刘彦佩选集

(Selected Publications of Y.P.Liu)

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On the average crosscap number II: Bounds for a graph

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Abstract The bounds are obtained for the average crosscap number. Let G be a graph which is not a tree. It is shown that the average crosscap number of G is not less than $\frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G)$ and not larger than $\beta(G)$. Furthermore, we also describe the structure of the graphs which attain the bounds of the average crosscap number.

Keywords: **average genus, average crosscap number, bounds.**

MSC(2000): **05C10**

A graph G in this paper is permitted to have both loops and multiple edges. A surface is equivalent to a compact 2-dimensional manifold without boundary. In topology, surfaces are classified into O_m, the orientable surface with $m(m \geqslant 0)$ handles and N_n, the nonorientable surface with $n(n > 0)$ crosscaps.

A representation $\mu(G)$ of a graph on a surface S with vertices as points and edges as curves which pairwise disjoint except possibility for endpoints is called an embedding of the graph in the surface S. The connected components of $S - \mu(G)$ are 2-cells and are called the faces of the embedding. In this paper, a surface embedding is also called a cellular embedding.

By a polygon with r sides, we shall mean a 2-cell which has its circumference divided into r arcs by r vertices. In fact, a surface can be seen as what is obtained by identifying each pair of edges on a polygon of even edges pairwise. According to refs. [1, 2], it is shown that the following three operations do not change the class of a surface.

Operation 1: $Aaa^- \Longleftrightarrow A$,

Operation 2: $AabBab \Longleftrightarrow AcBc$,

Operation 3: $AB \Longleftrightarrow (Aa)(a^-B)$.

Notice that A and B are all linear orders of letters with empty as a degenerate case in these operations. From the three operations, the following relations can be derived.

Relation 1: $AxByCx^-Dy^-E \sim ADCBExyx^-y^-$,

Relation 2: $AxBxC \sim AB^-Cxx$,

Relation 3: $Axxyzy^-z^- \sim Axxyyzz$.

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Relation 1 is also called the handle normalization, and Relation 2 and Relation 3 are also called the crosscap normalization. In the three relations, A, B and C are permitted to be empty. B^- is the inverse of B. By Relations 1, 2 and 3, we can always obtain the normal form of a surface as one of

$$O_0 = aa^-,$$
$$O_m = \prod_{i=1}^{m} a_i b_i a_i^- b_i^- (m > 0),$$
$$N_n = \prod_{i=1}^{n} a_i a_i (n > 0).$$

A connected graph without circuit is called a tree. A spanning tree of a graph is such a subgraph that is a tree with the same order as the graph. For a spanning tree of a graph G, the numbers of edges not on the tree are called the Betti number of the graph and denoted by $\beta(G)$.

A rotation at a vertex v of a graph G is a cyclic order of all semiedges incident with v. A pure rotation system P of the graph is the collection of rotations, one for each vertex of G. It is known that the total number of the pure rotation system P of G is given by the formula

$$\prod_{v \in V(G)} (d_v - 1)!.$$

A general rotation system is a pair (P, λ), where P is a pure rotation system and λ is a mapping $E(G) \to \{0, 1\}$. If $\lambda(e) = 1$, we mean that the edge e is twisted, otherwise e is untwisted. It is well known that every orientable embedding of a graph G can be described by a general rotation system (P, λ) where $\lambda(e) = 0$ for each edge e of G. By allowing λ to take the non-zero values we can describe the nonorientable embeddings of G. The details can be found in refs. [3, 4]. Let T be a spanning tree of G, a T-rotation system (P, λ) of G be a general rotation system such that $\lambda(e) = 0$, $e \in E(T)$.

The following theorem[3,4] is well known

Theorem 1.1. *Let T be a spanning tree of G and (P, λ) is a general rotation system. Then there exists a general T-rotation system (P', λ') such that (P', λ') yields the same embedding of G as (P, λ).*

Now, we fix a spanning tree T of a graph G. Let Φ_G^T be the set of all T-rotation systems of G. It is known that

$$|\Phi_G^T| = 2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!.$$

Suppose that in these $|\Phi_G^T|$ embeddings of G, there are a_i, for $i = 0, 1, \ldots$, embeddings on an orientable surface O_i and b_j, for $j = 1, 2, \ldots$, embeddings on a nonorientable surface N_j. We call the polynomial,

$$I_G^T(x, y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j,$$

the T-distribution polynomial of G.

By the total genus polynomial of G, we shall mean the polynomial

$$I_G(x,y) = \sum_{i=0}^{\infty} g_i x^i + \sum_{i=1}^{\infty} f_i y^i,$$

Note that g_i is the number of embeddings on the orientable surface O_i and f_i is the number of embeddings on the nonorientable surface N_i.

We call the first part of $I_G(x,y)$ the genus polynomial of G denoted by

$$g_G(x) = \sum_{i=0}^{\infty} g_i x^i.$$

Similarly,

$$f_G(y) = \sum_{i=1}^{\infty} f_i y^i$$

is the crosscap number polynomial of G. Of course $I_G(x,y) = g_G(x) + f_G(y)$.

The average genus of G is defined to be the value

$$\gamma_{\text{avg}}(G) = \frac{g'_G(1)}{g_G(1)}.$$

In the early years, the topological graph theorists focused their attention on the minimum and maximum genera of all the embeddings of G. Later, a lot of attention was given to the average genus of all the embeddings of G, for the details we can refer to [3, 5–13]. In this paper we study the average crosscap number which could be thought as a generalized average genus in a nonorientable surface. Now we have the following definition of the average crosscap number.

The average crosscap number of G is defined to be the value

$$\tilde{\gamma}_{\text{avg}}(G) = \frac{f'_G(1)}{f_G(1)}.$$

By the definition, we know the average crosscap number of a graph G is the ratio of the sum of all crosscaps of the nonorientable embeddings over the number of nonorientable embeddings of G. The investigation of the average genus and the average crosscap number will help us to understand the embeddings of the graphs.

Now we give an example to explain the above definition.

Example 1. $K_{3,3}$ and Q_3 are the two graphs shown in Fig. 1. The total genus polynomials of the two graphs are: $I_{K_{3,3}}(x,y) = 40 + 24x + 12y + 108y^2 + 432y^3 + 408y^4$, $I_{Q_3}(x,y) = 2 + 54x + 200x^2 + 24y + 192y^2 + 1288y^3 + 3264y^4 + 3168y^5$.

It is a routine task to compute the average genus and the average crosscap number of the two graphs $K_{3,3}$ and Q_3.

$$\gamma_{\text{avg}}(K_{3,3}) = 1\frac{3}{8}, \qquad \tilde{\gamma}_{\text{avg}}(K_{3,3}) = 2\frac{3}{8},$$
$$\gamma_{\text{avg}}(Q_3) = 1\frac{99}{128}, \qquad \tilde{\gamma}_{\text{avg}}(Q_3) = 4\frac{89}{496}.$$

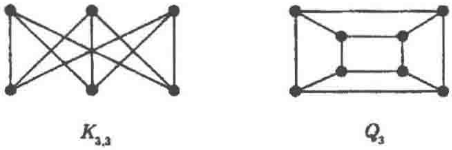

Fig. 1. The graphs $K_{3,3}$ and Q_3

The following theorem can be found in [3].

Theorem 1.2. *The total genus polynomial $I_G(x,y)$ is equal to the T-distribution polynomial $I_G^T(x,y)$, and the total number of the embeddings of the graph is*

$$2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!.$$

The above two theorems imply that the total genus polynomial of a graph independent of the choice of it is spanning tree.

The following method[2] can be used for finding an embedding on a surface (orientable and nonorientable). Firstly, we choose any spanning tree T of G and a general rotation system (P, λ) such that $\lambda(e) = 0, \forall\, e \in E(T)$. Secondly, we distinguish all co-tree edges of T by letters and replace each co-tree edge by two articulate edges with the same letter. Thirdly, we label the same letter with the same or distinct indices according as $\lambda(e) = 1$ or $\lambda(e) = 0$. Finally, according to the rotation, all lettered articulate edges of G_T form a polygon A with $\beta(G)$ pairs of edges. Then, we apply Relations 1, 2, and 3 and Operations 1, 2, and 3 to normalize the polygon A and get the handle or the crosscap number of the embedding.

From the above procedure, G is transformed into G_T without changing the rotation at each vertex except for the new vertices that are all articulate. Since G_T is a tree with a choice of the indices of pairs in the same letter, we called it the joint tree of G. Now we give an example to get the embedding of a graph by using the above procedure.

Example 2. Given the graph G=(V, E), $V = \{u, v, w, x, y\}, E = \{a, b, c, d, e, f, g, h\}$, h, e, f, and g are edges on T and a, b, c, and d are the co-tree edges. The rotation system P at each vertex is $u : (ehgf)$, $v : (had)$, $w : (bea)$, $x : (bfc)$, $y : (gcd)$.

Applying the above method, we get its joint tree G_T (see Fig. 2). We travel along the rotation (see the arrows) and get the polygon

$$aba^- b^- cd^- c^- d \sim O_2.$$

So the rotation system is an embedding on O_2.

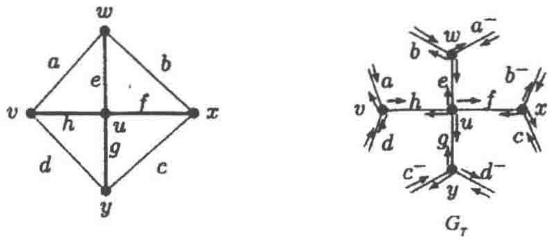

Fig. 2. The graph G and its joint tree G_T

Since the computing of the average crosscap number for a graph is NP-complete, it is not a easy task to compute the average crosscap number of a graph. Thus we may want to obtain the bounds for the average crosscap number of a graph. In the later sections, we will prove the following three theorems.

Theorem 1.3. *Let G be a connected graph which is not a tree, then*

$$\frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G) \leqslant \widetilde{\gamma}_{\text{avg}}(G) \leqslant \beta(G).$$

Furthermore the bounds are best possible.

We also describe the structures of a graph with the average crosscap numbers being $\frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G)$ and $\beta(G)$.

Theorem 1.4. *Let G be a connected graph, then $\widetilde{\gamma}_{\text{avg}}(G) = \frac{2^{\beta(G)-1}}{2^{\beta(G)}-1}\beta(G)$ if and only if G is a cactus.*

Theorem 1.5. *Let G be a connected graph, then $\widetilde{\gamma}_{\text{avg}}(G) = \beta(G)$ if and only if G is homeomorphic to a cycle.*

In other words, the cacti are the only graphs attaining the lower bound for the average crosscap number of Theorem 1.3. The loop B_1 is the only graph attaining the upper bound for the average crosscap number of Theorem 1.5.

2 Some basic results

For convenience, in a cyclic permutation P on Ω, i.e. every element of P belongs to Ω, if the two elements $x, y \in \Omega$ are in the form of $P = AxByCx^-Dy^-E$, they are said to be interlaced; otherwise, parallel.

We have the following

Lemma 2.1[2]**.** *If any two elements are parallel on P, then there is an element $x \in \Omega$ such that $\langle x, x^-\rangle \subseteq P$, i.e. $\langle x, x^-\rangle$ is a segment of P itself.*

Proof. Suppose to the contrary, if no such an element exists on Ω, then for any $x \in \Omega$ there is a nonempty linear order B_1 on Ω such that

$$P = A_1x_1B_1x_1^-C_1,$$

where A_1 and C_1 are some linear orders on Ω. Because B_1 is nonempty, for any $x_2 \in B_1$, on the basis of orientability and x_2 and x_1 being parallel, the only possibility is $x_2^- \in B_1$. From the known condition, there is also a nonempty linear order B_2 on Ω such that

$$B_1 = A_2x_2B_2x_2^-C_2,$$

where A_2 and C_2 are the segments on B_1, i.e. some linear orders on Ω. Such a procedure can go on to the infinity. This is a contradiction to the finiteness of the elements of P. Hence, the lemma is true.

Lemma 2.2. *Let P be a polygon on Ω, if $P \sim O_k$ ($k \geqslant 1$), then the two existing elements $x, y \in \Omega$ are interlaced.*

Proof. By contradication, any elements of P on Ω are parallel. By Lemma 2.1, we know that there exists an element $x \in \Omega$ such that $\langle x, x^-\rangle \subseteq P$, i.e. $P = Axx^-B$, where A and B are the linear orders on Ω. By Operation 1,

$$P = Axx^{-}B \sim AB.$$

Since any elements of AB are parallel too, by Lemma 2.1, there exists an element $y \in \Omega$ such that$< y, y^{-} >\subseteq AB$, i.e. $AB = Cyy^{-}D$, where C and D are the linear orders on Ω. By applying Operation 1 again, we have

$$AB = Cyy^{-}D \sim CD.$$

Such a procedure can go on to the infinity. Since the elements of P are finite, we at last have $P \sim O_0$. It contradicts to $P \sim O_k$ $(k \geqslant 1)$. Thus, there exist two elements $x, y \in \Omega$ which are interlaced.

Lemma 2.3. *Let a polygon $P = AB \sim O_0$, then we have the polygon $P_1 = aAaB \sim N_h$ where $2 \geqslant h \geqslant 1$.*

Proof. We may suppose the polygon

$$P = x_1x_2 \cdots x_{2n-1}x_{2n},$$

i.e. $\Omega = \{x_i | i = 1, 2, \dots, 2n\}$. We prove the lemma by induction on n.

If $n = 1$, then $P = x_1x_2 = x_1x_1^{-}$ and P_1 can be described as the following two forms

$$P_1 = aax_1x_1^{-} \quad \text{or} \quad P_1 = ax_1ax_1^{-}.$$

If P_1 is the former, by Operation 1, we have $P_1 \sim N_1$. If P_1 is the latter, by Relation 2, we have $P_1 \sim N_2$. So the lemma is immediately verified for $n = 1$.

Now we suppose the lemma is true for $n \leqslant m$ $(m \geqslant 1)$. If we prove the lemma for $n = m+1$, then we complete the proof. Since $P \sim O_0$, by Lemma 2.2 we know that any two elements are parallel on P. By Lemma 2.1, there is an element $x \in \Omega$ such that $\langle x, x^{-}\rangle \subseteq P$. So we can write $P = xx^{-}A_1A_2$, where $A_1, A_2 \subset \Omega - \{x\}$. If $P_1 = xx^{-}B_1aB_2aB_3$, by Operation 1 we know that $P_1 \sim B_1aB_2aB_3$, where $B_i \subset A_1 \cup A_2, i = 1, 2, 3$. By induction we know that $P_1 \sim B_1aB_2aB_3 \sim N_h$, where $2 \geqslant h \geqslant 1$. Otherwise $P_1 = xax^{-}C_1aC_2$. Since $C_1C_2 \sim O_0$, by the above discussion there exists an element $y \in \Omega - \{x\}$ such that $< y, y^{-} >\subseteq C_1C_2$. If $P_1 = xax^{-}D_1yy^{-}D_2aD_3$, where $D_i \subset C_1 \cup C_2 - \{y\}, i = 1, 2, 3$, by Operation 1 we know that $P_1 \sim xax^{-}D_1D_2aD_3$. By induction, we know that the theorem is true; otherwise we have $P_1 = xax^{-}E_1yay^{-}E_2$, where $E_i \subset C_1 \cup C_2 - \{y\}, i = 1, 2$. Since the polygon is $E_1E_2 \sim O_0$, such a procedure can go on to the infinity. Because the elements of P are finite, we at last have that the theorem is true, or the polygon P_1 can be described as $P_1 = x_{i_1} \cdots x_{i_k}ax_{i_k}^{-} \cdots x_{i_1}^{-}a$. By Relation 2, we have $P_1 \sim N_2$.

Thus, the proof is completed.

Lemma 2.4. *Let the polygon $P = AB \sim O_k$, then we have the polygon $P_1 = aAaB \sim N_h$ where $2k+2 \geqslant h \geqslant 2k+1$.*

Proof. We may assume that the polygon

$$P = x_1x_2 \cdots x_{2n-1}x_{2n},$$

i.e. $\Omega = \{x_i | i = 1, 2, \dots, 2n\}$. Then,we prove the lemma by induction on n.

If $n = 1$, then $P = x_1x_2 = x_1x_1^{-}$, by Lemma 2.3, we know that the theorem is true. Now we suppose the lemma is true for $n \leqslant m$ $(m \geqslant 1)$. If we prove the lemma for $n = m+1$, then we complete the proof.

Case 1. $P = AB \sim O_0$. In this case, by Lemma 2.3, it is true.

Case 2. $P = AB \sim O_k(k \geqslant 1)$. By Lemma 2.2, we can write P as the following form.

$$P = A_1xB_1yC_1x^-D_1y^-E_1,$$

where A_1, B_1, C_1, D_1 and E_1 are the linear orders on Ω.

Then, P_1 can be written as the form

$$A^1xB^1yC^1x^-D^1y^-E^1,$$

where $A^1 \cup B^1 \cup C^1 \cup D^1 \cup E^1 = A_1 \cup B_1 \cup C_1 \cup D_1 \cup E_1 \cup \{a\}$, $|A^1 \cup B^1 \cup C^1 \cup D^1 \cup E^1| = |A_1 \cup B_1 \cup C_1 \cup D_1 \cup E_1| + 2$ and $x, x^-, y, y^- \in \Omega$.

By Relation 1, we have

$$P \sim A_1D_1C_1B_1E_1xyx^-y^-,$$

$$P_1 \sim A^1D^1C^1B^1E^1xyx^-y^-.$$

Let $Q^1 = A^1D^1C^1B^1E^1$ and $Q_1 = A_1D_1C_1B_1E_1$. By the relation between Q^1 and Q_1, If we denote $Q^1 = aA'aB'$, then $Q_1 = A'B'$. Since $P = AB \sim O_k(k \geqslant 1)$ and by Relations 1, 2 and 3, we have $Q_1 \sim O_{k-1}$. Because $|Q_1| < m$, by induction

$$Q^1 \sim N_l,$$

where $2(k-1)+2 \geqslant l \geqslant 2(k-1)+1$.

By Relation 3, $P_1 \sim N_{l+2}$. Because

$$2k+2 = 2(k-1)+2+2 \geqslant l+2 \geqslant 2(k-1)+2+1 = 2k+1,$$

it is true for $n = m+1$.

Thus, the proof is completed.

Lemma 2.5. *Let $P = ABC \sim N_k$, $P_1 = AeBe^-C \sim N_l$ and $P_2 = AeBeC \sim N_m$. Then, we have $l \geqslant k+1$ or $m \geqslant k+1$.*

Proof. Suppose

$$P = x_1x_2\cdots x_{2n-1}x_{2n},$$

$$P_1 = x_1\cdots x_iex_{i+1}\cdots x_me^-x_{m+1}\cdots x_{2n},$$

$$P_2 = x_1\cdots x_iex_{i+1}\cdots x_mex_{m+1}\cdots x_{2n},$$

where $\Omega = \{x_1x_2\cdots x_{2n-1}x_{2n}\}$ and $n \geqslant 1$.

We prove the theorem by induction on n.

If $n = 1$, then $P = x_1x_2 = x_1x_1$. We have the following two cases.

Case 1. $P_1 = x_1ex_1e$, $P_2 = x_1ex_1e^-$. By Relation 2, $P_2 = x_1ex_1e^- \sim x_1x_1ee = N_2$.

Case 2. $P_1 = x_1x_1ee$, $P_2 = x_1x_1ee^-$. In this case, $P_1 = x_1x_1ee = N_2$.

So the theorem is verified for $n = 1$.

Now we assume that the theorem is true for $n \leqslant r$ $(r \geqslant 1)$. If we prove that it is true for $n = r+1$, we shall complete the proof. Since $P \sim N_k$, there exists a letter $a \in \{x_1, \ldots, x_{2n}\}$ with the same indices. By symmetry, there are the following two different forms of P, P_1 and P_2.

Case 1. $P = A_1aA_2aA_3BC, P_1 = A_1aA_2aA_3eBe^-C$ and $P_2 = A_1aA_2aA_3eBeC$ where A_1, A_2, A_3, B and C are the linear orders (or the polygons) on Ω.

By Relation 2

$$P = A_1aA_2aA_3BC \sim A_1A_2^- A_3BCaa,$$
$$P_1 = A_1aA_2aA_3eBe^-C \sim A_1A_2^- A_3eBe^-Caa,$$
$$P_2 = A_1aA_2aA_3eBeC \sim A_1A_2^- A_3eBeCaa,$$

where A_2^- is a linear order on Ω.

If $A_1A_2^- A_3BC \sim N_{k-1}$ $(k-1 \geqslant 1)$, we have

$$A_1A_2^- A_3eBe^-C \sim N_{l-1} \ \ (l-1) \geqslant 1,$$
$$A_1A_2^- A_3eBeC \sim N_{m-1} \ (m-1) \geqslant 1.$$

By induction, we have $(l-1) \geqslant (k-1)+1$ or $(m-1) \geqslant (k-1)+1$, i.e. $l \geqslant k+1$ or $m \geqslant k+1$. So it is true in this case.

Otherwise, $A_1A_2^- A_3BC \sim O_r$. Since

$$P = ABC \sim N_k,$$

by Relation 3, we have $2r+1 = k$. By Lemma 2.3,

$$A_1A_2^- A_3eBeC \sim N_{m-1}, \quad m-1 \geqslant 2r+1.$$

So $P_2 \sim N_m$ and $m \geqslant 2r+1+1 = k+1$.

Case 2. $P = A_1aA_2B_1aB_2C$, $P_1 = A_1aA_2eB_1aB_2e^-C$ and $P_2 = A_1aA_2eB_1aB_2eC$, where A_1, A_2, B_1, B_2 and C are the linear orders (or the polygons) on Ω. In this case, it is similar to Case 1 and the proof is omitted.

Thus, the proof is completed.

3 Main results

Let G' be a subgraph of a graph G and P be a pure rotation system on G. The induced rotation system P' on G' is obtained by deleting all edges of $G-G'$ from the rotation system P. Let Γ and Γ' are the sets of the pure rotation system on G and G' respectively. We denote $\Gamma_{P'}$ to be the set of all rotation systems on G that induce the rotation system P' on G'. The following Lemma is obvious[8].

Lemma 3.1. *Let G' be a subgraph of a graph G. Then the set Γ of all pure rotation systems on G is a disjoint union of the sets $\Gamma_{P'}$, taken over all pure rotation systems P' on G'. Moreover, $|\Gamma| = |\Gamma'| \cdot |\Gamma_{P'}|$, for any pure rotation system P' on the graph G'.*

Let $\Re_G^T$ be all T-rotation systems of G which are embeddable on the nonorientable surfaces. By the definition of the average crosscap number and Theorem 1.2, we have

$$\widetilde{\gamma}_{\rm avg}(G) = \frac{\sum_{R\in\Re_G^T}\widetilde{\gamma}(R)}{|\Re_G^T|}$$

and

$$|\Re_G^T| = (2^{\beta(G)}-1)\prod_{v\in V(G)}(d_v-1)!.$$

Note that $\widetilde{\gamma}(R)$ represents the crosscap number of R.

Let $e = uv$ which is not a cut-edge of G and $e \notin E(T)$, then we can choose the spanning tree T of G as the spanning tree of $G-e$. For any T-rotation system (P_G, λ) of $\Re_G^T$, let P_{G-e} be the induced rotation system of P_G. We define the T-rotation system (P_{G-e}, λ') of $G-e$ as follows:

(1) P_{G-e} is the induced rotation system of P_G,

(2) $\lambda'(f) = \lambda(f), f \in E(G-e) - E(T)$.

Let $\Re_{G-e}^T$ be all the set of the T-rotation systems (P_{G-e}, λ') of $G-e$ which are embeddable on the nonorientable surfaces and $\Im_{G-e}^T$ be all the pure T-rotation systems (P_{G-e}, λ') of $G-e$ which are embeddable on the orientable surfaces, i.e. for any $(P_{G-e}, \lambda') \in \Im_{G-e}^T$, $\lambda'(f) = 0$, $\forall\, f \in E(G-e)$. Let $\Gamma_{P_{G-e}}$ be the set of all rotation systems on G that induce the rotation system P_{G-e} on $G-e$. It is obvious that $|\Gamma_{P_{G-e}}| = (d_G(u)-1)(d_G(v)-1)$ or $(d_G(u)-1)(d_G(u)-2)$ according to $u \neq v$ or $u = v$. Since $\lambda(e)$ has two kinds of choice 0 or 1, we have the following.

Lemma 3.2.

$$|\Re_G^T| = 2|\Gamma_{P_{G-e}}||\Re_{G-e}^T| + |\Gamma_{P_{G-e}}||\Im_{G-e}^T|.$$

Proof of Theorem 1.1. Let $G = (V, E)$. By the definition of the average crosscap number, the inequality $\widetilde{\gamma}_{avg}(G) \leqslant \beta(G)$ is obvious. Now we prove the left part of the inequality. The theorem is proved by induction on $|\beta(G)|$. If $\beta(G) \leqslant 2$, G is homeomorphic to one of B_1, B_2, D_3 and $B_1 \oplus_e B_1$ (see Fig. 3).

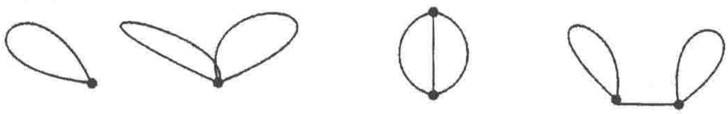

Fig. 3. Graphs whose betti number is less than 3.

The crosscap number polynomials of the graphs D_3, B_1 and $B_1 \oplus_e B_1$ are

$$f_{B_1}(y) = y, \qquad f_{B_2}(y) = 10y + 8y^2,$$
$$f_{D_3}(y) = 6y + 6y^2, \qquad f_{B_1 \oplus_e B_1}(y) = 8y + 4y^2.$$

It is a routine task to compute the average crosscap number of these graphs and we get

$$\widetilde{\gamma}_{\mathrm{avg}}(B_1) = 1, \qquad \widetilde{\gamma}_{\mathrm{avg}}(B_2) = \frac{13}{9} > \frac{4}{3},$$
$$\widetilde{\gamma}_{\mathrm{avg}}(D_3) = \frac{3}{2} > \frac{4}{3}, \qquad \widetilde{\gamma}_{\mathrm{avg}}(B_1 \oplus_e B_1) = \frac{4}{3}.$$

So, it is a easy task to check that it is true for $\beta(G) \leqslant 2$.

We may suppose the theorem is true for the graphs with $\beta(G) = k \geqslant 3$. Since the average crosscap number is a homeomorphic invariance of a graph, we can suppose $\delta(G) \geqslant 3$ and hence there exists an edge $e = uv$ which is not a cut-edge of G. i.e. $G-e$ is connected. Then we can choose a spanning tree T of G such that $e \notin E(T)$. We delete an edge e of G, and get the graph $G-e$. It is obvious that T is also a spanning tree of $G-e$. Let $\Re_G^T$ be all T-rotation systems (P_G, λ) of G which are embeddable on the nonorientable surfaces. Let $\Re_{G-e}^T$ be all the set of