Francis Borceux

Geometric Trilogy I

An Axiomatic Approach to Geometry

几何三部曲 第1卷

几何的公理化方法

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An Axiomatic Approach to Geometry

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Geometric Trilogy I: An Axiomatic Approach to Geometry
by Francis Borceux

To Christiane

Preface

The reader is invited to immerse himself in a "love story" which has been unfolding for 35 centuries: the love story between mathematicians and geometry. In addition to accompanying the reader up to the present state of the art, the purpose of this *Trilogy* is precisely to tell this story. The *Geometric Trilogy* will introduce the reader to the multiple complementary aspects of geometry, first paying tribute to the historical work on which it is based and then switching to a more contemporary treatment, making full use of modern logic, algebra and analysis. In this *Trilogy*, Geometry is definitely viewed as an autonomous discipline, never as a sub-product of algebra or analysis. The three volumes of the *Trilogy* have been written as three independent but complementary books, focusing respectively on the axiomatic, algebraic and differential approaches to geometry. They contain all the useful material for a wide range of possibly very different undergraduate geometry courses, depending on the choices made by the professor. They also provide the necessary geometrical background for researchers in other disciplines who need to master the geometric techniques.

The present book leads the reader on a walk through 35 centuries of geometry: from the papyrus of the Egyptian scribe *Ahmes*, 16 centuries before Christ, to Hilbert's famous axiomatization of geometry, 19 centuries after Christ. We discover step by step how all the ingredients of contemporary geometry have slowly acquired their final form.

It is a matter of fact: for three millennia, geometry has essentially been studied via "synthetic" methods, that is, from a given system of axioms. It was only during the 17th century that algebraic and differential methods were considered seriously, even though they had always been present, in a disguised form, since antiquity.

After rapidly reviewing some results that had been known empirically by the Egyptians and the Babylonians, we show how Greek geometers of antiquity, slowly, sometimes encountering great difficulties, arrived at a coherent and powerful deductive theory allowing the rigorous proof of all of these empirical results, and many others. Famous problems—such as "squaring the circle"—induced the development of sophisticated methods. In particular, during the fourth century BC, *Eudoxus* overcame the puzzling difficulty of "incommensurable quantities" by a method which is essentially that of Dedekind cuts for handling real numbers. Eudoxus also proved the validity of a "limit process" (the *Exhaustion theorem*) which allowed him to answer questions concerning, among other things, the lengths, areas or volumes related to various curves or surfaces.

We first summarize the knowledge of the Greek geometers of the time by presenting the main aspects of *Euclid's Elements*. We then switch to further work by *Archimedes* (the circle, the sphere, the spiral, ...), *Apollonius* (the conics), *Menelaus* and *Ptolemy* (the birth of trigonometry), *Pappus* (ancestor of projective geometry), and so on.

We also review some relevant results of classical *Euclidean geometry* which were only studied several centuries after Euclid, such as additional properties of triangles and conics, Ceva's theorem, the trisectors of a triangle, stereographic projection, and so on. However, the most important new aspect in this spirit is probably the theory of inversions (a special case of a conformal mapping) developed by Poncelet during the 19th century.

We proceed with the study of projective methods in geometry. These appeared in the 17th century and had their origins in the efforts of some painters to understand the rules of perspective. In a good perspective representation, parallel lines seem to meet "at the horizon". From this comes the idea of adding "points at infinity" to the Euclidean plane, points where parallel line eventually meet. For a while, projective methods were considered simply as a convenient way to handle some Euclidean problems. The recognition of projective geometry as a respectable geometric theory in itself—a geometry where two lines in the plane always intersect—only came later. After having discussed the fundamental ideas which led to projective geometry—we focus on the amazing *Hilbert theorems*. These theorems show that the very simple classical axiomatic presentation of the projective plane forces the existence of an underlying field of coordinates. The interested reader will find in [5], Vol. II of this *Trilogy*, a systematic study of the projective spaces over a field, in arbitrary dimension, fully using the contemporary techniques of linear algebra.

Another strikingly different approach to geometry imposed itself during the 19th century: non-Euclidean geometry. Euclid's axiomatization of the plane refers—first—to four highly natural postulates that nobody thought to contest. But it also contains the famous—but more involved—"fifth postulate", forcing the uniqueness of the parallel to a given line through a given point. Over the centuries many mathematicians made considerable efforts to prove Euclid's parallel postulate from the other axioms. One way of trying to obtain such a proof was by a *reductio ad absurdum*: assume that there are several parallels to a given line through a given point, then infer as many consequences as possible from this assumption, up to the moment when you reach a contradiction. But very unexpectedly, rather than leading to a contradiction, these efforts instead led step by step to an amazing new geometric theory. When actual models of this theory were constructed, no doubt was left: mathematically, this "non-Euclidean geometry" was as coherent as Euclidean geometry. We recall first some attempts at "proving" Euclid's fifth postulate, and then develop the main characteristics of the non-Euclidean plane: the limit parallels and some properties of triangles. Next we describe in full detail two famous models of non-Euclidean geometry: the *Beltrami–Klein disc* and the *Poincaré disc*. Another model—the famous *Poincaré half plane*—will be given full attention in [6], Vol. III of this *Trilogy*, using the techniques of Riemannian geometry.

We conclude this overview of synthetic geometry with Hilbert's famous axiomatization of the plane. Hilbert has first filled in the small gaps existing in Euclid's axiomatization: essentially, the questions related to the relative positions of points and lines (on the left, on the right, between, ...), aspects that Greek geometers considered as "being intuitive" or "evident from the picture". A consequence of Hilbert's axiomatization of the Euclidean plane is the isomorphism between that plane and the Euclidean plane $\mathbb{R}^2$: this forms the link with [5], Vol. II of this *Trilogy*. But above all, Hilbert observes that just replacing the axiom on the uniqueness of the parallel by the requirement that there exist several parallels to a given line through a same point, one obtains an axiomatization of the "non-Euclidean plane", as studied in the preceding chapter.

To conclude, we recall that there are various well-known problems, introduced early in antiquity by the Greek geometers, and which they could not solve. The most famous examples are: squaring a circle, trisecting an angle, duplicating a cube, constructing a regular polygon with n sides. It was only during the 19th century, with new developments in algebra, that these ruler and compass constructions were proved to be impossible. We give explicit proofs of these impossibility results, via field theory and the theory of polynomials. In particular we prove the transcendence of π and also the Gauss–Wantzel theorem, characterizing those regular polygons which are constructible with ruler and compass. Since the methods involved are completely outside the "synthetic" approach to geometry, to which this book is dedicated, we present these various algebraic proofs in several appendices.

Each chapter ends with a section of "problems" and another section of "exercises". Problems generally cover statements which are not treated in the book, but which nevertheless are of theoretical interest, while the exercises are designed for the reader to practice the techniques and further study the notions contained in the main text.

A selective bibliography for the topics discussed in this book is provided. Certain items, not otherwise mentioned in the book, have been included for further reading.

The author thanks the numerous collaborators who helped him, through the years, to improve the quality of his geometry courses and thus of this book. Among them, the author particularly wishes to thank *Pascal Dupont*, who also gave useful hints for drawing some of the illustrations, realized with *Mathematica* and *Tikz*.

The Geometric Trilogy

Contents

Chapter 1
Pre-Hellenic Antiquity

This very short chapter is intended only to give an overview of some of the first geometric ideas which arose in various civilizations before the influence of the systematic work of the Greek geometers. So *pre-Hellenic* should be understood here as "before the Greek influence".

From this "pre-Hellenic antiquity", we know of various works due to the Egyptians and the Babylonians. Indeed, these are the only pre-Hellenic civilizations which have produced written geometric documents that have survived to the present day.

It should nevertheless be mentioned here that some works in China and India—posterior to the golden age of geometry in Greece—are considered by some historians as "pre-Hellenic" in the sense of being "absent of Greek influence". But not all specialists agree on this point. Therefore we choose in this book to mention these developments at their chronological place, after the Greek period.

1.1 Prehistory

Prehistory is characterized by the absence of writing. In those days, the transmission of knowledge was essentially oral. But nowadays, we no longer hear those voices. Therefore prehistory remains as silent about geometry as it is about all other aspects of human life. The best that we can do is to rely on archaeological discoveries and try to interpret the various cave pictures and objects that have been found.

The first geometric pictures date from 25000 BC. They already indicate some mastering of the notions of symmetry and congruence of figures. Some other objects of the same period show evidence of the first arithmetical developments, such as the idea of "counting".

Particularly intriguing is the picture in Fig. 1.1: it seems to be evident that the prehistoric artist did not just want to draw a nice picture: he/she wanted to emphasize some mathematical discovery. Indeed, looking at this picture, we notice at once that:

- doubling the side of the triangle multiplies the area by 4; tripling the side of the triangle multiplies the area by 9;

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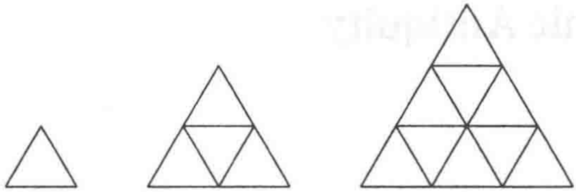

Fig. 1.1

- counting the number of small triangles on each "line" we observe that

$$1 = 1, \qquad 1 + 3 = 4, \qquad 1 + 3 + 5 = 9.$$

The oldest written documents that we know concerning geometry already mention the corresponding general results:

- *multiplying the lengths by n results in multiplying the areas by n^2;*
- *the sum of the n first odd numbers is equal to n^2.*

To what extent was the prehistoric artist aware of these "theorems"? We shall probably never know.

A tradition claims that the origin of arithmetic and geometry is to be found in the religious rituals of our ancestors: they were fascinated by the properties of some forms and some numbers, to which they attributed magical powers. By introducing such magical forms and numbers into their rituals they might perhaps draw the benediction of their gods.

Another tradition, reported by *Herodotus* (c. 484 BC–c. 425 BC) presents geometry as the precious daughter of the caprices of the Nile. Legendary Pharaoh *Sesostris* (around 1300 BC; but probably a compound of *Seti* and *Ramesses II*) had, claims Herodotus, distributed the Egyptian ground between "the" (by which we understand "some few privileged") inhabitants. The annual floods of the Nile valley, the origin of its fertility but also of many dramatic events, made it necessary to devise practical methods of retracing the limits of each estate after each flood. These methods were based on triangulation and probably made use of some special instances of Pythagoras' theorem for constructing right angles. For example, the fact that a triangle with sides 3, 4, 5 has a right angle seems to have been known at least since 2000 BC.

But the Nile valley certainly does not have the hegemony of early developments of mathematics, not even in Africa: the discovery in 1950 of the *Ishango* bone in Congo, dating from 22000 BC, is one of the oldest testimonies of some mathematical activity. And various discoveries in Europe, India, China, Mesopotamia, and so on, indicate that—at different levels of development—mathematical thought was present in many places in the world during antiquity.

However, up to now, modest prehistory has unveiled very little of its personal relations with geometry.

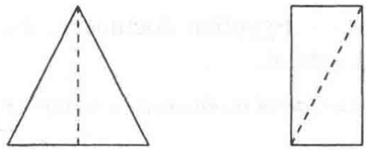

Fig. 1.2

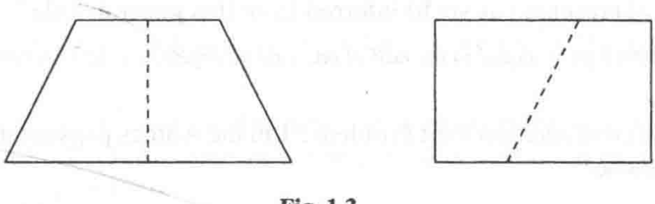

Fig. 1.3

1.2 Egypt

The oldest mathematical papyrus which has reached us is the so-called *Moscow papyrus*, most likely written around 1850 BC. But our main knowledge of Egyptian mathematics during high antiquity comes from a more extended papyrus copied by the scribe *Ahmes* around 1650 BC. These papyri contain the solutions to many arithmetical and geometrical problems whose elaboration, according to Ahmes, dates back to 2000 BC.

The *Moscow* papyrus is also called the *Golenischev Mathematical Papyrus*, after the Egyptologist *Vladimir Golenishchev* who bought the papyrus in Thebes around 1893. The papyrus later entered the collection of the *Pushkin State Museum of Fine Arts* in Moscow, where it remains today. The Ahmes papyrus is often referred to as the *Rhind papyrus*, so named after *Alexander Henry Rhind*, a Scottish antiquarian, who purchased the papyrus in 1858 in Luxor, Egypt. The papyrus was apparently found during illegal excavations on the site of the mortuary temple of Pharaoh Ramesses II. It is kept at the *British Museum in London*.

For example, Problem 51 of the Ahmes papyrus shows that

The area of an isosceles triangle is equal to the height multiplied by half of the base.

The explanation is a "cut and paste" argument as in Fig. 1.2. Cut the triangle along its height; reverse one piece, turn it upside down and glue both pieces together; you get the rectangle on the right.

An analogous argument is used in Problem 52 to show that

The area of an isosceles trapezium is equal to the height multiplied by half the sum of the bases.

See Fig. 1.3, which is again "a proof" in itself. At least, it is a "proof" in the spirit of the time: in any case, an argument based on congruences of figures.

However, let us stress that the Egyptians did not have a notion of what a "theorem" or a "formal proof" is, in the mathematical sense of the term. In particular, they did not make a clear distinction between a precise result and an approximative