刘彦佩

半闲数学集锦

Semi-Empty Collections in Mathemetics by Y.P.Liu

第十五编

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第十五编序

虽然多面体从柏拉图(Plato)年代就有了, 抽象多面形的概念, 尽目前所知, 在 110 余年前, Heffter 开始成形. 上世纪 60 年代, Edmonds 提供了其对偶形式. 经过这一个多世纪的研究, 一直采用 Heffter-Edmonds 模型.

由于 Heffter-Edmonds 模型在识别多面形不可定向性时, 遇到的困境, 导致 Tutte 在将一条棱视为四元胞腔的基础上, 建立了用所有四元胞腔并上的的一类置换, 表示多面形. 因为这一模型被证明, 与 Klein 在三角化的基础上, 所建立的模型, 别无二致, 允许我们称之为 Klein-Tutte 模型.

又由于 Klein-Tutte 模型, 仍存在不能完全区别多面形的弊病, 即存在一个多面形有多个表示的情形, 促使我们, 终于发现了联树模型, 克服了上述表示的不足. 这就为多面形理论的形成, 打下了最简洁表示论方面的根基.

鉴于我已经出版的专著, 特别是 [292](12.39—12.62) 和 [354](13.27—13.56), 都体现了一些零碎的多面形基础知识, 虽暗示了其理论意义, 但尚不系统.

在反省和批判以往有关论题, 尤其我本人的已有专著中, 萌孕一本新专著的诞生. 这就是本编专著 *Theory of Polyhedra* [396](15.01—15.21). 其中, 15.01—15.02 为本书的序和目录; 在 15.17 和 15.18 中, 前者给出边数不超过 15 三连通三正则图的手柄多项式与叉帽多项式, 边数不超过 5 环束的手柄多项式与叉帽多项式, 节点数不超过 8 轮图的手柄多项式与叉帽多项式, 边数不超过 6 杆束的手柄多项式与叉帽多项式, 节点数不超过 7 完全二部图的手柄多项式与叉帽多项式和节点数 8 完全二部图的手柄多项式, 以及边数不超过 8 四正则图的手柄多项式与叉帽多项式. 记 β 为 Betti 数, 后者提供边缘长 2β 不超过 10, 可定向单面形的叉帽多项式. 15.19—15.21 为参考文献, 术词索引和引用过的作者索引.

实际上, 专著 [137](2.01—2.20) 的 2.03, 以及专著 [141](3.11—3.29) 的 3.13 中, 就提到多面形, 由于不可定向性判别不够清晰, 常给读者带来不便, 甚至误解, 在此后的专著中, 如 [292](12.43) 和 [354](13.31), 虽然改善了上面的不足, 却以图的曲面嵌入, 这样直观的对象为出发点.

在 15.03 中, 则从任何一个集合出发, 不必考虑带任何直观背景. 以集合上的剖分和置换为代数基础, 建立抽象的多面形. 因此, 更具有普遍性.

在 15.04 中, 将这种抽象的多面形, 即一种二维流形, 特别是曲面, 进行分类, 为图的嵌入在曲面上分类, 奠定理论基础.

在 15.05—15.06 中, 以不同的角度, 讨论图在曲面上的嵌入. 引进联树模型, 导致对于嵌入与地图的, 最简捷的表示.

在 15.07 中, 建立了多面形上, 对于棱, 以及各种拓扑(基础), 甚至非拓扑运算, 的对偶性. 讨论曲面上四角化的对偶性.

在 15.08 中, 确定在基础(拓扑)运算之下, 多面形等价类的, 完全不变量.

实际上, 这就是多面形在曲面上的一种分类. 也就是将曲面上的所有多面形, 划分为基础等价类.

一个多面形, 往往带有各种对称性, 为了区别多面形, 必须考虑这些对称性. 但这样一来, 就会带来一些麻烦. 不如先破坏这些对称性, 待区别破坏了对称性之后的多面形, 再按照原来的对称性, 进行分类, 以达到区分带对称性多面形的目的. 这就引出 15.09 的非对称化的过程.

在此基础上, 15.10, 15.11 和 15.12, 分别对于普通多面形, 瓣丛和一些其它类型的球面多面形, 非对称化了的情形, 提供不同多面形的数目. 前两款都是考虑单个计数参数和最后一款, 则是多参数, 以致无穷计数参数.

在 15.13 中, 提供了, 如何从已经得到的被非对称化了的多面形的同构分类, 转换到考虑对称性多面形的同构分类. 揭示了只需要求出非对称多面形, 按照带对称性多面形的自同构群阶的分布即足.

通过在圈或上圈上的定向, 将多面形分裂为若干部分, 确立这个多面形的亏格与各部分亏格的关系, 以便从小亏格多面形, 导出大亏格的多面形. 建立图与多面形的关系. 这就是 15.14 和 15.15 所处理的问题, 以及由此所导出的一些图的亏格多项式 (包括手柄多项式和叉帽多项式). 特别是后者, 提供了, 用联树, 确定图的亏格多项式, 的一个普遍准则.

值得一提地, 这里所提供的普遍准则, 既不同于 Walsh 和 Lehman[1], 也不同于 Jackson[2]. 这里的方法, 适用于任何类型的地图集. 而他们的, 则都只适用于, 本身具有合适对称性, 的地图集.

在 15.16 中, 提供了多面形理论的一些应用. 例如, 图的凸嵌入, 纵横嵌入, 特别是第一次提出曲面一个纵横模型, 使得有条件, 将平面上的纵横嵌入, 发展到曲面上, 边缘厚度, 纽结的 Dehn 图式, 以及理论物理中的 Potts 模型等.

其余部分, 即 15.17—15.21, 都是为读者阅读正文方便而设立的. 前两者, 提供遍数小多面形的示例. 后三者, 为查阅文献的方便.

综上所述, 这本专著为高等图论(包括拓扑图论, 几何图论, 代数图论, 以及解析图论), 开始建立一个共同的理论基础.

文 15.22, 15.27, 15.29—15.35 都是关于一类多面形—图在曲面上嵌入, 包括亏格多项式, 平均亏格, 在小亏格曲面上, 以及连带的整体性质. 其余的, 即文 15.23—15.26, 15.28, 以及文 15.36—15.37, 都是与另一类多面形—-地图有关.

刘彦佩
2015 年 8 月
於北京上园村

[1]Walsh, T.R.S. and Lehman, A.B., Conting rooted maps by genus, *J. Combin. Theory*: I, B13(1972), 192—218; II, B13(1972), 122—144; III, B18(1975), 222—259

[2]Jackson, D.M., The genus series for maps, *J. Pure Appl. Algebra*, 105(1995), 293—297.

第十五编目录

Mathematics Monograph Series 9

Yanpei Liu

Theory of Polyhedra

(多面形理论)

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Preface

Polyhedra arose probably from the Platonic age(400–300 B.C.) and far from then the Eulerian formula on the relationship among the numbers of vertices, edges and faces of a polyhedron in the mid of 18th century[Ore1]. However, maps are now treated as polyhedra and appeared from the four color problem[Ore2] and the more general map coloring problem[Rin2–3] in the mid of 19th century. This book is only intended to present a theoretical foundation of polyhedra which have been developed mostly by the author himself only in recent years.

The first formal definition of a polyhedron was done by Heffter[Hef1] in the 19th century. However, it was not paid an attention by mathematicians until 1960 when Edmonds published a note in the AMS Notices as the dual form of Heffter's[Edm1,Liu9]. Now, it is named as Heffter-Edmonds' model of a polyhedron.

Although this concept was widely used in literature[Liu6–7,Liu10–11,Rin1–3, Sta1, *etc*], its disadvantages for identifying the nonorientability and distinguishing polyhedra involved do not bring with convenience for clarifying related mathematical thinking.

Since Tutte described the nonorientability of a map as a polyhedron in a new way [Tut1–3], a number of authors began to develop it as combinatorization of continuous objects[Lit1, Liu4–5, Vin1, *etc*]. However, Tutte's can be seen from the Klein triangulation of an embedding as the origin. Now, it is named as Klein-Tutte's model of a map.

Because of the generality that in any asymmetric object there is some kind of local symmetry, the concepts of maps are just put in such a rule. In fact, the former is corresponding to that a group of two elements stick on an edge and the later is that a group of four elements stick on an edge such that a graph, or a map, without symmetry at all is in company with local symmetry. This treatment will bring more advantages for observing the combinatorial structures of an object. Of course, the former is a specific case of the later.

On the other hand, in early papers[Liu6–7] published in 1979, an embedding of a graph as a polyhedron was dealt with a tree as a starting point for its construction. However by standard surfaces an embedding can not often be constructed easily. A surface was further adopted as a polyhedral polygon in monographs[Liu58,62] in 1994 and 1995. Although these have implied the joint tree model, it did not occur in literature until the monograph[Liu65] and related articles[Liu67–68,71] were published a few years ago. This book is for reflecting the present developments of the joint tree model in a variety of aspects as a fundamental theory of polyhedra.

The contents of the book covers the following subjects. In Chapter I, known knowledge about sets, relations and permutation groups are briefly described for the usage. Particularly, graphs and networks are with a new view based on partition on a set. The second chapter is on surfaces as sets of polyhedra with topological classification and combinatorization. The third chapter is for embeddings of a graph as polyhedra on surfaces. The joint tree model is deduced from the Heffter-Edmonds' model with clarification and simplification. The fourth chapter is on maps as a mathematical subject from permutation with conjugate and transitive axioms. The fifth chapter concentrates on duality not only for maps themselves but also for operations on maps from one surface to another and for edges of a map with an explanation by diagrams of knots. One can see how the duality is much simply deduced from the mathematical concept of a map described in the fourth chapter. The sixth chapter is for the classification of maps based on basic operations. The seventh chapter is on the asymmetrization of a map with symmetry for providing an affective procedure of determining the automorphism group of a map. In the eighth chapter, a method for determining the number of asymmetrized maps on surfaces is established by extracting a functional equation. Chapter IX considers petal bundles which can be seen by contracting a spanning tree of a graph on surfaces. Chapter X discusses maps with the same under graph in vertex, or face partition via a direct method based on joint tree. In Chapter XI, the number of nonisomorphic maps with consideration of symmetry is investigated by building up principles from asymmetrized maps. Chapter XII is for maps with certain orientation on a submap, in particular a circuit, of a map to get all maps of given size on a surface of given genus. Chapter XIII carries on the distribution of maps with given size by genus on the basis of joint tree model. The final chapter, Chapter XIV, offers applications of the theory to distinct types of objects in mathematics, theoretical physics and engineering.

Each chapter has a section for notices on background and historical remarks, complements of the context and problems for further research.

In addition, two appendices are designed for the reader's convenience to get and to check preresults. Calculations are done via the computer programs compiled by Wang Tao based on the theory of joint trees.

On this occasion, some of my former and present graduates such as Dr. Ying Liu, Dr. Yuanqiu Huang, Dr. Junliang Cai, Dr. Han Ren, Dr. Deming Li, Dr. Tongyin Liu, Dr. Rongxia Hao, Dr. Linfan Mao, Dr. Zhaoxiang Li, Dr. Erling Wei, Dr. Liangxia Wan, Dr. Yichao Chen *et al* should be particularly mentioned for their successful work in related topics. Meanwhile,

I should express my heartiest appreciation of the financial support by KOSEF of Korea from the Com^{2}MaC (Combinatorial and Computational Mathematics Research Center) of the Pohang University of Science and Technology in the summer of 2001. In that period, the intention of this book was initiated. Moreover, I should also be appreciated to the Natural Science Foundation of China under Grants 60373010 and 10571013 for the research development reflected in this book.

Y.P. Liu
Beijing, China
March, 2007

Contents

Chapter I

Preliminaries

Basic concepts and known related results on sets, partitions, permutations and groups are stated for the coming usages. A graph is dealt with a partition of a set and a semi-arc automorphism is defined. The relationship between automorphism group and semi-arc automorphism of a graph is given. A network is a graph with a weight on its edge set. Two types of finite recursion principles are exploited. The efficacity is exposed for a theorem with efficiency of recognization.

I.1 Sets and mappings

A *set* is a collection of distinct objects called *elements* which can be arbitrary things material or mental, concrete or abstract, except only for the set itself. If some elements are allowed with repetition in a set, then it is called a *multi-set*. Let X be a finite set. By $x \in X$ is meant that x is an element of a set X. For two sets A and B, if $x \in A \Rightarrow$(implies) $x \in B$, then A is called a *subset* of B, denoted by $A \subseteq B$. Of course, any set has the *empty*(a set without element), denoted by $\varnothing$, and itself as subsets. The set(or family) of all subsets of a set X is denoted by 2^X. A set of all its elements in A or B is called the *union* of A and B, denoted by $A \cup B$. A set of all its elements in both A and B is called the *intersection* of A and B, denoted by $A \cap B$.

Theorem I.1.1 For any sets A and B, the two operations on sets: union and intersection satisfy the following three laws:

Idempotent law $A \cup A = A \cap A = A$.

Absorption law $A \cup (A \cap B) = A \cap (A \cup B) = A$.

Commutative law $A \cup B = B \cup A$, $A \cap B = B \cap A$. □

Theorem I.1.2 For any sets A, B and C, the two operations on sets: union and intersection satisfy the following two laws:

Associative law $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

Distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □