万哲先文选

万哲先/著

科学出版社

华罗庚—吴文俊数学出版基金资助项目

万哲先文选

万哲先 著

科学出版社
北京

内 容 简 介

万哲先院士著述甚丰，许多系统性的成果已总结在他的专著中. 除此以外，还有一些有意义的零散的研究工作. 本书挑选了其中的9篇论文，加上10部专著的序言和6篇回忆文章集结出版，以此庆祝万哲先院士九十华诞.

本书适合数学专业的研究人员、大学数学系的教师和学生，以及其他对数学有兴趣的年轻人阅读.

图书在版编目(CIP)数据

万哲先文选/万哲先著. —北京：科学出版社，2017.6
ISBN 978-7-03-053025-7

Ⅰ. ①万…　Ⅱ. ①万…　Ⅲ. ①数学-文集　Ⅳ. ①O1-53

中国版本图书馆 CIP 数据核字（2017）第 116886 号

责任编辑：王丽平／责任校对：张凤琴
责任印制：张　伟／封面设计：黄华斌

科学出版社 出版
北京东黄城根北街16号
邮政编码：100717
http://www.sciencep.com

北京东华虎彩印刷有限公司 印刷
科学出版社发行　各地新华书店经销
*
2017年6月第 一 版　开本：720×1000 1/16
2017年6月第一次印刷　印张：14 1/4
字数：280 000

定价：89.00 元

（如有印装质量问题，我社负责调换）

目 录

A Proof for a Graphic Method for Solving the Transportation Problem*

Zhexian Wan (万哲先)**

(Institute of Mathematics, Academia Sinica)

1. The Method

In September, 1958, we learned a graphic method for solving the transportation problem from a certain transportation department in Peking. And we were asked to work out a mathematical proof for this method. The method may be stated as follows:

Suppose it is desired to set up a transportation schedule to distribute a certain goods from several sources to several destinations. The total amount of supply of all the sources is assumed to be equal to the total amount of demand of all the destinations. The schedule should be such that the total "transportation cost" (in terms of ton-kilometres, for instance) for realizing it would be a minimum.

Before giving the transportation schedule, we first draw a map showing all the sources and destinations, and their connecting lines (railways for instance), with the sources denoted by small circles "○" and the destinations by crosses "×". The numbers given beside the sources and those beside the destinations are their respective amounts (in terms of tons, say) of supply and demand.

* First reported in Chinese in *Shuxue Tongbao* (*Bulletin of Chinese Mathematics*), No. 11, pp. 478—482, 1958.

** In collaboration with Comrades Shen Xin-yao, Xu Yi-chao, Gan Dan-yan, Zhu Yong-jin, and Yang Xi-an.

Along the connecting lines are given the distances (in terms of kilometres, say) between the neighbouring cities or towns. Such a distribution map is shown in Fig. 1.

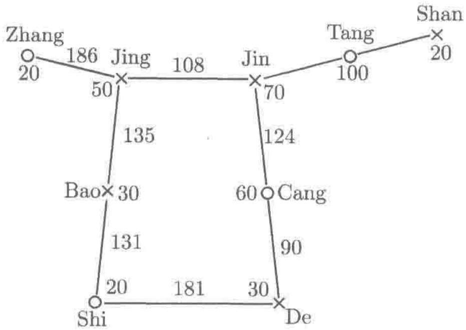

Fig. 1

Table 1 gives an illustrative schedule.

Table 1

Sources \ Destinations	Shan	Jin	Jing	De	Bao
Tang	0	70	30	0	0
Zhang	0	0	0	0	20
Cang	20	0	0	30	10
Shi	0	0	20	0	0

For realizing this schedule, a sort of flow diagram is required. Suppose there are goods, m tons say, flowing from source A to destination B. A flow vector is drawn alongside of the line segment representing the railway from A (on the right) to B (on the left) with the amount of flow m marked above the flow vector (see Fig. 2). If along the same line there are more than one flow vector in the same sense, a single flow vector is drawn with the total amount of transportation indicated, as illustrated in Fig. 3. If this is done for all the section lines, a complete flow diagram of the schedule can be drawn as in Fig. 4. It is evident that the total "transportation costs" (in terms of ton-kilometres, say) of schedules with the same flow diagram are equal.

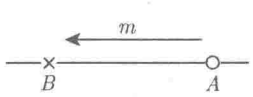

Fig. 2

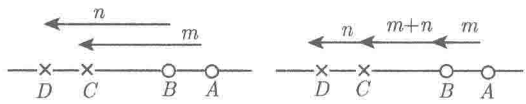

Fig. 3

Clearly, when counter flow vectors occur in the flow diagram, for example, in the case of Fig. 4, when counter flows occur between the Jing-Bao and Jin-Tang lines, the transportation schedule is certainly not the best. In this case, the schedule should be changed so as to obtain a new one, of which the flow diagram contains no counter flows. For example, Fig. 4 may be changed into Fig. 5 in which counter flows disappear. The corresponding transportation schedule is shown in Table 2.

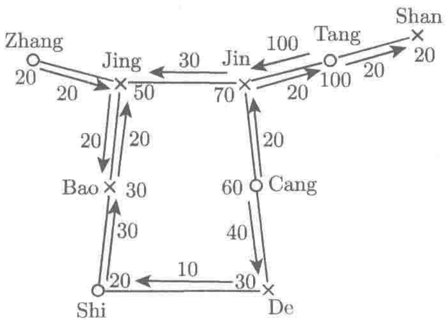

Fig. 4

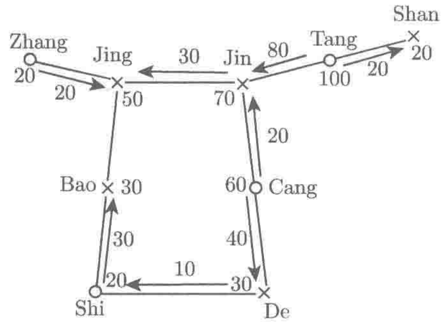

Fig. 5

Table 2

Sources \ Destinations	Shan	Jin	Jing	De	Bao
Tang	20	70	10	0	0
Zhang	0	0	20	0	0
Cang	0	0	20	30	10
Shi	0	0	0	0	20

In the following we shall consider only schedules with flow diagrams that contain no counter flows.

Now let us consider a cycle C in a flow diagram; by a cycle is meant a closed path without self-intersecting points. There may be flow vectors in the positive sense of C (i.e. in the counterclockwise sense) and in the negative sense of C (i.e. in the clockwise sense). For distinction, flow vectors in the positive sense are placed outside the cycle, while those in the negative sense inside the cycle. Denote the total length of C by $S(C)$, that of the flow vectors in the positive sense of C by $S^+(C)$, and that of the flow vectors in the negative sense of C by $S^-(C)$. A cycle C is said to be normal if both $S^+(C) \leqslant \frac{1}{2}S(C)$ and $S^-(C) \leqslant \frac{1}{2}S(C)$ hold. A flow diagram is said to be normal, if each of its cycles is normal. Then the graphic method asserts: "A transportation schedule is an optimal one if and only if its flow diagram is normal"[1)].

2. The Proof

Before we come to the proof of the above mentioned criterion, it may be remarked that branching lines of the distribution map can be omitted in the following manner. At a branching point is recorded only the net supply or the net demand, as the case may be, of the whole branching line (including the branching point), instead of the original amount of supply or demand. For instance, in Table 2, the Jin-Shan line can be omitted by changing Jin to a source with a net supply of 10 units and the Jing-Zhang line can also be omitted by considering Jing to be a destination with a net demand of 30 units. Thus we obtain the following distribution map that contains no branching lines (Fig. 6).

First let us consider the case where the flow diagram contains only a single cycle. For simplicity, we regard the demand of each destination to be 1 unit. In fact, if a destination demands m units, we may imagine a series of m destinations, each with a demand of 1 unit, situated at the original destination with distance 0 between any two such destinations.

---

1) The original criterion, as we learned from the transportation company, states that a transportation schedule is optimal if and only if all those cycles which contain no other cycles inside themselves and all those which are not contained in any other cycle are normal. But this condition is not sufficient as may be shown by examples. The present criterion was suggested during the proof.

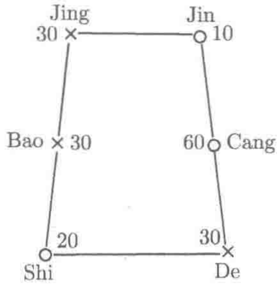

Fig. 6

Suppose that there is a transportation schedule with a non-normal flow diagram G (see Fig. 7). For definiteness, assume that $S^+(G) > \frac{1}{2}S(G)$. Then we may draw a new flow diagram G' which differs from G only in that the end point of each flow vector in the positive sense in G is now supplied by the flow vector in the negative sense in G', where other destinations are supplied by the same sources in G' as in G (see Fig. 8). (The process of change from G to G' is called the process of shrinking the flow vector in the positive sense by one station.) The "transportation cost" for transporting the goods to the end points of the flow vectors in the positive sense of G is equal to $S^+(G)$ according to G, and equal to $S^-(G')$ according to G'. Since $S^+(G) + S^-(G') \leqslant S(G)$ and $S^+(G) > \frac{1}{2}S(G)$, we have

$$S^-(G') < S^+(G).$$

Thus the transportation schedule with flow diagram G' is better than that with flow diagram G. This indicates that a schedule with a nonnormal map is not optimal.

Next, let us prove that transportation schedules with normal flow diagrams are always optimal. By the above proof, we know that normal flow diagrams always exist. If there is only one flow diagram, the statement just made is self-evident. Suppose now there are more than one flow diagram. Then we may arrange all the flow diagrams in an order $G_1, G_2, \cdots, G_n$, such that G_1 is the flow diagram without flow vectors in the negative sense, G_n is the flow diagram without flow vectors in the positive sense, and each G_{i+1} is obtained from the preceding G_i by the process of shrinking the flow vectors of G_i in the positive sense by one station $(i = 1, 2, \cdots, n-1)$. Assume that G_i is the normal diagram with the smallest subscript and G_i is the normal diagram with the greatest subscript, then $i \leqslant j$, and the flow diagrams between G_i and G_j are all normal. If $i = j$, this is quite evident. Suppose now $i < j$. It is sufficient to show that any two neighbouring flow diagrams G_k and $G_{k+1} (i \leqslant k \leqslant j)$ have the same total "transportation cost". Notice that G_k and G_{k+1} differ only in the end points of the flow vectors in the positive sense of G_k. Since $S^+(G_k) \leqslant \frac{1}{2}S$, $S^-(G_{k+1}) \leqslant \frac{1}{2}S$, $S^-(G_{k+1}) \leqslant S - S^+(G_k)$, we have $S^+(G_k) = S^-(G_{k+1}) = \frac{1}{2}S$, S being the total length of the cycle. This proves that G_k and G_{k+1} have the same "transportation cost". Consequently, normal flow diagrams have the same "transportation cost". This finishes the proof for the case that the communication map consists of a single cycle.

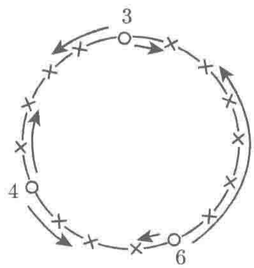

Fig. 7

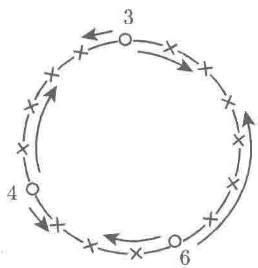

Fig. 8

Now consider the general case. By the same argument as in the case of a single cycle, we can prove that transportation schedules with nonnormal flow diagrams are not optimal. It remains to show that schedules with normal flow diagrams have the same "transportation cost". Let G and G' be two distinct normal flow diagrams (Fig. 9), which, of course, belong to different schedules. In comparing G with G', we may assume that there are no common flow vectors in G and G'. In fact, if there is a common flow vector, we may remove a certain amount of transportation from both G and G' so as to obtain flow diagrams

G_1 and G'_1 without common flow vectors[2)].

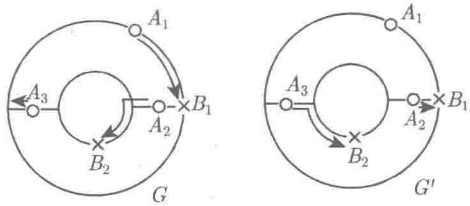

Fig. 9

Now we apply induction to the total amount of supply of all the sources of G. Let us start with a source A_1. Assume that, in G, one ton of goods is transported from A_1 to destination B_1. Then in G', one ton needed by B_1 is supplied by another source A_2 . Then is G, A_2 supplies another destination B_3. Continuing in this way, and assuming that B_i is the first destination which coincides with a further destination $B_k (i < k)$, we obtain a closed path Z:

$$(*) \qquad B_i, A_{i+1}, B_{i+1}, \cdots, A_k, B_k = B_i.$$

In general, this is a self-intersecting closed path. Along this closed path, goods supplied according to G go round in one sense, while those supplied according to G' go round in another (see Fig. 9). It is evident that the closed path Z may be a single cycle or may be decomposed into several cycles $C_1, C_2, \cdots$. For example, the closed path shown in Fig. 10 may be decomposed into two cycles as shown in Fig. 11.

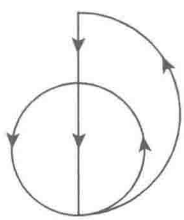

Fig. 10

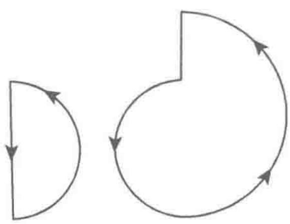

Fig. 11

2) This idea is due to Professor Tian Fang-zeng. We are indebted to Professor Tian for his valuable suggestions.

Then in each cycle, the flow vectors of G are all of the same sense, and those of G' are all of the same sense too. Since both G and G' are normal, the "transportation costs" for carrying one ton of goods according to G and G' in each of these cycles $C_i(i = 1, 2, \cdots)$ forming Z, are the same, i.e. both are equal to $\frac{1}{2}S(C_i)$. Thus the "transportation costs" for carrying one ton of goods according to G and G' in the closed path Z are also the same. Removing one ton of goods along the closed path Z from both G and G', we obtain two new normal flow diagrams G_1 and G'_1 with a smaller total amount of supply of all sources. By induction hypothesis, G_1 and G'_1 have the same amount of transportation, so do G and G'

The proof is now complete.

Remarks 1. After we finished the above proof in September 1958, we learned that Professor Xu Guo-zhi and Comrade Gui Xiang-yun had also obtained a proof for the case of a communication map consisting of a single cycle by showing that the criterion of the graphic method is the same as that given by the simplex method. In 1959, the general case was also considered by them with the same method. But their proof has not yet been published.

2. That the transportation problem can be solved by the simplex method is well known. Recently, we learned that the problem had also been solved by Л. Б. Канторович and М. К. Гавурин by another method [1].

References

[1] Канторович, Л. В., Гавурин, М. К. Применение математических методов в вопросах анализа грузопотоков, *Сб. "проблемы поєыщения эффектиєности работы транспорта"*, *АН СССР*. 1949, стр. 110–138.

Automorphisms and Isomorphisms of Linear Groups over Skew Fields

Hongshuo Ren, Zhexian Wan, and Xiaolong Wu

Let K be a skew field and $n \geqslant 2$ be an integer. Notations $\mathrm{GL}(n, K)$, $\mathrm{SL}(n, K)$, $\mathrm{PGL}(n, K)$ and $\mathrm{PSL}(n, K)$ are defined as usual. If $X \in \mathrm{SL}(n, K)$, we write $\overline{X}$ for its projection image in $\mathrm{PSL}(n, K)$. An automorphism Λ of $\mathrm{PSL}(n, K)$ is called standard if there exist a matrix $A \in \mathrm{GL}(n, K)$ and an automorphism σ or an antiautomorphism τ of K such that

$$\Lambda\overline{X} = \overline{AX^{\sigma}A^{-1}} \text{ for all } \overline{X} \in \mathrm{PSL}(n, K)$$

or

$$\Lambda\overline{X} = \overline{A(X'^{\tau})^{-1}A^{-1}} \text{ for all } \overline{X} \in \mathrm{PSL}(n, K),$$

where X^{σ} (resp. X^{τ}) is the matrix obtained by the action of σ (resp. τ) on each entry of X, and X' is the transpose of X.

We have completed the proof of the following theorem [5, 6]:

Theorem *For any skew field K and any integer $n \geqslant 2$, all automorphisms of $\mathrm{PSL}(n, K)$ are standard.*

Remarks (1) The theorem can be generalized to any group Δ with $\mathrm{PSL}(n, K) \subseteq \Delta \subseteq \mathrm{PGL}(n, K)$.

(2) The theorem can be generalized to any group Δ with $\mathrm{SL}(n, K) \subseteq \Delta \subseteq$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20G35.

Projects supported by the Science Fund of the Chinese Academy of Sciences nos. 85188 and 85204.

$\mathrm{GL}(n, K)$, where the definition of "standard" is modified by a homomorphism $\xi : \Delta \to$ the center of the multiplicative group of K.

(3) With minor modification of the proof of the theorem, we proved in [7] that if K and K_1 are skew fields, $n, n_1 \geqslant 2$ are integers and $\mathrm{PSL}(n, K)$ is isomorphic to $\mathrm{PSL}(n_1, K_1)$, then $n = n_1$ and K is isomorphic or anti-isomorphic to K_1, with the only exceptions: $\mathrm{PSL}(2, \mathbf{F}_4) \cong \mathrm{PSL}(2, \mathbf{F}_5)$ and $\mathrm{PSL}(2, \mathbf{F}_7) \cong \mathrm{PSL}(3, \mathbf{F}_2)$.

Proof of the Theorem O. Schreier and van der Waerden proved the theorem for commutative K in 1928 [8]. L. Hua corrected a mistake in their proof in 1948 [2]. J. Dieudonné proved the theorem for all $n \neq 2, 4$ in 1951 [1]. L. Hua solved the case $n = 4$ in 1951 [3]. L. Hua and Z. Wan solved the case $n = 2$, char $K > 0$ in 1953 [4]. Therefore, only the case $n = 2$, char $K = 0$ was left open.

Now write K^* for the multiplicative group of K, Z for the center of K, K^c for the commutator subgroup of K^*, and I for the identity matrix of order 2. Assume from now on that $n = 2$ and char $K = 0$. If S is a subset of $\mathrm{PSL}(2, K)$, denote by CS the centralizer of S in $\mathrm{PSL}(2, K)$ and by C^2S the set $\{X^2 | X \in CS\}$. If S is a group, denote by DS the commutator subgroup of S.

First assume $-1 \in K^c$. Write $\Lambda\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \overline{A}$. Then we may assume $A = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ for some $\alpha \in Z$. Write $\overline{B} = \Lambda^{-1}\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$. Then B is of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$. If $B = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $CC^2\overline{A}$ contains an element $\overline{\begin{pmatrix} x & \alpha y \\ y & x \end{pmatrix}}$, $xy \neq 0$, then $\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \notin CCC^2\overline{A}$. But $\overline{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}} \in CCC^2\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$, so we get a contradiction. If $B = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $CC^2\overline{A}$ contains no element of the form $\overline{\begin{pmatrix} x & \alpha y \\ y & x \end{pmatrix}}$, $xy \neq 0$, then $CC^2\overline{A}$ has at most four elements whereas $CC^2\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ has infinitely many elements. So we again get a contradiction. Therefore, we must have $B = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. Conjugating by $\overline{\begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix}}$, we may assume $B = \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}$, $\beta \in Z$. Hence

$$\begin{aligned}\mathrm{PSL}(2,Z) =&DCC^2\left\{\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}}\right\}\\ =&DCC^2\left\{\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix}}\right\}\\ =&\mathrm{PSL}(2,Z).\end{aligned}$$

By the known result for the commutative case, we may assume

$$\Lambda\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} = \overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \quad \text{and} \quad \Lambda\overline{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} = \overline{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}.$$

Now by the well-known method we can show that Λ is standard.

Now we assume $-1 \notin K^c$. The main steps of the proof are:

(1) Write $\Lambda\overline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \overline{S}$, $\Lambda\overline{\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}} = \overline{W}$. The we may assume

$$S^2 = \alpha I, \quad (SW^2)^3 = \beta I, \quad (W^4SWS)^3 = \gamma I, \quad \text{for some } \alpha, \beta, \gamma \in Z,$$

since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$ satisfy similar relations. A study of the elements of order three in $\mathrm{PSL}(2,K)$ shows $\beta, \gamma \in \{x^3 | x \in Z\}$. Hence by choosing representatives of S and W, we may assume $\beta = \gamma = 1$.

(2) We show that W is triangulizable. Define $M(2,K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in K \right\}$. We may regard $M(2,K)$ as an extension ring over K by the map $a \mapsto aI$, for all $a \in K$.

If $\alpha \notin K^2 = \{a^2 | a \in K\}$, we may assume $S = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, $\alpha \in A$. Write $W = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Define

$$A = \begin{pmatrix} y & -ywy^{-1} \\ 0 & 1 \end{pmatrix}, \quad T = AMA^{-1} = \begin{pmatrix} 0 & q \\ 1 & s \end{pmatrix} \text{(say)}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & T \end{pmatrix}.$$

Then conjugate by BA in $\mathrm{GL}(2, M(2, K))$. We get

$$BASA^{-1}B^{-1} = \begin{pmatrix} * & * \\ R & * \end{pmatrix}, \quad \text{where } R = -\alpha y^{-1}((ywy^{-1} + yT)^2 - \alpha^{-1} I),$$
$$BAWA^{-1}B^{-1} = \begin{pmatrix} s - T & 1 \\ 0 & T \end{pmatrix}.$$

When R is invertible, we can conjugate by a matrix of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ in $\mathrm{GL}(2, M(2, K))$ to send S and W to matrices of the form $\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} X & Y \\ 0 & T \end{pmatrix}$, respectively. If $X^2 + T^2$ is invertible, then we deduce from (1) that $(T^3 - I)(T^5 - I) = 0$. So T^{15} and hence W is triangulizable in $\mathrm{GL}(2, K)$. If $X^2 + T^2$ is not invertible, then neither is $T^4 - \alpha^{-1} I$ by (1). So T^4 and hence W is triangulizable. If R is not invertible, then we must have $R = 0$ since $\alpha \notin K^2$. Hence $(ywy^{-1} - yT)^2 - \alpha^{-1} I = 0$. But this means $WS = SW$, Which is impossible.

Now assume $\alpha \in K^2$. Then $\alpha = \theta^2$ for some $\theta \in Z$. We can assume $S = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$. Write W, A, T, B as in the last paragraph. We have

$$BASA^{-1}B^{-1} = \begin{pmatrix} -\theta & 0 \\ R & \theta \end{pmatrix}, \quad \text{where } R = 2\theta(T - ywy^{-1}),$$
$$BAWA^{-1}B^{-1} = \begin{pmatrix} s - T & 1 \\ 0 & T \end{pmatrix}.$$

If R is invertible, we can proceed as in the case $\alpha \notin K^2$. If R is not invertible, then $T - ywy^{-1}$ is not invertible. So T and hence W is triangulizable in $\mathrm{GL}(2, K)$.

(3) Write $S = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$, $\alpha \in Z, W = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. If $y = 0$, we can use equalities in (1) to get $w = z^{-1} \in Z$ and $x = (w + w^{-1})^{-1}$. So

$$\mathrm{PSL}(2, Z) = DC\left\{\overline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}, \overline{\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}}\right\} = DC\left\{\overline{S}, \overline{W}\right\} = \mathrm{PSL}(2, Z)$$

and we are through. If $x = 0$, we can conjugate by $\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$ and return to the case $y = 0$. If $xy \neq 0$, then we must have $uxu + zu - uw - y = 0$ for some $u \in K$ since W is triangulizable. When $u^2 \neq \alpha$, we can conjugate by $\begin{pmatrix} 1 & \alpha^{-1}u \\ u & 1 \end{pmatrix}$ and return to the case $y = 0$, When $u^2 = \alpha$, we can conjugate by $\begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix}$ and then use equalities in (1) to get a contradiction. This completes the proof.

References

[1] J. Dieudonné, *On the automorphisms of classical groups*, Mem. Amer. Math. Soc. No. 2 (1951).

[2] L. Hua, *On the automorphisms of the symplectic groups over any field*, Ann. of Math. (2) **49** (1948), 739–759.

[3] L. Hua, *Supplement to the paper of Dieudonné on the automorphisms of classical groups*, Mem. Amer. Math. Soc. No. 2 (1951), 96–122.

[4] L. Hua and Z. Wan, *Automorphisms ans isomorphisms of linear groups*, Acta. Math. Sincia **2** (1953), 1–32. (Chinese)

[5] H. Ren and Z. Wan, *Automorphisms of* $\mathrm{PSL}_2^{\pm}(K)$ *over any skew field* K, Acta Math. Sinica **25** (4)(1982), 484–492.

[6] H. Ren, Z. Wan, and X. Wu, *Automorphisms of* $\mathrm{PSL}(2, K)$ *over skew fields*, Acta Math. Sinica (to appear).

[7] H. Ren, Z. Wan, and X. Wu, *Isomorphisms of linear groups over skew fields*, (to appear).