刘彦佩选集

(Selected Publications of Y.P.Liu)

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第十一编 目录

Enumerative Theory Of Maps

by

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Preface

Combinatorics as a branch of mathematics studies the arts of counting. Enumeration occupies the foundation of combinatorics with a large range of applications not only in mathematics itself but also in many other disciplines. It is too broad a task to write a book to show the deep development in every corner from this aspect. This monograph is intended to provide a unified theory for those related to the enumeration of maps.

For enumerating maps the first thing we have to know is the symmetry of a map. Or in other words, we have to know its automorphism group. In general, this is an interesting, complicated, and difficult problem. In order to do this, the first problem we meet is how to make a map considered without symmetry. Since the beginning of sixties when Tutte found a way of rooting on a map, the problem has been solved. This forms the basis of the enumerative theory of maps. As soon as the problem without considering the symmetry is solved for one kind of map, the general problem with symmetry can always, in principle, be solved from what we have known about the automorphism of a polyhedron, a synonym for a map, which can be determined efficiently according to another monograph of the present author [Liu58].

Now, the problems facing us are how to find a functional equation satisfied by the enumerating function of one kind of map given and how to find a way to determine the coefficients in the power series form of the enumerating function by solving the equation. Then, a further problem is to investigate the stochastic behaviors of the kind of maps we have already enumerated.

For extracting an equation, a crucial trick is suitably to decompose the set of maps we are concerned with into several parts such that each of them can be generated by some operations on the set itself. The starting operations usually employed are the so called deletion and contraction of a specific edge properly chosen. Along this line one can see how different kinds of operations are constructed for enumerating a variety of types of maps. The way of decomposition is very closely related to the choice of parameters which the enumeration is according to. There are tricks which have to be exploited to avoid the complicatedness involved in deriving the functional equation from the decomposition.

As soon as a functional equation has been built up, the problem that follows is to find a suitable way to solve it, or to transform and simplify it for clarifying the solution. Here, we present a number of methods for solving the equations directly, or converting them into some special ones which are solvable in certain cases. The most interesting part is to try to find a way by which the Lagrangian inversion can be suitably applied for determining the coefficients in the power series form of the solution. Hopefully, many simpler formulae for enumerating a variety of maps have been obtained via a series of subtle treatments in this way.

In spite of whether the functional equation is completely solved or not, one is always allowed to investigate stochastic behaviors by estimating asymptotic properties of the solution as an enumerating function of certain kind of maps, when the order of maps is big enough, up to tending to infinity.

According to the basic theoretical idea as described above, the whole book is divided into three main parts. The first part, from Chapter 2 through Chapter 8, is on the ordinary theory of enumerating maps. The second, from Chapter 9 through Chapter 11, is on chromatic and dichromatic sums which can be seen as a kind of generalization of enumeration with much complication and much difficulty. And the third which consists of only one chapter, Chapter 12 is on the stochastic behaviors. Of course, Chapter 1 provides the necessary knowledge and basic techniques for the requirements of the whole book. In order to save space, the last section of each chapter is designed to be notes in which some historical remarks, new progress and unsolved problems with clues for possibly solving them are indicated in corresponding areas.

On this occasion, I should express my heartiest thanks to all those having made contributions themselves directly or indirectly to this book.

The initiation of this theory was established by Professor W. T. Tutte whose articles and directions were invoked for me to enter the field when I was working in the Department of Combinatorics and Optimization at the University of Waterloo, Canada, in the period of 1982–1984. Without these, I would be doing something else at the present time.

Many other friends of mine including Professors R. Cori, P.L. Hammer, D.M. Jackson, R. C. Mullin, R.C. Read, L.B. Richmond, P. Rosenstiehl, B. Simeone, T.T.S. Walsh, W. Xu and J. Yan are constantly concerned with me in material and spirit. For the final version, many people including J.L. Cai (PhD), Y.X. Chang (PhD), F.M. Dong (PhD), J.Q. Dong, R.X. Hao, Y.Q. Huang (PhD), S. Lawrencenko (PhD), A.P. Li (PhD), D.M. Li (PhD), X. Liu (PhD), Yi. Liu (PhD), T.Y. Liu, T.J. Lu (PhD), K. Ouyang (PhD), X.R. Sun (PhD), H. Ren, E.L. Wei, F.E. Wu (PhD) and M.L. Zheng (PhD) provide errata in part or whole.

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Y.P. Liu

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Chapter 1

Preliminaries

Throughout, for the sake of brevity, we adopt the following logical conventions: disjunction, conjunction, negation, implication, equivalence, universal quantification and existential quantification denoted by the symbols: $\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\forall$ and $\exists$, respectively.

In the context, $(i.j.k)$ (or $i.j.k$) refers to item k of section j in chapter i for formulae (or theorems and the like).

A reference $[k]$ refers to item k in the bibliography where k consists of the first few letters(initials) of the last name(s) of the author(s) in alphabetical order followed by a number to distinguish publications of the same author(s).

Terminologies not explained in this book can be found in [Liu58], or probably in [GoJ1] or [Tut39].

§1.1 Maps

A *map*, denoted by M, is a mathematical concept which can, of course, be seen as a kind of abstraction from that appearing in geography, is defined to be a basic permutation $\mathcal{P}$ on a disjoint union $\mathcal{X}$ of quadricells with Axiom 1 and Axiom 2 bellow.

Let X be a finite set, and K the Klein group of four elements which are denoted by 1, α, β, and $\alpha\beta$. For $x \in X$, the set $Kx=\{x, \alpha x, \beta x, \alpha\beta x\}$ is said to be a *quadricell*.

We may write $\mathcal{X} = \sum_{x\in X} Kx$. Naturally, both α and β themselves are permutations on $\mathcal{X}$. A permutation $\mathcal{P}$ on $\mathcal{X}$ is said to be *basic* if for any $x \in \mathcal{X}$ there does not exist an integer k such that $\mathcal{P}^k x = \alpha x$.

Axiom 1 $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$.

Axiom 2 *The group* Ψ_J *which is generated by* $J = \{\alpha, \beta, \mathcal{P}\}$ *is transitive on* $\mathcal{X}$.

Thus, we may write the map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$. From Axiom 1, α and β are asymmetric, *i.e.*, $(\mathcal{X}_{\alpha,\beta}(X), \mathcal{P}) \neq (\mathcal{X}_{\beta,\alpha}(X), \mathcal{P})$ in general. Sometimes, α is called the *first operator* and β, the *second operator*. Generally, for a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$, it is not necessary that $(\mathcal{X}_{\beta,\alpha}(X), \mathcal{P})$ is also a map because it is not guaranteed to have Axiom 1 for β. In fact, Axiom 1 allows us to define the *vertices* of a map as the pairs of *conjugate* orbits of $\mathcal{P}$ on $\mathcal{X}$. We always write $\mathcal{P}$ as the product of the orbits (in cyclic order) obtained by choosing exactly one, which represents a vertex as well, in each *conjugate pair* determined by x and αx for $x \in \mathcal{X}$.

For a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ given, from the definition it is easy to check that $M^* = (\mathcal{X}_{\beta,\alpha}(X), \mathcal{P}\alpha\beta)$ is also a map with β as the first operator and α, the second. We call M^* the *dual* (map) of M. From the duality, the *faces* of M are defined to be the vertices of M^*. Moreover, an *edge* of M is defined to be the quadricell $Kx = \{x, \alpha x, \beta x, \alpha\beta x\}$ for $x \in X$. An edge $\{x, \alpha x, \beta x, \alpha\beta x\}$ can be seen as a pair of *semiedges* $\{x, \alpha x\}$ and $\{\beta x, \alpha\beta x\}$, or $\{x, \beta x\}$ and $\{\alpha x, \alpha\beta x\}$ as well.

The graph whose vertices and edges are those of a map M is said to be the *underlying graph* of M and is denoted by $G(M)$. From Axiom 2, $G(M)$ has to be connected. Conversely, a map M whose vertices and edges are those of a graph G is said to be an *underlain map* of G and denoted by $M(G)$. Of course, M is an underlain map of $G(M)$. Although any map has a unique underlying graph, a graph is in general allowed to have many underlain maps.

In fact, any underlain map of a graph (connected of course) is an embedding on a surface. This enables us to denote a map M by (G, F) such that $G = (V, E) = G(M)$ where V, E and F are the vertex, edge and face sets respectively. Only one vertex without edge is always defined to be a map, which is called the *trivial map*, or the *vertex map*. If a map has a single edge, then it is called an *edge map* denoted by L. If an edge map has a loop, then it is called a *loop map*; otherwise, the *link map*. Apparently, only two possible loop maps exist. They are $L_1 = (\mathcal{X}, (x, \alpha\beta x))$ and $L_2 = (\mathcal{X}, (x, \beta x))$. The unique link map is $L_0 = (\mathcal{X}, (x)(\alpha\beta x))$.

Let ν, ϵ and ϕ be the numbers of vertices, edges and faces of a map M respectively. The number

$$\mathrm{Eul}(M) = \nu - \epsilon + \phi \tag{1.1.1}$$

is said to be the *Euler characteristic* of M.

Further, if a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ satisfies the following Axiom 3, then it is said to be *nonorientable*; otherwise, *orientable*.

Axiom 3 *The group Ψ_L generated by $L = \{\alpha\beta, \mathcal{P}\}$ is transitive on $\mathcal{X}_{\alpha,\beta}(X)$.*

Because it can be shown that if the group Ψ_L is not transitive on $\mathcal{X}_{\alpha,\beta}(X)$ then it has exactly two orbits one of which is conjugate of the other on $\mathcal{X}_{\alpha,\beta}(X)$, a map $(\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ is orientable iff the group Ψ_L has two orbits on $\mathcal{X}_{\alpha,\beta}(X)$.

Let $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ be a map and $e_x = \{x, \alpha x, \beta x, \alpha\beta x\}$ be the edge incident with $x \in \mathcal{X}_{\alpha,\beta}(X)$. For convenience, we always write $\mathcal{X}$ instead of $\mathcal{X}_{\alpha,\beta}(X)$ without specific indication and see that

$$\mathcal{X} = X + \alpha X + \beta X + \alpha\beta X \tag{1.1.2}$$

where $\gamma X = \{\gamma x | \forall x \in X\}$ for $\gamma = \alpha$, β, or $\alpha\beta$. Moreover, an edge e_x for $x \in X$ is simply denoted by e.

Now, we introduce two kinds of operations for an edge e on a map M. By the *deletion* of e on M, denoted by $M - e$, is meant that

$$M - e = (\mathcal{X} - e, \ \mathcal{P}\langle e\rangle), \tag{1.1.3}$$

where $\mathcal{P}\langle e\rangle$ is the restriction of $\mathcal{P}$ on $\mathcal{X} - e$. The other, called *contraction* of e on M and denoted by $M \bullet e$, is

$$M \bullet e = (\mathcal{X} - e, \ \mathcal{P}[e]) \tag{1.1.4}$$

where $\mathcal{P}[e]$ is obtained by composing the two vertices u and v incident to e as

$$\{AB, \ \alpha B^{-1}A^{-1}\}$$

when

$$u = \{xA, \alpha x\alpha A^{-1}\} \text{ and } v = \{\alpha\beta xB, \beta x\alpha B^{-1}\}$$

while all other vertices are in agreement with those for $\mathcal{P}$.

Theorem 1.1.1 *Any map* M *has* $\text{Eul}(M) \leq 2$.

Proof Because the deletion of an edge on the common boundary of two faces in a map M reduces one in the face number of M, we can always find a map M', $\nu(M') = \nu(M)$, such that M' has only one face and $\text{Eul}(M') = \text{Eul}(M)$ by a series of the operations. From the connectedness, $\nu(M') \leq \epsilon(M') + 1$. Therefore,

$$\text{Eul}(M) = \nu(M') - \epsilon(M') + 1 \leq 2.$$

The theorem is proved. □

Two more operations which are often used have to be explained. Suppose $v = (AB)$ is a vertex of a map M. Let $\mathcal{P}'$ be obtained by substituting (Ax) and $(\alpha\beta xB)$ for (AB) in $\mathcal{P}$ where x is incident with the new edge. Then, the map $M' = (\mathcal{X}+Kx, \mathcal{P}')$ is said to be obtained by *splitting* the vertex v on M. If $v = (xy)$ is a vertex in a map M, then the map

$$M' = (\mathcal{X} - Kx - Ky + Kz, \mathcal{P}'),\ z = x = y,$$

where $\mathcal{P}'$ is the resultant one of deleting (xy) from $\mathcal{P}$, is said to be obtained by *missing* the vertex v on M. The inverse of deletion of an edge is called the *addition* of an edge and the inverse of missing a vertex, *subdividing* an edge.

Of course, the inverse of contraction of an edge is splitting a vertex as defined above. It is easily seen that the Euler characteristic is unchanged under the contraction of an edge, missing a vertex and their inverses: splitting a vertex, subdividing an edge.

However, the invariance of the Euler characteristic under edge deletion and its inverse, the edge addition, is only for an edge on the common boundary of two faces, or say, under *standard* deletion and addition.

Because any map can be transformed into another which has only one face such that the Euler characteristic is unchanged by virtue of what appears in the proof of Theorem 1.1.1, we are allowed to consider one face maps for the sake of finding the simplest one with a given Euler characteristic. For brevity, a map is represented by its faces with the convention: $x^{-1} = \alpha\beta x$ and hence $(\alpha x)^{-1} = \beta x$. On the whole, we are allowed to realize $x = \alpha x$ and hence $\beta x = \alpha\beta x$.

For orientable maps we have the following two properties: Orien.1 and Orien.2 which can be derived from the operations mentioned above.

Orien.1 If a one face map $M = (Rxx^{-1}Q)$, $R, Q \neq \emptyset$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(RQ).$$

Orien.2 If a one face map $M = (PxQyRx^{-1}Sy^{-1}T)$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(PSRQTxyx^{-1}y^{-1}).$$

For nonorientable maps, we have the following two properties: Norien.1 and Norien.2 which can be derived from the operations as well.

Norien.1 If a one face map $M = (PxQxR)$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(PQ^{-1}Rxx).$$

Norien.2 If a one face map $M = (Axxyzy^{-1}z^{-1})$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(Ax_1x_1x_2x_2x_3x_3).$$

Theorem 1.1.2 *If a map M is orientable, then we have* $\mathrm{Eul}(M) = 0 \pmod 2$. *Moreover, M is on the surface of genus p (orientable), $p \geq 0$, iff*

$$\mathrm{Eul}(M) = 2 - 2p$$

where $\mathrm{Eul}(M)$ *is the Euler characteristic of M defined by* (1.1.1).

Proof From the orientability, each edge $e = Kx$ is only allowed to have $\{x, \alpha\beta x\} = \{x, x^{-1}\}$ (or $\{\alpha x, \beta x\}$ as well) in one of the two orbits of the group Ψ_L on $\mathcal{X}$. By using the properties: Orien.1 and Orien.2 as far as possible, we may finally find that $\mathrm{Eul}(M)$ is equal to

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