刘彦佩

# 半闲数学集锦

## Semi-Empty Collections in Mathematics by Y.P.Liu

第三编

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**半闲数学集锦**（第三编）

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# 第三编序

这一编所反映的是有关专著 *Embeddability in Graphs*[141](3.11—3.29) 的形成与发展.

这本书原稿中的主要内容, 都在写专著 **图的可嵌入性理论**[137](2.01—2.20) 时选用. 在后者出版之后, 又重新修订前者的全部内容, 特别是英文的表述. 出版后, 在国内外同时发行.

这两本书, 就是我早期培养研究生, 主要是博士研究生, 所授专业课之用. 根据研究方向, 选取其中有关的一些内容.

文 3.01[028]—3.06[138] 都为专著 [141] 提供了素材, 和文 3.07[149]—3.10[155] 都是由专著 [141] 引起的.

例如, 文 3.01[028] 提供的三个不等式, 由于它们的普遍性, 而被纳入 3.16 的一节之中. 文 3.02[048]—3.05[096] 反映在研究图的平面性和平面嵌入时, 如何逐步提高理论结果的有效性, 和易读性的过程. 它们在 3.17 和 3.19 中有所利用. 文 3.06[138], 主要在于揭示, 在图上, 一对有关同调与上同调定理, 如何一举导出 Whitney, MacLane, 以及 Lefschetz 三个, 被当时人们认为, 独立的不同定理. 事实上, 这个定理还意味 Euler 必要条件的充分性. 形成 3.15 和 3.24 中的, 一部分内容.

例如, 文 3.07[148], 从 3.24, 有关上可嵌入性, 的内容引起. 文 3.08[152], 是由 3.16 中, 对于以多面形观点, 看 Jordan 定理, 提出的三个等价条件所引起. 文 3.09[153] 和文 3.10[155], 都是与 3.25 中讨论的, 双循环有关.

在专著 [141] 中, 贯穿着以下三条主线:

- 根据导出的一些原则, 以平面性为例, 构建一个边带二元权的图, 称之为演生网, 最初称派生图, 或平面性辅助图. 通过演生网上的变换, 一步一步地, 减少问题的复杂度, 以图将问题有效化, 使得形成一种有效的, 或相对有效的理论.

这一路线, 不仅适于判定一个图的平面性, 产生 3.17, 还导致求一个可平面图的一个平面嵌入 (见 3.18), 对纽结在平面上投影的识别(即 Gauss 平面闭曲线猜想的证明, 见 3.19), 图的平面分解(见 3.23), 拟阵图性与上图性的识别(见 3.26), 以及建立纽结的一类新的多项式不变量(见 3.27)等.

- 将图视为多面形的集合. 在多面形上, 建立不同维数的空间. 通过这些空间间的边缘与上边缘运算, 产生各种有利用价值的子空间. 在此基础上, 形成同调与上同调, 从而通过它们, 揭示出图上蕴藏着的对偶性, 以及其它全局性质.

例如, 沿着这一路线, 对于图的平面性的判别, 得到了用同调与上同调描述的一对定理, 从这对定理, 一并直接导出了 Whitney 定理, MacLane 定理和 Lefschetz 定理, 见 3.15. 对于拟阵, 提供图性与上图性的识别, 见 3.26.

在此基础上, 同样导出 Kuratowski 定理. 还可以从多面形的对偶出发, 揭示 Wu(吴文俊)-Tutte 定理的对偶形式. 更值得注意地, 沿着这一路线, 还导出了一个, 表述十分简洁的, 判定一个图是, 或否, 可以嵌入到, 任意给定亏格(包括可定向与不可定向!)曲面上, 的一个准则. 这些都是在 10 余年后, 才澄清的, 例如 5.16[348], 18.54[453], 19.07[473], 19.30[486], 23.01[487] 等.

- 为了揭示一个组合构形的某种性质, 这种性质具有遗传性, 先调查没有此性质的极小构形, 如果找到这种极小构形的一个集合, 使得任何一个, 不具备这一性质组合构形, 必有一个子构形, 与这个集合中的, 一个构形同构.

在图论, 或网络理论中, 上面的一条主线, 就是用一个完备禁用次形集, 刻画一种全局性质的扩充. 之谓扩充是因为在 3.20, 3.21 和 3.25 中, 所用到的, 都不是次形. 实际上, 它们都是由半边(不是边!)所组成的, 伴之以圈, 或上圈, 之类.

上面没有提及的, 都是为了阅读方便, 而设立的. 例如, 3.11 和 3.12 分别为专著[141]的序和目录. 3.28 和 3.29 分别为专著[141]的参考文献与名词索引等.

刘彦佩<br>2015 年 5 月<br>於北京上园村

# 第三编目录

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# $k$-VALENT MAPS ON THE SURFACES*

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**Abstract**

In this paper, the works of Kotzig[1], Zaks[3] and the author[2] have been generalized and unified. A number of new results have also been obtained.

## I. Introduction

A map $M$ means an embedding of a graph $G=(V,E)$ on the surface $S$. $G$ is said to be the underlying graph of $M$. For convenience, we consider $G$ to be simple (with no loops or multi-edges) and 2-connected. And all the valencies of vertices and faces are at least 3.

Let $\nu$, $\varepsilon$ and $\varphi$ denote the number of vertices, edges and faces of $M$ respectively, and let $\nu_i$, $\varphi_i$ denote the number of vertices and faces of valencies $i$, $i\geqslant 3$, respectively.

A map $M$ is said to be a $k$-valent map, or $k$-map for short, if all the valencies of vertices are $k$, $k\geqslant 3$. In this paper, a number of properties related to the valencies of vertices and faces for $k$-maps on the surfaces are considered. Part of the results obtained here are, in fact, a kind of generalization and unification of [1], [3] and [2].

## II. The First Inequality

For a $k$-map $M$, the following relations are known:

$$k\nu=2\varepsilon=\sum_{j\geqslant 3}j\varphi_j; \tag{2.1}$$

$$\varphi=\sum_{j\geqslant 3}\varphi_j; \tag{2.2}$$

$$\nu=\sum_{j\geqslant 3}\nu_j=\nu_k. \tag{2.3}$$

**Lemma 2.1.** For any $k$-map $M$ on the surface $S$, we always have

$$\sum_{j\geqslant 3}(2k-(k-2)j)\varphi_j=2k\delta(S), \tag{2.4}$$

where $S=S_p$, $p\geqslant 0$, or $N_q$, $q\geqslant 0$, with $N_0=S_0$, which are the orientable, non-orientable surfaces of genus $p$, $q$ respectively, and

$$\delta(S)=\begin{cases}2-2p, & \text{if } S=S_p,\ p\geqslant 0;\\ 2-q, & \text{if } S=N_q,\ q\geqslant 0.\end{cases} \tag{2.5}$$

*Proof.* From (2.1), (2.2) and the Euler formula,

$$\sum_{j\geqslant 3}(2k-(k-2)j)\varphi_j=2\sum_{j\geqslant 3}j\varphi_j-k\sum_{j\geqslant 3}j\varphi_j+2k\sum_{j\geqslant 3}\varphi_j=2k(\nu-\varepsilon+\varphi)=2k\delta(S).$$

A face whose valency is $t$ is said to be a $t$-gon. Let $m_i^t$ be the number of vertices

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each of which is incident to $i$ $t$-gons, $i$, $t\geqslant 3$. Let $\lambda^t$ be the number of pairs $(v, T)$, where $v$ is a vertex incident to a $t$-gon $T$. Then

$$t\varphi_t=\lambda^t=\sum_{i=1}^{k} im_i^t. \tag{2.6}$$

In addition,

$$\nu=\sum_{i=0}^{k} m_i^t, \quad \text{for any } t\geqslant 3. \tag{2.7}$$

Thus, for any $l$, $3\leqslant l\leqslant k$,

$$t\varphi_t\leqslant l\nu+\sum_{j=1}^{k-l} jm_{l+j}^t; \tag{2.8}$$

the equality holds iff $m_0^t=m_1^t=\cdots=m_{l-1}^t=0$.

**Theorem 2.1.** For any $k$-map $M$ on the surface $S$, and any $t\geqslant 3$, $t\neq\frac{2k}{k-2}$, $3\leqslant l<k$, we have

$$\sum_{s=1}^{k-l} sm_{l+s}^t\geqslant\frac{2(k-l)t\delta(S)}{2k-(k-2)t}+\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{((k-2)t-2l)j-2(k-l)t}{2k-(k-2)t}\varphi_j. \tag{2.9}$$

*Proof.* From (2.8), (2.1) and (2.4),

$$\sum_{s=1}^{k-l} sm_{l+s}^t\geqslant t\varphi_t-l\nu=\left(1-\frac{l}{k}\right)t\varphi_t-\frac{l}{k}\sum_{\substack{j\geqslant 3\\ j\neq t}} j\varphi_j$$

$$=\left(\frac{k-l}{k}\right)t\left(\frac{2k\delta(S)}{2k-(k-2)t}-\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{2k-(k-2)j}{2k-(k-2)t}\varphi_j\right)-\frac{l}{k}\sum_{\substack{j\geqslant 3\\ j\neq t}} j\varphi_j$$

$$=\frac{2(k-l)t\delta(S)}{2k-(k-2)t}-\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{2(k-l)t-((k-2)t-2l)j}{2k-(k-2)t}\varphi_j.$$

For convenience, (2.9) is called the first inequality. The right hand side of the inequality is denoted by

$$\Phi=\Phi(S;\varphi_3,\varphi_4,\cdots). \tag{2.10}$$

## III. Deductions from the First Inequality

First, we investigate the upper bound of $k$.

**Corollary 3.1.** For any $k$-map $M$ on the surface $S$ with $\delta(S)>0$, it is only possible that $k\leqslant 5$. That is to say, on the sphere or the projective plane, there are at most five maps, or the connectivity of the underlying graph of any $k$-map on the sphere or the projective plane is at most 5.

*Proof.* Even if all the faces were triangles, the first term of (2.4) would be $2k-(k-2)3\leqslant 0$, if $k\geqslant 6$.

Similarly, we have the following

**Corollary 3.2.** For a $k$-map $M$ on the torus or the Klein bottle, we have $k\leqslant 6$. If $k=6$, all the faces are triangles.

**Conjecture 3.1.** For any surface $S$, $\delta(S)<0$, there exists a $k$-map, $k\geqslant 3$.

When $k$ is fixed, the bound of the minimum valency of faces may be determined.

**Corollary 3.3.** On the sphere or the projective plane, in any 3-map, there exists either a triangle, or a quadrangle, or a pentagon; in any 4- or 5-map, there exists a triangle. On the torus or the Klein bottle, in any 3-map, there exists either a triangle, or a quadrangle, or a pentagon, or a hexagon; in any 4-map, there exists either a triangle or a quadrangle; in any 5- or 6-map, there exists a triangle.

**Corollary 3.4.** On the surface $S$ with $\delta(S)<0$, in any $k$-map, there exists a polygon (i.e. a face) with valency not greater than $\lfloor 2k(\delta(S)-1)/(2-k)\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer not greater than $x$ and $\lceil x\rceil$ the least integer not less than $x$.

In the following we shall discuss the bound of $m_l^t$, $t\geqslant 3$, $3\leqslant l<k$.

**Lemma 3.1.** For $k=5$, $t=3$, $l=3$ or 4, all the coefficients of $\varphi_j$, $j\geqslant\lceil 2(k-l)t/((k-2)t-2l)\rceil$, in (2.9) are nonnegative.

*Proof.* Since $j\geqslant\lceil 2(k-l)t/((k-2)t-2l)\rceil$, the numerators of $\dfrac{\partial\Phi}{\partial\varphi_j}$ are nonnegative; and $t<\dfrac{2k}{k-2}$, so are the denominators.

Let

$$N=\lfloor 2(k-l)t/((k-2)t-2l)\rfloor+1 \quad \text{if} \quad (k-2)t-2l\neq 0.$$

**Corollary 3.5.** If $M$ is a 5-map on the surface $S$ with $\delta(S)>0$, or if in case $\delta(S)=0$ there exists $j_0\geqslant N$ such that $\varphi_{j_0}>0$ and $\varphi_j=0$, $3\leqslant j<N$, then for $l=3$ or 4, there exists a vertex incident to at least $l+1$ triangles.

*Proof.* Using Lemma 3.1 and Theorem 2.1 for $k=5$, $t=3$, we have

$$\sum_{s=1}^{k-l} s m_{l+s}^t\geqslant\begin{cases}\dfrac{2(k-l)t\delta(S)}{2k-(k-2)t}>0, & \text{if } \delta(S)>0,\\[2ex] \dfrac{((k-2)t-2l)j_0-2(k-l)t}{2k-(k-2)t}\varphi_{j_0}>0, & \text{if } \delta(S)=0,\end{cases}$$

i.e. there exists $m_{l+s}^t>0$, for some $s$, $1\leqslant s\leqslant k-l$.

**Remark 3.1.** This corollary contains Theorem 8 of [1], Corollary 1 of [3] and Corollary 2.1 of [2] as special cases.

## IV. The Second Inequality and Its Applications

From (2.9), (2.7) and (2.10), we have

$$m_k^t\geqslant-(k-l-1)\nu+\Phi(S;\ \varphi_3,\ \varphi_4,\ \cdots). \tag{4.1}$$

**Theorem 4.1.** For any $k$-map $M$ on the surface $S$, and any $t\geqslant 3$, $t\neq\dfrac{2k}{k-2}$, we have

$$m_k^t\geqslant\Psi=\Psi(S;\ \varphi_3,\ \varphi_4,\ \cdots), \tag{4.2}$$

where

$$\Psi=\frac{2t\delta(S)}{2k-(k-2)t}+2\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{((k-2)t-2(k-1))j-2t}{2k-(k-2)t}\varphi_j. \tag{4.3}$$

*Proof.* From (2.1) and (2.4), we may obtain

$$\nu=\frac{2t\delta(S)}{2k-(k-2)t}+2\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{j-t}{2k-(k-2)t}\varphi_j. \tag{4.4}$$

Then, by substituting (4.4) for $\nu$ in (4.1), inequality (4.2) may be derived.

**Lemma 4.1.** If $k=3$, $t=5$, or $k=5$, $t=3$, then in (4.3),

$$\frac{\partial\Psi}{\partial\varphi_j}>0 \tag{4.5}$$

for all $j\geqslant N_0=\lfloor 2t/((k-2)t-2(k-1))\rfloor+1$.

*Proof.* Since $t<\frac{2k}{k-2}$, the denominators of $\frac{\partial\Psi}{\partial\varphi_j}$, $j\geqslant N_0$, are positive, so are the numerators, for $j\geqslant N_0$.

**Lemma 4.2.** For $k=3$, $t=5$, or $k=5$, $t=3$, we have

$$\frac{\partial\Psi}{\partial\varphi_j}\Big/\frac{\partial\Psi}{\partial\varphi_n}\geqslant\frac{\partial\nu}{\partial\varphi_j}\Big/\frac{\partial\nu}{\partial\varphi_n},\quad \frac{\partial\Psi}{\partial\delta}\Big/\frac{\partial\Psi}{\partial\varphi_n}\geqslant\frac{\partial\nu}{\partial\delta}\Big/\frac{\partial\nu}{\partial\varphi_n}\tag{4.6}$$

for $j\geqslant n\geqslant N_0$.

*Proof.* By (4.3) and (4.4),

$$\begin{aligned}&\left(\frac{\partial\Psi}{\partial\varphi_j}\Big/\frac{\partial\Psi}{\partial\varphi_n}\right)-\left(\frac{\partial\nu}{\partial\varphi_j}\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\\&\quad=\frac{1}{\lambda}((n-t)(((k-2)t-2(k-1))j-2t)-(j-t)(((k-2)t-2(k-1))n-2t))\\&\quad=\frac{1}{\lambda}(2k-(k-2)t)t(j-n)\geqslant 0,\quad\text{if}\quad \lambda\geqslant 0.\end{aligned}$$

However, $\lambda=(n-t)(((k-2)t-2(k-1))n-2t)\geqslant(N_0-t)(((k-2)t-2(k-1))N_0-2t)>0$. Similarly for the other part.

Inequality (4.2) is called the second inequality. Using it, we may determine a lower bound of $m_k^t$.

**Corollary 4.1.** For any $k$-map $M$ on the surface $S$, $\delta(S)\geqslant 0$, $k=3$, $t=5$, or $k=5$, $t=3$, if there exists an integer $n\geqslant N_0$ such that $\varphi_j=0$, $3\leqslant j\leqslant n-1$, $j\neq t$, $\varphi_n\neq 0$, then

$$m_k^t\geqslant\frac{((k-2)t-2(k-1))n-2t}{2(n-t)}\nu;\tag{4.7}$$

the inequality is strict when $\delta(S)\neq 0$.

*Proof.* From (4.2), (4.5) and Lemma 4.2,

$$\begin{aligned}m_k^t&\geqslant\frac{\partial\Psi}{\partial\varphi_n}\left(\left(\frac{\partial\Psi}{\partial\delta}\Big/\frac{\partial\Psi}{\partial\varphi_n}\right)\delta(S)+\sum_{j\geqslant n}\left(\frac{\partial\Psi}{\partial\varphi_j}\Big/\frac{\partial\Psi}{\partial\varphi_n}\right)\varphi_j\right)\\&\geqslant\frac{\partial\Psi}{\partial\varphi_n}\left(\left(\frac{\partial\nu}{\partial\delta}\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\delta(S)+\sum_{j\geqslant n}\left(\frac{\partial\nu}{\partial\varphi_j}\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\varphi_j\right)\\&=\frac{\partial\Psi}{\partial\varphi_n}\left(\nu\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\\&=\frac{((k-2)t-2(k-1))n-2t}{2(n-t)}\nu.\end{aligned}$$

**Remark 4.1.** Corollary 4.1 contains Corollaries 3.0 and 3.1 in [3] and Corollary 2.2 in [2] as special cases.

Similarly to Lemmas 4.1, 4.2, for any $j\geqslant n\geqslant N$, we may obtain

$$\frac{\partial\Phi}{\partial\varphi_j}\Big/\frac{\partial\Phi}{\partial\varphi_n}\geqslant\frac{\partial\nu}{\partial\varphi_j}\Big/\frac{\partial\nu}{\partial\varphi_n},\quad \frac{\partial\Phi}{\partial\delta}\Big/\frac{\partial\Phi}{\partial\varphi_n}\geqslant\frac{\partial\nu}{\partial\delta}\Big/\frac{\partial\nu}{\partial\varphi_n}.\tag{4.8}$$

**Corollary 4.2.** For any $k$-map $M$ on the surface $S$, $\delta(S)\geqslant 0$, $k=5$, $t=3$, $l=3$ or 4, if there exists an integer $n\geqslant N$ such that $\varphi_j=0$, $3\leqslant j\leqslant n-1$, $j\neq t$, and $m_{s+l}^t=0$, $1\leqslant s\leqslant k-l$, except for $s=r$, $\varphi_n\neq 0$, then

$$m_r^t\geqslant\frac{1}{2r}\,\frac{((k-2)t-2l)n-2(k-l)t}{n-t}\,\nu.\tag{4.9}$$

*Proof.* By (2.9) and (4.8)

$$m_{r+l}^t \geqslant \frac{1}{r}\frac{\partial\Phi}{\partial\varphi_n}\left(\left(\frac{\partial\Phi}{\partial\delta}\Big/\frac{\partial\Phi}{\partial\varphi_n}\right)\delta(S)+\sum_{j>n}\left(\frac{\partial\Phi}{\partial\varphi_j}\Big/\frac{\partial\Phi}{\partial\varphi_n}\right)\varphi_j\right)$$
$$\geqslant \frac{1}{r}\frac{\partial\Phi}{\partial\varphi_n}\left(\left(\frac{\partial\nu}{\partial\delta}\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\delta(S)+\sum_{j>n}\left(\frac{\partial\nu}{\partial\varphi_j}\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\varphi_j\right)$$
$$=\frac{1}{r}\left(\frac{\partial\Phi}{\partial\varphi_n}\Big/\frac{\partial\nu}{\partial\varphi_n}\right)\nu=\frac{1}{2r}\,\frac{((k-2)t-2l)n-2(k-l)t}{n-t}\,\nu.$$

**Remark 4.2.** Corollary 4.2 contains Corollaries 4.0 and 4.1 in [3] and Corollary 2.2 for $S=S_0$, $N_1$, $N_2$ in [2] as special cases.

## V. An Identity

For any $k$-map $M$ on the surface $S$, let $s_{i,j}$ be the number of edges incident to two faces one of which has valency $i$, and the other, $j$. Then

$$j\varphi_j=s_{j,j}+\sum_{i>3}s_{i,j}. \tag{5.1}$$

From (2.4) we have

$$\varphi_t=\frac{2k\,\delta(S)}{2k-(k-2)t}-\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{2k-(k-2)j}{2k-(k-2)t}\left(\frac{1}{j}s_{j,j}+\sum_{i>3}\frac{1}{j}s_{i,j}\right), \tag{5.2}$$

provided $2k-(k-2)t\neq 0$.

For convenience, let

$$A_{t,j}=\begin{cases}\dfrac{2k-(k-2)j}{2k-(k-2)t}, & j\neq 0,\ j\neq t;\\[2ex] \dfrac{2k}{2k-(k-2)t}, & j=0.\end{cases} \tag{5.3}$$

Thus

$$\varphi_t=A_{0,t}\delta(S)-2\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{A_{t,j}}{j}s_{j,j}-\sum_{\substack{j\geqslant 3\\ j\neq t}}\frac{A_{t,j}}{j}s_{t,j}-\sum_{\substack{j\geqslant 3\\ j\neq t}}\sum_{\substack{i\geqslant j+1\\ i\neq t}}\left(\frac{A_{t,i}}{i}+\frac{A_{t,j}}{j}\right)s_{i,j}=A(s_{i,j}). \tag{5.4}$$

Substituting (5.1) for $\varphi_t$ in (5.4), we obtain

$$\frac{2}{t}s_{t,t}+\sum_{\substack{i\geqslant 3\\ i\neq t}}\frac{1}{t}s_{i,t}=A(s_{i,j}). \tag{5.5}$$

**Theorem 5.1.** For any $k$-map $M$ on the surface $S$, $t\geqslant 3$, $t\neq\frac{2k}{k-2}$, satisfying $\varphi_j=0$, $j<t$, and $s\geqslant 0$, we have

$$\left(\frac{2k}{t}-(k-2)\right)s_{t,t}+\sum_{l=1}^{s}\left(\frac{k}{t}+\frac{k}{t+l}-(k-2)\right)s_{t,t+l}$$
$$=k\,\delta(S)+\sum_{j\geqslant t+1}\left(k-2-\frac{2k}{j}\right)s_{j,j}+\sum_{j\geqslant t+s+1}\left(k-2-\frac{k}{t}-\frac{k}{j}\right)s_{t,j}$$
$$+\sum_{j\geqslant t+1}\sum_{i\geqslant j+1}\left(k-2-\frac{k}{i}-\frac{k}{j}\right)s_{i,j}. \tag{5.6}$$

*Proof.* By multiplying the two sides of (5.5) by $2k-(k-2)t$, identity (5.6) may be obtained.

**Lemma 5.1.** If $k=3$, $t=5$, $s=2$, or $k=4$, $t=3$, $s=2$, or $k=5$, $t=3$, $s=0$, then all the coefficients of $s_{i,j}$, $i$, $j\geqslant t$, in (5.6) are nonnegative.

*Proof.* Suffice it to show that in (5.6)

$$\begin{cases} (2k/t)-(k-2)>0; \\ \dfrac{k}{t}+\dfrac{k}{t+s}-(k-2)\geqslant 0; \\ k-2-\dfrac{k}{t}-\dfrac{k}{t+s+1}\geqslant 0; \\ k-2-\dfrac{2k}{t+1}\geqslant 0. \end{cases} \tag{5.7}$$

It is just guaranteed by the conditions.

**Corollary 5.1.** For any $k$-map on the surface $S$, $\delta(S)\geqslant 0$, if the conditions of Lemma 5.1 hold, then there exists an edge $e$ such that when $\delta(S)>0$,

$$\sigma(e)\leqslant 2t+s. \tag{5.8}$$

Furthermore, let $\sigma(e)$ denote the sum of the valencies of the two faces incident to $e$. If $k=4$, $t=3$, $s=2$, $\delta(S)=0$, and $\sigma(e)\geqslant 9$ for all the edges $e$, then for each $e$

$$\sigma(e)=9$$

and $e$ is incident to a triangle and a hexagon.

*Proof.* The first part may be proved as usual. As for the second part, the only thing we should pay attention to is that in this case, all the coefficients of $s_{i,j}$, $i, j\geqslant t$, are positive except those of $s_{t,t+s+1}$.

**Remark 5.1.** Corollary 5.1 is a more general form of Theorem 3 in [1], Corollary 5.1 in [3] and Corollaries 4.1, 4.2, 5.3 in [2].

**Note.** All the results obtained have their dual (planar or surface) forms.

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# ON THE LINEARITY OF TESTING PLANARITY OF GRAPHS

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**Abstract**

In 1978, the author published a paper in which a characteristic theorem of planarity of a graph was provided as determining if another graph has a fundamental circuit with a certain property. However, the new graph is with, at worst, quadratic order of the vertex number of the original graph[2].

This paper presents a new criterion of testing planarity of a graph based on what the author obtained before. Fortunately, it is equivalent to finding a spanning tree in another graph with only linear order of the vertex number of the original one in the worst case.

## § 1. Introduction

In the seventies, W. Wu discovered that testing planarity of a graph can be transformed into solving linear equations on $GF(2)$ based on cohomology theory in algebraic topology[8]. Then, Y. Liu found a criterion of planarity which seemed to be much simpler[2]. In fact, the only thing that remained for testing planarity was to solve such linear equations on $GF(2)$ in each of which there were at most two variables. Furthermore, the problem was transformed into finding a circuit with a certain property or a tree in another graph $H$ related to $G$, the original one.

In the 1979 Montreal Conference on Combinatorics, P. Rosenstiehl proved the result again in an algebraic way[4]. In a private communication, P. Rosenstiehl told Y. Liu that he and his colleague obtained an algorithm in linear time. However, he had not mentioned what method they used. Of course, the first linear time algorithm on this topic was due to J. E. Hopcroft and R. E. Tarjan whose paper was published in 1974[1]. The depth-first search tree technique they used plays a substantial role in the simplification.

This raper provides a new criterion which, in fact, is a simplified form of the one we obtained in [2]. Fortunately, from this criterion, a linear time algorithm for testing planarity of a graph and embedding a planar graph into the plane can be

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deduced. However, the procedure has been described for 3-regular graphs with a depth-first search tree being a path.

## § 2. A Criterion of Planarity

Let $G=(V, E)$ be a graph with $V$ being the vertex set, $E$ the edge set. Or $G$ is treated as a 1-complex in Euclidean space with $G^0=V$ as 0-simplex set, $G^1=E$, 1-simplex set. $T$ denotes a spanning tree of $G$, $T=(T^0, T^1)$, $T^0=G^0$. Here, only depth-first search trees are considered[1]. $\bar{T}=(\bar{T}^0, \bar{T}^1)=(G^0, G^1-T^1)$ is the cotree corresponding to $T$. Let $T_D(G)$ be the set of all the depth-first search trees of $G$. And, for $T\in T_D(G)$, let $\prec$ be the partial order on $G^0$ determined by $T$.

**Proposition 2.1**[5]. *For any $T\in T_D(G)$, there exists a unique orientation of the edges of $G$, e. g., $e=\langle u, v\rangle$ representing $u$ to $v$ such that*

(i) $u\prec v$, if $e\in T^1$;

(ii) $u\succ v$, if $e\in \bar{T}^1$.

Thus the vertices of $G$ can be labelled so that $\prec$ becomes $<$. In what follows, all the vertices are treated as non-negative integers. For $e\in T^1$, there is a unique cocircuit $\bar{C}_e(T)$, called a fundamental cocircuit, with all the edges in $\bar{C}^1_e(T)$ being in $\bar{T}^1$ save only for one edge. And, for $\alpha\in\bar{T}^1$, there is a unique circuit $C_\alpha(T)$, called a fundamental circuit, with all the edges in $C^1_\alpha(T)$ being in $T^1$ except for $\alpha$. A circuit $C$ or path $P$ with all edges in $C^1$ or $P^1$ having the same direction is said to be a dicircuit or dipath respectively.

**Proposition 2.2.** *For $T\in T_D(G)$, all the fundamental circuits of $G$ are dicircuits and each fundamental cocircuit of $G$ has all its edges with the same direction saving only for one edge which belongs to $T^1$.*

The vertex with the minimum label is said to be the root. The minimum label is always set to be 0,

**Proposition 2.3.** *For $\langle u, v\rangle\in T^1$, we always have*

(iii) $0\leqslant\nu\leqslant u$ and $v\in P^0\langle 0, u\rangle$, *for all $\langle\mu, \nu\rangle\in\bar{C}^1_{\langle u,v\rangle}(T)$. At each vertex, there is exactly one incoming edge in $T^1$ except for the root. And, for 2-connected graph $G$, the root has valency 1 on $T$.*

For $v\in V$, let $E_v=\{e\mid e\in E$ and $e$ is incident to $v\}$. From Proposition 2.3, we have

$$E_v=e_v\cup E_v(T)\cup E_v(\bar{T}), \tag{2.1}$$

where $e_v$ is the tree edge coming to $v$, $E_v(T)=E_v\cap T^1$, $E_v(\bar{T})=E_v\cap\bar{T}^1$. Now, let us introduce variables on $GF(2)$

$$x_{t,s}=x_{s,t},\ t\neq s, \tag{2.2}$$

for $s$, or $t\in E_v(T)$ and the other in $E_v(T)\cup E_v(\bar{T})$, at each vertex $v\in V$. If both

$s, t \in E_v(T)$, then $x_{t,s}$ is said to be a tree variable; otherwise, a cotree variable.

For any two fundamental circuits $C_\alpha(T)$, $C_\beta(T)$, a variable $x_{t,s}$ with $s \in C_\alpha^1(T)$, $t \in C_\beta^1(T)$ or vice versa is said to be covariable of $C_\alpha(T)$ and $C_\beta(T)$.

**Proposition 2.4.** *For any two fundamental circuits $C_\alpha(T)$, $C_\beta(T)$ with $\alpha, \beta \in \bar{T}^1$ having no end in common, there are, if any, exactly two covariables*

*Proof* If $C_\alpha^1 \cap C_\beta^1 = \emptyset$, then no covariable exists; otherwise, there are only two possible cases.

*Case* 1. $C_\alpha^1 \cap C_\beta^1 = \{v\}$. By symmetry, we may suppose the tree edge coming to $v$ to be in $C_\alpha$. According to Proposition 2.2, only two configurations possibly appear as follows. In both configurations, $x_{s,t}$ and $x_{s,\beta}$ are the two covariables (in Fig. 2.1).

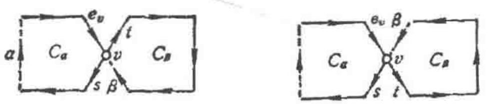

Fig. 2.1

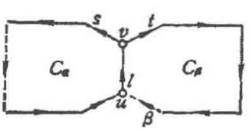

Fig. 2.2

*Case* 2. $C_\alpha \cap C_\beta = P\langle u, v\rangle$. Similarly to Case 1, we may also suppose the tree edge coming to $u$ to be in $C_\alpha$. From Proposirion 2.2, the only possible configuration is as in Fig. 2.2. In this configuration, only $x_{s,t}$ and $x_{l,\beta}$ are the covariables.

For $T \in T_D(G)$, we define a $T$-immersion of $G$ as such a plane representation that two edges $\alpha$, $\beta$ cross only if $\alpha, \beta \in \bar{T}^1$ and have no end in common. According to Jordan Axiom, it does exist. Let $D = \{(\alpha, \beta) \mid \alpha, \beta \in \bar{T}^1$ and no end in common$\}$ and $w_{\alpha\beta}$ be the characteristic of $\alpha$, $\beta$ crossing, i.e., $w_{\alpha,\beta} = 1$, or 0 according as $\alpha$, $\beta$ cross, or do not for $(\alpha, \beta) \in D$.

**Criterion I**[2]. *A graph $G$ is planar iff for any given $T \in T_D(G)$ and a $T$-immersion, the equation system on $GF(2)$*

$$x(\alpha, \beta) + y(\alpha, \beta) = w_{\alpha,\beta}, \text{ for } (\alpha, \beta) \in D, \tag{2.3}$$

*has a solution, where $x(\alpha, \beta)$, $y(\alpha, \beta)$ are the covariables of $C_\alpha$ and $C_\beta$,*

$$x(\alpha, \beta) \in X = \{x \mid \text{corresponding to an angle with two edges having different directions}\},$$

$$y(\alpha, \beta) \in Y = \{y \mid \text{corresponding to an angle with two edges having the same direction}\}$$

*are said to be a forward, backward variable, respectively.*

**Remarks.** 1. Equation (2.3) is defined by Proposition 2.4.

2. The existence of a solution of (2.3) does not depend on the choice of $T \in T_D(G)$ and a $T$-immersion. Thus, a proper choice of $T$ and a $T$-immersion are allowed to make the system simpler.

## § 3. Some Results Derived from the Criterion

Let $Z$ be the set of all variables which occur in (2.3), i.e. $Z=X\cup Y$ is the vertex set, and two vertices are adjacent iff the two corresponding variables appear in one equation, or, say, are covariables. The resultant graph, denoted by $H^1_T(G)$, is said to be the first auxiliary graph of $G$ for the $T$-immersion. Each edge of $H^1_T(G)$ is assigned a weight as the constant term of the corresponding equation.

A circuit in $H^1_T(G)$ is called a 1-circuit if the sum of the weights of all edges on it is 1 (mod 2).

**Lemma 3.1.** Equation (2.3) has a solution iff there is no 1-circuit in $H^1_T(G)$.

*Proof* Necessity. If not, suppose $C=x_1h_1x_2\cdots x_sh_sx_1$ to be a 1-circuit in $H^1_T(G)$, i. e., $\sum_{i=1}^{s} w_{h_i}=1$ (mod 2). However

$$0=\sum_{i=1}^{s}(z_i+z_{i+1})=\sum_{i=1}^{s}h_i=1 \pmod 2,$$

a contradiction appears.

Sufficiency. Let $\Lambda(H)$ be a spanning tree of $H^1_T(G)$. Then since no 1-circuit appears in $H^1_T(G)$, any solution of the equations determined by $\Lambda(H)$ can be extended into a solution of (2.3) determined by the whole $H^1_T(G)$.

**Theorem 3.2**[2]. *$G$ is planar iff $H^1_T(G)$ has no fundamental 1-circuit.*

*Proof* Since the sum of 0-circuits does not contain a 1-circuit, from Lemma 3.1, it follows.

**Lemma 3.3.** *$H^1_T(G)$ has no 1-circuit iff the set of all the edges with weight 1 is a cocycle of $H^1_T(G)$.*

*Proof* Let $W_1$ be the set of all the edges with weight 1 in $H^1_T(G)$.

Necessity. Since no 1-circuit occurs, for any spanning tree $\Lambda(H)$ of $H^1_T(G)$, there exists an edge $h\in\Lambda^1$ and $w_h=1$ except for the trivial case of no edge with weight 1, in which $W_1=\emptyset$ is a cocycle. And we have

$$W_1=\sum_{h\in\Lambda^1 \text{ and } w_h=1} C^1_h(\Lambda).$$

Therefore, $W_1$ is a cocycle of $H^1_T(G)$.

Sufficiency. From $W_1$ being a cocycle, for any circuit $C$, we have

$$|W_1\cap C|=0 \pmod 2,$$

i.e., $C$ is not a 1-circuit.

**Theorem 3.4**[4]. *$G$ is planar iff $W_1$ is a cocycle of $H^1_T(G)$.*

*Proof* A direct conclusion of Lemma 3.3 and Theorem 3.2.

Generally, as it is only need to consider the number of edges of $G$ not greater