-1}\gamma\beta a^{-1}). \end{cases}$$




(a) $T$ (b) $\check{T}$

**Fig.2**

§4. **Research Notes**

**A.** For the embeddability of a graph on the torus, double torus *etc* or in general orientable surfaces of genus small, more efficient characterizations are still necessary to be further contemplated on the basis of Theorem 1.
**B.** For the embeddability of a graph on the projective plane(1-crosscap), Klein bottle(2-crosscap), 3-crosscap *etc* or in general nonorientable surfaces of genus small, more efficient characterizations are also necessary to be further contemplated on the basis of Theorem 2.
**C.** Tree-travels can be extended to deal with all problems related to embedings of a graph on surfaces as joint trees in a constructive way.

**References**

[1] Archdeacon, D., A Kuratowski theorem for the projective plane, *J. Comb. Theory*, 5 (1981), 243–246.
[2] Liu, Yi. and Liu, Y.P., A characterization of the embeddability of graphs on the surface of given genus, *Chin. Ann. Math.*, 17B(1996), 457–462.
[3] Liu, Y.P., The nonorientable maximum genus of a graph(in Chinese with English abstract), *Scintia Sinica*, Special Issue on Math. I(1979), 191–201.
[4] Liu, Y.P., The maximum orientable genus of a graph, *Scientia Sinica*, Special Issue on Math. II(1979), 41–55.
[5] Liu, Y.P., *Topological Theory on Graphs*, USTC Press, 2008.

# Flexibility of Circular Graphs $C(2n, 2)$ on the Projective Plane*

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**Abstract** In this paper, we study the flexibility of embeddings of circular graphs $C(2n, 2)$, $n \geq 3$ on the projective plane. The numbers of (nonequivalent) embeddings of $C(2n, 2)$ on the projective plane are obtained, and by describing structures of these embeddings, the numbers of (nonequivalent) weak embeddings and strong embeddings of $C(2n, 2)$ on the projective plane are also obtained.

**Keywords** Circular graph; embedding; weak embedding; strong embedding; joint tree.

**Mathematics Subject Classification** 05C10, 05C30

## 1 Introduction

A *surface* is a compact 2-dimensional manifold without boundary. It can be represented by a polygon of even edges in the plane. Furthermore, it can be also written by words, for example, the plane is written as $O_0 = aa^-$, the projective plane $N_1 = aa$. See [8,13] for more detail. In this way, some topological transformations and operations on surfaces can be represented by words easily. For example, the following relations can be deduced, as shown in, e.g.,[8].

Relation 1: $(AxByCx^-Dy^-) \sim ((ADCB)(xyx^-y^-))$,

Relation 2: $(AxBx) \sim ((AB^-)(xx))$,

Relation 3: $(Axxyzy^-z^-) \sim ((A)(xx)(yy)(zz))$.

In which $A, B, C$, and $D$ are all linear orders of letters and permitted to be empty. Parentheses are always omitted when the letters in parentheses represent surfaces. $\sim$ means topological equivalence on surfaces.

An *embedding* of a graph $G$ on a surface $S$ is a homeomorphism $h : G \to S$ of $G$ into $S$ such that every component of $S - h(G)$ is a 2-cell. Two embeddings $h : G \to S$ and $g : G \to S$ of $G$ on a surface $S$ are said to be *equivalent* if there is an homeomorphism $f : S \to S$ such that $f \circ h = g$. The connected components of $S - h(G)$ are called *faces* of the embedding. A *weak embedding* of a graph $G$

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