刘彦佩

半闲数学集锦

Semi-Empty Collections in Mathemetics by Y.P.Liu

第十三编

时代文化出版社

刘彦佩
半闲数学集锦

Semi-Empty Collections in Mathemetics by Y.P.Liu

第十三编

时代文化出版社

半闲数学集锦（第十三编）

作　　者：刘彦佩
出版单位：时代文化出版社
地　　址：香港湾仔骆克道骆基中心23楼C座

编辑设计：北京时代弄潮文化发展有限公司
地　　址：北京市海淀区中关村创业大街25号家谱传记楼
电　　话：010-68920114　13693651386
网　　址：www.grcsw.com

印　　刷：京冀印刷厂
开　　本：880×1230　1/16
版　　次：2016年9月第1版

书　　号：ISBN 978-988-18455-0-4
定　　价：全套 1890.00元（共22编）

第十三编序

本编分两个部分. 第一部分由文 13.01[131]—13.26[248] 组成, 反映在地图的各类计数中, 对已有结果的进一步简化. 特别是在小亏格曲面上, 的计数方程, 及其求解; 考虑到地图上的多项式, 以及以点剖分向量, 或面剖分向量为参量的一些新计数等.

第二部分为专著[354] *地图的代数原理*(13.27—13.56).

在上一编的序中提到, 对于各种地图同构分类的计数, 包括色和, 范色和, 以及梵和等的基础理论研究, 分三个重要阶段.

第一阶段是在求得给定地图集合, 计数函数所满足方程的基础上, 通过特征曲线法(重根法, 或二次法, 可视为特例), 引出参数方程组, 通过消元和反演, 或演化为可资利用的差分方程, 甚至微分方程. 这里, 有两个任务.

任务 1. 发现尽可能多的, 具有相当普遍性的地图类, 使得能够通过对于每类地图合适的分解, 对于适合的参数, 导出确定任何给定参数下, 这类地图根同构类数的, 计数函数所满足的方程.

任务 2. 求出这个满足给定方程的所需要的解. 这些方程的未定元, 一般总是至少两个变元的函数, 同时还带至少一个由这个未定元决定, 但少一个变元的函数, 我们称*侧函数*, 由此导致直接求解的困难.

例如, 文 13.02[128]—13.03[130], 13.05[161]—13.07[184], 13.12[196], 13.14[207]—13.15[209], 13.19[219]—13.20[225], 13.23[235], 13.25[240] 等都是考虑在平面, 或球面, 的情形. 其中, 13.03 伴随地图的多项式.

文 13.08[185]—13.10[191], 13.13[202], 13.16[210], 13.18[218], 13.21[226]—13.22[228], 13.24[239], 13.26[248] 等, 都是考虑在亏格 (不超过 3) 非零的曲面上. 其中, 13.25 伴随一个多项式.

以上, 都是第一阶段的产物.

下面的两个阶段, 都是讨论, 从无穷计数参数, 所演化来的方程, 或者说, 介子泛函方程, 简称介子方程.

第二阶段是平面型的介子方程的出现, 用无穷维的矩阵分析和/或 Lagrange 反演, 求解其中的一些方程.

例如, 文 13.01[131] 和 13.17[211]. 在前者中, 只综述了已经取得的研究成果, 和由此引发的需要进一步研究的问题. 后者, 则是指讨论与树等价的一些介子方程, 及它们的解.

第三阶段是发现曲面型介子方程, 以及通过计数理论, 导出其中一些方程解的显式, 为直接求解提供了明确的目标. 这些都是围绕地图的计数.

本编所收录的文章, 只与第一阶段和第二阶段有关.

另外, 文 13.04[144], 13.11[193] 都是与图在曲面上的可嵌入性有关.

因为专著[292](12.39—12.62)出版之后, 又取得了足够的新进展, 示意急待完善. 这就迫使我, 继续完成专著[354](13.27—13.56).

第一、重提图的一种代数模型, 为从理论上, 澄清图的嵌入与地图的关系, 建立了一个顺理成章的基础.

第二、明确了一系列代数化的原则, 特别是将有限递归原理、强有限递归原理, 和曲面闭曲线公理, 作为起步点. 这些都是从专著[292]开始提及的.

第三、完善了地图棱之间的对偶性, 由此导致在图上添一棱、删一棱、伸一棱和缩一棱等, 非拓扑运算之间的对偶性.

第四、论证了由联树所确定的, 图在曲面上的嵌入, 不依赖支撑树的选择.

第五、引入图的关联曲面. 建立了关联曲面与联树之间的, 一个 1—1 对应. 由此, 在面上的联树问题, 变成了在线上的关联曲面问题.

第六、通过一个关联曲面的按层分解, 构建图的曲面嵌入与联树之间的, 一个 1—1 对应.

第七、提供了一个, 既不遗漏, 也不重复, 列出一个图, 所有曲面嵌入, 的最有效的算法.

第八、提出确定非树图的任何一个可定向嵌入, 由它产生的所有不可定向嵌入的亏格多项式, 的问题. 或者等价地, 确定一个可定向关联曲面, 由它产生所有不可定向关联曲面的, 亏格多项式.

第九、基于这个算法, 通过计算机, 得以实现. 在书中提供了一批, 计算结果.

第十、在课外活动中, 思考题、练习题, 特别是研究题, 都增到约 150 道.

刘彦佩
2015 年 7 月
於北京上园村

第十三编目录

Discrete Mathematics 123 (1993) 93–109
North-Holland

On functional equations arising from map enumerations*

Yanpei Liu

Institute of Applied Mathematics and Institute of Mathematics, Academia Sinica, Beijing 100080, China

Received 8 March 1990

Abstract

This paper summarizes with a number of new results, a variety of functional equations which arise from the enumerations of planar maps. A few applications are also discussed.

1. Introduction

1.1. A map M here is meant a permutation $\mathscr{J}$ on a set $\mathscr{X}$ with the following conditions.

Condition 1. The set $\mathscr{X}=\bigcup_{x\in X}\mathscr{K}x$, where $\mathscr{K}x=\{x,\alpha x,\beta x,\alpha\beta x\}$ is called a quadricell, $X=\{x_1,x_2,\ldots,x_m\}$, and $\mathscr{K}=\{1,\alpha,\beta,\alpha\beta\}$ is the Klein transformation group of four elements.

We usually write $\mathscr{X}=\mathscr{X}_{\alpha,\beta}(X)$ when it is not necessary to notify X,α and β.

Condition 2. The permutation $\mathscr{J}$ on $\mathscr{X}$ has to obey the following two axioms.

Axiom 1. $\alpha\mathscr{J}=\mathscr{J}^{-1}\alpha$.

Axiom 2. *The group* Ψ_J generated by $J=\{\alpha,\beta,\mathscr{J}\}$ *is transitive on* $\mathscr{X}$.

Thus, we may write the map $M=(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})$. From Axiom 1, α,β are asymmetric, i.e. $(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})\neq(\mathscr{X}_{\beta,\alpha}(X),\mathscr{J})$. Generally, for a map $M=(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})$, it is not necessary that $(\mathscr{X}_{\beta,\alpha}(X),\mathscr{J})$ is also a map because for β, it is not guaranteed to have

Correspondence to: Yanpei Liu, Institute of Applied Mathematics, Academia Sinica, Beijing, China.
* Supported by the National Natural Science Foundation of China.

Axiom 1. In fact, Axiom 1 allows us to define the vertices of a map as the pairs of conjugate orbits of $\mathscr{J}$ on $\mathscr{X}$.

1.2. For a map $M=(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})$ given, from the asymmetry of α and β, we may call α the first operator and β, the second. It is easy to show that $M^*=(\mathscr{X}_{\beta,\alpha}(X),\mathscr{J}_{\alpha\beta})$ is also a map with β as the first operator and α, the second. We call M^* the dual map of M. From the duality, we may define the faces of M to be the corresponding vertices of M^*. In addition, the edges of a map $M=(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})$ are defined to be the quadricells $\mathscr{K}x=\{x,\alpha x,\beta x,\alpha\beta x\}$ for $x\in X$. For an edge $\{x,\alpha x,\beta x,\alpha\beta x\}$, $\{x,\alpha x\}$ and $\{\beta x,\alpha\beta x\}$, $\{x,\beta x\}$ and $\{\alpha x,\alpha\beta x\}$ as well, are said to be semi-edges. Let ν, ε, and φ be the number of vertices, edges, and faces of M, respectively. The number

$$E(M)=\nu-\varepsilon+\varphi$$

is said to be the Eulerian characteristic of M.

1.3. For a map $M=(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})$, if it satisfies the following axiom, then we call it orientable; otherwise, nonorientable.

Axiom 3. *The group Ψ_L generated by $L=\{\alpha\beta,\mathscr{J}\}$ is not transitive on $\mathscr{X}_{\alpha,\beta}(X)$.*

In fact, if Ψ_L is not transitive, then it will have exactly two orbits on $\mathscr{X}_{\alpha,\beta}(X)$. When M is orientable, we always have $E(M)\leqslant 2$ and $E(M)=0 \pmod 2$. If $E(M)=2-2p$, then M is a map on the surface of genus $p\geqslant 0$. If $p=0$, then M is said to be planar. Here we mainly discuss planar maps. When M is nonorientable, we always have $E(M)\leqslant 0$. If $E(M)=1-q$, then M is a map on the nonorientable surface of genus q. When $q=1$, the surface is the projective plane.

1.4. A map $M=(\mathscr{X}_{\alpha,\beta}(X),\mathscr{J})$ is said to be rooted if an element of $\mathscr{X}_{\alpha,\beta}(X)$ is chosen to be the one specially marked, which is called the root. We denote the root by r and the rooted map by $M^{(r)}=(\mathscr{X}^{(r)}_{\alpha,\beta}(X),\mathscr{J})$. In a rooted map, the vertex, the edge, and the face which involve the root are said to be the root-vertex, the root-edge, and the root-face denoted by v_r, e_r and f_r, respectively. For two rooted maps $M_1^{(r_1)}=(\mathscr{X}^{(r_1)}_{\alpha,\beta}(X_1),\mathscr{J}_1)$ and $M_2^{(r_2)}=(\mathscr{X}^{(r_2)}_{\alpha,\beta}(X_2),\mathscr{J}_2)$, if there exists a bijection $\tau:\mathscr{X}^{(r_1)}_{\alpha,\beta}(X_1)\to\mathscr{X}^{(r_2)}_{\alpha,\beta}(X_2)$ with $\tau(r_1)=r_2$ such that the diagrams

$$\begin{array}{ccc}\mathscr{X}^{(r_1)}_{\alpha,\beta}(X_1) & \xrightarrow{\tau} & \mathscr{X}^{(r_2)}_{\alpha,\beta}(X_2)\\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2}\\ \mathscr{X}^{(r_1)}_{\alpha,\beta}(X_1) & \xrightarrow{\tau} & \mathscr{X}^{(r_2)}_{\alpha,\beta}(X_2)\end{array}$$

are commutative for $\gamma_1=\gamma_2=\alpha$, for $\gamma_1=\gamma_2=\beta$, and for $\gamma_1=\mathscr{J}_1$ and $\gamma_2=\mathscr{J}_2$, then we say that $M_1^{(r_1)}$ and $M^{(r_2)}$ are isomorphic, denoted by $M_1^{(r_1)}\sim M_2^{(r_2)}$. We only consider rooted maps here and all isomorphic maps as the same.

1.5. In a map M, the valency $val(v)$ of a vertex v is the number of semi-edges involved in v. For a map M given, a vertex partition of the non-root-vertex set $\mathscr{V}-v_r$ is determined by the valencies of vertices as

$$\mathscr{V}(M)-\{v_r\}=\sum_{i\geq 1}\mathscr{V}_i(M),\quad \mathscr{V}_i(M)=\{v\,|\,val(v)=i\}.$$

Let $m(M)$ be the valency of the root-vertex in M, and let $n_i(M)=|\mathscr{V}_i(M)|$, $i\geq 1$. In order to enumerate nonisomorphic maps in a set $\mathscr{N}$, we have to investigate the function

$$g_{\mathscr{N}}(x;y_1,y_2,\ldots)=\sum_{M\in\mathscr{N}}x^{m(M)}\prod_{i\geq 1}y_i^{n_i(M)},\tag{1.1}$$

which is said to be the vertex partition function of $\mathscr{N}$. Let $\mathscr{R}(x;y_1,y_2,\ldots)=\{f\,|\,f=\sum_{m\geq 0}\sum_{(n_1,n_2,\ldots)\geq 0}a_{n_1,n_2,\ldots}x^m\prod_{i\geq 1}y_i^{n_i},\ a_{n_1,n_2,\ldots}\in R^+\}$. In dual form, the face partition function is defined as

$$f_{\mathscr{N}}(x;y_1,y_2,\ldots)=\sum_{M\in\mathscr{N}}x^{s(M)}\prod_{i\geq 1}y_i^{s_i(M)},$$

where $s(M)$ is the valency of the root-face and $s_i(M)$ is the number of non-root-faces of valency i, $i\geq 1$, in M.

1.6. We treat a function $f(z)$ as a member of the function space $\mathscr{F}$ which has $\{1,z,z^2,z^3,\ldots\}$ as a basis. On the function space $\mathscr{F}$, we introduce a functional, denoted by $\int_z$ such that

$$\int_z z^i=z_i,\quad i\geq 1,\qquad \int_z 1=1.\tag{1.2}$$

Therefore, $\int_z$ is a transformation from $\mathscr{F}$ to the vector space $\mathscr{V}$ which has $\{1,z_1,z_2,\ldots\}$ as a basis. It is easy to see that $\int_z$ is linear.

In addition, we also introduce two operators on $\mathscr{F}$. For $f(z)\in\mathscr{F}$, let

$$\delta_{x,y}f=\frac{f(x)-f(y)}{x-y},\qquad \partial_{x,y}f=\frac{yf(x)-xf(y)}{x-y}\tag{1.3}$$

which are said to be (x,y)-, $\langle x,y\rangle$-deference of f, respectively. It is easy to check that the following relation holds:

$$\partial_{x,y}(zf)=xy\delta_{x,y}f.\tag{1.4}$$

The vertex partition function $g_{\mathscr{N}}(x;y_1,y_2,\ldots)$ can be treated as a member of $\mathscr{F}$ as $g_{\mathscr{N}}(z)=g_{\mathscr{N}}(z;y_1,y_2,\ldots)$ in the context.

The purpose of this paper is to discuss a number of functional equations which involve the functional $\int_z$.

1.7. Terminologies not explained here refer to [27] on combinatorial maps and to [5] on combinatorial enumeration.

2. Equations solved

2.1. A map M is called a tree map if the underlying graph which consists of the vertices and the edges of M, is a tree. If the valency of the root-vertex of a tree map is 1, then the tree map is said to be planted.

It is easy to see that any planted tree map, except the trivial case of the link map, can be uniquely produced by several planted tree maps in the following way. Suppose the valency of the nonrooted end, which is incident with the semi-edge $\{\beta r, \alpha\beta r\}$, of the root-edge in M be $k \geqslant 2$, then M can be uniquely produced by identifying the root-vertices of $k-1$ planted tree maps and then identifying the nonrooted end of the link map with the root-vertex of the resultant map. Let $\mathscr{T}_1$ be the set of all planted tree maps and let $g_{\mathscr{T}_1} = \sum_{T \in \mathscr{T}_1} \prod_{i \geqslant 1} y_i^{n_i(T)}$ be the vertex partition function of $\mathscr{T}_1$.

Theorem 2.1 (Liu [11]). *The functional equation*

$$f = y_1 + \int_y \left(\frac{y^2 f}{1 - yf} \right) \tag{2.1}$$

is well defined in $\mathscr{R}(x; y_1, y_2, \ldots)$ *and the solution is*

$$f = g_{\mathscr{T}_1} = \sum_{n \geqslant 1} \sum_{(n_1, n_2, \ldots) \in \Gamma} \frac{1}{n} \binom{n}{n_1, n_2, \cdots} \prod_{i \geqslant 1} y_i^{n_i}, \tag{2.2}$$

where $\Gamma = \{(n_1, n_2, \cdots) \geqslant 0 \mid \sum_{i \geqslant 1} n_i = n, \sum_{i \geqslant 1} in_i = 2n - 1\}$.

Equation (2.1) can be derived from what was discussed in *2.1*. Thus, we see that the solution $f = g_{\mathscr{T}_1}$. The formula (2.2) can be obtained by using Lagrangian inversion to solve (2.1).

Let $\mathscr{T}$ be the set of all tree maps. Then by (2.2), $g_{\mathscr{T}} = g_{\mathscr{T}_1} + g_{\mathscr{T}_1}^2 + \cdots = g_{\mathscr{T}_1}(1 - g_{\mathscr{T}_1})^{-1}$ can be determined. Here we have to pay attention that a vertex P without an edge is not a map. Of course, $P \notin \mathscr{T}$.

2.2. A map M which has the root-vertex v_r adjacent to all other vertices such that $M - v_r$ is also a map is said to be a superwheel. Let $\mathscr{S}$ be the set of all superwheels in which the root-vertices are not cut-vertices. Then $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$, where $\mathscr{S}_1 = \{L\}$, L is the loop map $(\{r, \alpha r, \beta r, \alpha\beta r\}, (r, \alpha\beta r)(\beta r, \alpha r))$. We may find that $g_{\mathscr{S}_1} = x^2$ and

$$g_{\mathscr{S}_2} = x \int_y \left(\sum_{k \geqslant 1} y^{k+1} \prod_{i=0}^{k-1} \left(\sum_{M_i \in \mathscr{S}} x^{m(M_i)-1} \prod_{j \geqslant 2} y_j^{n_j(m_i)} \right) \right)$$

$$= x^2 \int_y \left(\frac{y}{x - y g_{\mathscr{S}}} \right).$$

Theorem 2.2. *The functional equation*

$$f=x^2+x^2\int_y\left(\frac{y}{x-yf}\right) \tag{2.3}$$

is well defined in $\mathscr{R}(x;\ y_1,y_2,\dots)$ *and the solution is*

$$f=g_{\mathscr{S}}=x^2+\sum_{m\geq 2}x^m\sum_{N\in S}\frac{1}{s+1}\left(\frac{(s+1)!}{(m-1)!\prod_{j\geq 2}n_j!}\right)\prod_{k\geq 1}y_{k+1}^{n_{k+1}}, \tag{2.4}$$

where $S=\{N=(n_2,n_3,\dots)\mid\sum_{j\geq 2}n_j=m,\ s=\sum_{k\geq 1}kn_{k+1},\ 0\leqslant n_j\leqslant m,\ j\geqslant 2\}$.

Here we can also employ the Lagrangian inversion to solve (2.3) for obtaining the solution given by (2.4).

In fact, superwheels here are duals of nonseparable outerplanar maps. Therefore, the face partition function of nonseparable outerplanar maps is also determined by (2.4) in the dual form. Further, we may also find the summation-free formulae for the simple, the bipartite, and the simple bipartite cases of this kind of maps [18].

Theorem 2.3 (Liu [11]). *The functional equation*

$$f=x\int_y(y\delta_{x,y}(zf+z)) \tag{2.5}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\dots)$ *and the solutions can be expressed as*

$$f=g_{\mathscr{T}}=X\left(\sum_{i\geq 0}y_1\,Y_{T_r}^{i}e_1^{\mathrm{T}}\right), \tag{2.6}$$

where

$$X=(x,x^2,x^3,\dots),\qquad e_1=(1,0,0,\dots),$$

$$Y_{T_r}=\begin{pmatrix} y_2 & y_3 & y_4 & y_5 & \cdots\\ y_1 & y_2 & y_3 & y_4 & \cdots\\ & y_1 & y_2 & y_3 & \cdots\\ & & y_1 & y_2 & \cdots\\ 0 & & & \ddots & \ddots \end{pmatrix}.$$

Because the function given by (2.6) is the vertex partition function of general tree maps, the coefficient of x in (2.6) is just $g_{\mathscr{T}_1}$ which is given by (2.2).

2.3. A tree map which is allowed to have one circuit at each articulated vertex is called a wintersweet map. The root-edge is not chosen on a circuit. Let $\mathscr{W}$ be the set of all the wintersweet maps.

Theorem 2.4 (Liu [11]). *The functional equation*

$$\left(1-\frac{xy_3}{1-y_2}\right)f=x\int_y(y\,\delta_{x,y}(z+zf)) \tag{2.7}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\dots)$ *and the solution can be expressed by*

$$f=g_{\mathscr{W}}=X\left(\sum_{i\geqslant 0}\left(y_1+\frac{y_3}{1-y_2}\right)Y_W^i e_1^{\mathrm{T}}\right) \tag{2.8}$$

where we denote

$$Y_W=\begin{pmatrix} y_2 & y_3 & y_4 & y_5 & \cdots \\ c & y_2 & y_3 & y_4 & \cdots \\ & c & y_2 & y_3 & \cdots \\ & & c & y_2 & \cdots \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

and $c=y_1+y_s/(1-y_2)$.

2.4. Let $\mathscr{U}$ be the set of all the maps in each of which there is only one circuit. The root-edge is chosen to be on the circuit, and the circuit is the boundary of a face.

Theorem 2.5 (Liu [11]). *The functional equation*

$$f=x^2g_{\mathscr{T}}+x\int_y(y\,\partial_{x,y}f) \tag{2.9}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\dots)$ *and the solution has the form*

$$f=g_{\mathscr{U}}=X\left(\sum_{i\geqslant 0}\sum_{j\geqslant 0}y_1\,Y_U^i\,Y_{T_r}^j e_1^{\mathrm{T}}\right) \tag{2.10}$$

where $g_{\mathscr{T}}$ *and* Y_{T_r} *are indicated in* (2.6) *and*

$$Y_U=\begin{pmatrix} y_2 & y_3 & y_4 & y_5 & y_6 & \cdots \\ & y_2 & y_3 & y_4 & y_5 & \cdots \\ & & y_2 & y_3 & y_4 & \cdots \\ & & & y_2 & y_3 & \cdots \\ & & & & y_2 & \cdots \\ 0 & & & & & \ddots \end{pmatrix}.$$

For some kinds of general unicyclic maps, similar results can also be obtained in this way.

Theorem 2.6 (Liu [13]). *Let $\mathscr{A}$ be the set of all outerplanar maps and let*

$$\varphi(x)=\sum_{n\geq 0}\frac{(2n)!}{(n+1)!n!}x^{2n}=\frac{1}{2x^2}(1-\sqrt{1-4x^2}).$$

Then the functional equation

$$f=1+x^2\varphi(x)f+x\int_y(y\delta_{x,y}(zf)) \tag{2.11}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\ldots)$ and the solution can be expressed as

$$f=g_{\mathscr{A}}=X\left(\sum_{i\geq 0}Y_0^i\left(y_1e_1^{\mathrm{T}}+\sum_{n\geq 1}\frac{(2n-2)!}{n!(n-1)!}e_{2n}^{\mathrm{T}}\right)\right), \tag{2.12}$$

where

$$Y_0=\begin{pmatrix} y_2 & y_3 & y_4 & y_5 & y_6 & \cdots \\ y_1 & y_2 & y_3 & y_4 & y_5 & \cdots \\ & y_1 & y_2 & y_3 & y_4 & \cdots \\ & & y_1 & y_2 & y_3 & \cdots \\ & & & y_1 & y_2 & \cdots \\ * & & & & \ddots & \ddots \end{pmatrix}$$

and the asterisk '' denotes*

$$y_{i,j}=\begin{cases}(2n-2)!/n!(n-1)!, & i-j=2n\\ 0, & i-j=2n+1,\end{cases}$$

with $n=1,2,3,\ldots$

Theorem 2.7 (Liu [14]). *Let $\mathscr{B}$ be the set of all nonseparable outerplanar maps. Then the functional equation*

$$f=x^2+\int_y\left(\frac{xy\partial_{x,y}(zf)}{1-xy}\right) \tag{2.13}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\ldots)$ and the solution is

$$f=g_{\mathscr{B}}=X\left(\sum_{i\geq 0}Y_S^ie_1^{\mathrm{T}}\right), \tag{2.14}$$

where

$$Y_S=\begin{pmatrix} y_2 & y_3 & y_4 & y_5 & \cdots \\ & y_2+y_4 & y_3+y_5 & y_4+y_6 & \cdots \\ & & y_2+y_4+y_6 & y_3+y_5+y_7 & \cdots \\ & & & y_2+y_4+y_6+y_8 & \cdots \\ & * & & & \ddots \end{pmatrix}$$

and asterisk '$*$' is determined by the symmetry

$$y_{i,j}=y_{j,i},\quad i,j\geqslant 1.$$

3. Equations unsolved

3.1. Let $\mathcal{M}$ be the set of all planar maps. For convenience, we have to put the graph ϑ which consists of a single vertex in $\mathcal{M}$ as the degenerate case. $\mathcal{M}$ can be partitioned into three parts: one consists of the single map ϑ, and one of the other two consists of all the planar maps in which the root-edge is a loop. For a map M, let $R=e_r$. If R is a loop, then M can be uniquely reformed by two maps $M_1, M_2\in\mathcal{M}$ through identifying the two root-vertices of M_1, M_2 and then adding the root-loop.

For $\vartheta\neq M\in\mathcal{M}$, and R not a loop, we have that $M\cdot R$, the resultant map of contracting R from M, is a member of $\mathcal{M}$. However, for a map $M\in\mathcal{M}$, we have exactly $m(M)+1$ maps: $M_{(i,m(M)-i+2)}$, $i=1,2,\ldots,m(M)+1$, in $\mathcal{M}$. They are obtained by splitting the root-vertex, v_r in M into v_{r_i} and $v_{\beta r_i}$ such that the valency of v_{r_i} is i, $i=1,2,\ldots,m(M)+1$. Their root-edges are not loops.

Theorem 3.1. *The functional equation*

$$f=1+x^2f^2+\int_y(xy\delta_{x,y}(zf))\tag{3.1}$$

is well defined in $\mathcal{R}(x;y_1,y_2,\ldots)$ and the solution is $f=g_{\mathcal{M}}$.

From [25], the dual form of (3.1) can be found.

3.2. Let $\mathcal{L}$ be the set of all loopless planar maps [8]. We may partition $\mathcal{L}$ into three parts as $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3$, where $\mathcal{L}_1$ consists of the single map ϑ and $\mathcal{L}_2$ consists of all the maps whose root-edges are not multiple. We may further see that $\mathcal{L}_2$ can be produced by splitting the maps in $\mathcal{L}$. On $\mathcal{L}_3$, we have to introduce new kind of maps, called inner maps, in each of which, there is only one loop, the root-edge with the root-face $\{(r),(\beta r)\}$. Then we may construct maps in $\mathcal{L}_3$ by inner maps in a proper way. However, an inner map can be exactly expressed by adding the root-loop in a loopless map. By this procedure, we may finally find the following theorem.

Theorem 3.2 (Liu [19]). *The functional equation*

$$f=1+\int_y\left(\frac{\partial_{x,y}(z^2f)}{1-\partial_{x,y}(z^2f)}\right)\tag{3.2}$$

is well defined in $\mathcal{R}(x;y_1,y_2,\ldots)$ and the solution is $f=g_{\mathcal{L}}$.

3.3. Let $\mathscr{S}_{\text{im}}$ be the set of all simple maps, the planar maps whose underlying graphs are simple [9]. Here we have to consider the case of face partition, the dual vertex partition. Let $f_{\mathscr{S}_{\text{im}}}$ be the face partition function of $\mathscr{S}_{\text{im}}$. In this case, we take $\mathscr{S}_{\text{im}}=\mathscr{S}_{\text{im}1}+\mathscr{S}_{\text{im}2}+\mathscr{S}_{\text{im}3}$, where $\mathscr{S}_{\text{im}1}=\{\vartheta\}$, $\mathscr{S}_{\text{im}2}$ consists of all the maps in $\mathscr{S}_{\text{im}}$ such that the root-edges are separable. The contribution of $\mathscr{S}_{\text{im}1}$ to $f_{\mathscr{S}_{\text{im}}}$ is 1. The contribution of $\mathscr{S}_{\text{im}2}$ to $f_{\mathscr{S}_{\text{im}}}$ can also be easily found because a map in $\mathscr{S}_{\text{im}2}$ is determined by a pair of maps in $\mathscr{S}_{\text{im}}$. The more complicated case is to find the contribution of $\mathscr{S}_{\text{im}3}$ to $f_{\mathscr{S}_{\text{im}}}$. In order to do this, we have to consider that $\mathscr{S}_{\text{im}_3}=\widetilde{\mathscr{S}_{\text{im}}}-\mathscr{S}_{\text{im}}(loop)-\mathscr{S}_{\text{im}}(mult)$, where $\widetilde{\mathscr{S}_{\text{im}}}$ consists of all the maps which can be seen as the resultant maps of adding the root-edges in the root-faces of maps in $\mathscr{S}_{\text{im}}$. Of course, not all maps in $\widetilde{\mathscr{S}_{\text{im}}}$ are simple. Therefore, we have to leave off $\mathscr{S}_{\text{im}}(loop)$ and $\mathscr{S}_{\text{im}}(mult)$ from $\widetilde{\mathscr{S}_{\text{im}}}$, where $\mathscr{S}_{\text{im}}(loop)=\{M \mid M\in\widetilde{\mathscr{S}_{\text{im}}}$ *with only one loop which is rooted*$\}$, $\mathscr{S}_{\text{im}}(mult)=\{M \mid M\in\widetilde{\mathscr{S}_{\text{im}}}$ *with only the root-edge multiple*$\}$. Further, we may find the relation of $\mathscr{S}_{\text{im}}(loop)$ and $\mathscr{S}_{\text{im}}(mult)$ to $\mathscr{S}$. Thus, we have the following theorem.

Theorem 3.3 (Liu [20]). *The functional equation*

$$\left(x^2 f+\int_y((1-xy)f)\right)f=\int_y(xy\delta_{x,y}(zf)+f) \tag{3.3}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\ldots)$ *and the solution is* $f=f_{\mathscr{S}_{\text{im}}}$.

3.4. Let $\mathscr{M}_{\text{ns}}$ be the set of all nonseparable maps and let $\mathscr{M}_{\text{ns}}=\mathscr{M}_{\text{ns}1}+\mathscr{M}_{\text{ns}2}$, where $\mathscr{M}_{\text{ns}1}=\{L\}$, the loop map. The main step is to investigate the relationship between $\mathscr{M}_{\text{ns}2}$ and $\mathscr{M}$. First, we may see that for $M\in\mathscr{M}_{\text{ns}2}$, $M\cdot R$ can be constructed by identifying the root-vertices of several maps in $\mathscr{M}$. For several maps in $\mathscr{M}$, if the ways of splitting each of the root-vertices are given, then a map in $\mathscr{M}_{\text{ns}2}$ can be determined by identifying the root-edges which are obtained from splitting. From this consideration, in consequence we may find the following theorem.

Theorem 3.4 (Liu [10]). *The functional equation*

$$f=x^2+x\int_y\left(\frac{y\partial_{x,y}f}{1-\partial_{x,y}f}\right) \tag{3.4}$$

is well defined in $\mathscr{R}(x;y_1,y_2,\ldots)$ *and the solution is* $f=g_{\mathscr{M}_{\text{ns}}}$.

In what follows, we present three functional equations related to the Eulerian cases.

3.5. Let $\mathscr{E}$ be the set of all Eulerian maps. As soon as we note that no Eulerian map has an edge separable, we may find that $\mathscr{E}$ can be partitioned into three parts: $\mathscr{E}_0,\mathscr{E}_1$