刘彦佩选集

（Selected Publications of Y.P.Liu）

第二十一编

时代文献出版社

刘彦佩选集

（Selected Publications of Y.P.Liu）

第二十一编

时代文献出版社

刘彦佩选集（第二十一编）

作　　者：刘彦佩
出版单位：时代文献出版社

编辑设计：北京时代弄潮文化发展公司
地　　址：北京中关村海淀图书城25号家谱传记楼二层
电　　话：010-62525116　13693651386
网　　址：www.grcsw.com

印　　刷：京冀印刷厂
开　　本：880×1230　1/16
版　　次：2016年3月第1版

书　　号：ISBN 978-988-18772-5-3
定　　价：全套 1978.00元（共计23编）

第二十一编 目录

Liu Yanpei

Selected Papers

Volume 20

Beijing Jiaotong University
2011

Contents

Applied Mathematics Letters 22 (2009) 738–742

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On the embedding genus distribution of ladders and crosses

Liangxia Wan [a,b,*], Yanpei Liu [a]

[a] Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China

[b] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

ARTICLE INFO

Article history:
Received 4 February 2007
Received in revised form 15 May 2008
Accepted 28 July 2008

Keywords:
Joint tree
Surface
Embedding surface
Genus distribution
Genus

ABSTRACT

In this work, the relations between ladder surface sets and cross surface sets are found. The embedding genus distribution of ladders can be obtained by using the genus distribution of cross type surface sets.

1. Introduction

Throughout this work a surface, a graph and an embedding always imply an orientable cycle, a connected graph and an orientable embedding. The concepts can be found in [2,3,8].

A *linear sequence* is a letter sequence with the relation $\prec$. Since an orientable closed surface can be regarded as forming by gluing the edges of a directed polygon as a direction, a surface can be regarded as an *orientable cycle S* which contains one a and one a^- for each $a \in S$. $\gamma(S)$ denotes the genus of the surface S and $\mathcal{S}$ denotes the set containing all of the surfaces. An equivalence $\sim$ (for example [2]), defined on $\mathcal{S}$, is as follows:

Op1. $AB \sim (Ax)(x^-B)$ where $AB \in \mathcal{S}$ and $x \notin AB$;

Op2. $Ax_1x_2Bx_2^-x_1^- \sim AxBx^- = Ax^-Bx$ where $Ax_1x_2Bx_2^-x_1^- \in \mathcal{S}$ and $x \notin AB$;

Op3. $Axx^-B \sim AB$ where $Axx^-B \in \mathcal{S}$ and $AB \neq \emptyset$.

Lemma 1 (*For Example [9]*). *Let A, B, C and D be linear sequences and let $xABx^-CD$ be a surface. Then*

$$xABx^-CD \sim xBAx^-CD \sim xABx^-DC$$

where $x, x^- \notin ABCD$.

Let U be a surface set. The *genus distribution* of U is

$$g_0(U), g_1(U), g_2(U), \ldots$$

The *genus polynomial* of U is $f_U(x) = \sum_{i=0}^{\infty} g_i(U)x^i$ where $g_i(U)$ denotes the number of distinct surfaces of U with genus i ($i \geq 0$). Given a graph G and a surface S, if there is a homeomorphism $\phi : G \to S$ such that each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then G has a two-cell embedding on S. The *genus of a graph G* is the minimum

* Corresponding author at: Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China.
E-mail address: wanliangxia@126.com (L.X. Wan).

0893-9659/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.07.008

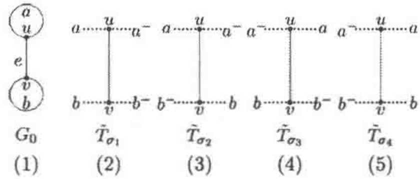

Fig. 1. G_0 and its four joint trees.

genus of the surface which it can be embedded on. The *embedding genus distribution* of G, also called the *genus distribution*, is

$$g_0(G), g_1(G), g_2(G), \ldots.$$

The *embedding polynomial*, also called the *genus polynomial*, of a graph G is $f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i$ where $g_i(G)$ denotes the number of distinct embeddings of G with genus i. Since determining the genus of a graph is NP-complete [6], it is NP-complete to determine the embedding genus distribution of a graph.

Given a graph G, a *rotation* at a vertex v of G is a cyclic permutation of edges incident with v. A *rotation system* of G is obtained by assigning a rotation at each vertex of G. Let T be a spanning tree of G. A *joint tree* $\tilde{T}$ is formed by splitting each cotree edge a into two semi-edges a and a^-. Given a spanning tree, a joint tree is determined by a rotation system and the associated embedding surface is a cyclic permutation which is formed by the semi-edges and which is determined by the joint tree. For example, four joint trees $\tilde{T}_{\sigma_k}(1 \le k \le 4)$ of G_0 are obtained by letting a and b of G_0 be cotree edges and letting each vertex have a clockwise rotation (see Fig. 1). Embedding surfaces of $\tilde{T}_{\sigma_k}$ for $i = 1, 2, 3$ and 4 are respectively aa^-b^-b, aa^-bb^-, a^-ab^-b and a^-abb^-.

We obtained explicit expressions for the genus distribution for ladder surface sets and cross surface sets [7,9,10]. In this work we get the relations between genera of ladder surface sets and genera of cross surface sets. Since the embedding genus distribution of ladders and crosses can be calculated by using the genus distribution for ladder surface sets and cross surface sets respectively [9,10], the embedding genus distribution of ladders can be obtained by using the genus distribution of cross surface sets. Consequently, explicit expressions for genus distribution for closed-end ladders [1], Ringel ladders [5], circular ladders and Möbius ladders [4] are deduced.

2. Main theorem

Let e_1 and e_2 be edges of a graph G. A *ladder* GL_n is obtained by adding n ($n \ge 1$) vertices $u_1, u_2, u_3, \ldots, u_n$ on e_1 in sequence, n vertices $v_1, v_2, v_3, \ldots, v_n$ on e_2 in sequence and edges u_lv_l such that u_1v_1 and u_2v_2 are parallel edges. A *cross* GC_n is obtained by adding n vertices $u_1, u_2, u_3, \ldots, u_n$ on e_1 in sequence, n vertices $v_1, v_2, v_3, \ldots, v_n$ on e_2 in sequence and edges u_lv_l such that u_1v_1 and u_2v_2 are not parallel edges. Denote u_lv_l by a_l for $1 \le l \le n$.

Suppose that a_l are distinct letters for $l \ge 1$. The *ladder surface sets* S_k^n are as follows for $1 \le k \le 11$:

$$S_1^n = \{R_1^nR_2^nR_3^nR_4^n\} \quad S_2^n = \{R_1^nR_2^nR_4^nR_3^n\} \quad S_3^n = \{R_1^nR_3^nR_2^nR_4^n\}$$
$$S_4^n = \{aR_1^nR_2^na^-R_3^nR_4^n\} \quad S_5^n = \{aR_1^nR_3^na^-R_2^nR_4^n\}$$
$$S_6^n = \{aR_1^nR_4^na^-R_2^nR_3^n\} \quad S_7^n = \{aR_1^na^-R_3^nR_2^nR_4^n\}$$
$$S_8^n = \{R_1^nR_2^naR_3^na^-bR_4^nb^-\} \quad S_9^n = \{R_1^nR_3^naR_2^na^-bR_4^nb^-\}$$
$$S_{10}^n = \{R_1^nR_4^naR_2^na^-bR_3^nb^-\} \quad S_{11}^n = \{R_1^naR_2^na^-bR_3^nb^-cR_4^nc^-\}$$

where $R_1^n = a_{k_1}a_{k_2}a_{k_3}\cdots a_{k_r}$, $R_2^n = a_{k_{r+1}}a_{k_{r+2}}a_{k_{r+3}}\cdots a_{k_n}$, $R_3^n = a_{t_1}^-a_{t_2}^-a_{t_3}^-\cdots a_{t_s}^-$, $R_4^n = a_{t_{s+1}}^-a_{t_{s+2}}^-a_{t_{s+3}}^-\cdots a_{t_n}^-$, $n \ge k_1 > k_2 > k_3 > \cdots > k_r \ge 1$, $1 \le k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_n \le n$, $n \ge t_1 > t_2 > t_3 > \cdots > t_s \ge 1$, $1 \le t_{s+1} < t_{s+2} < t_{s+3} < \cdots < t_n \le n$ and $1 \le r, s \le n$, $k_p \le k_q$, $t_p \ne t_q$ for $p \ne q$.

The *cross surface sets* U_k^n are as follows for $1 \le k \le 11$:

$$U_1^n = \{K_1^nK_2^nK_3^nK_4^n\} \quad U_2^n = \{K_1^nK_2^nK_4^nK_3^n\} \quad U_3^n = \{K_1^nK_3^nK_2^nK_4^n\}$$
$$U_4^n = \{aK_1^nK_2^na^-K_3^nK_4^n\} \quad U_5^n = \{aK_1^nK_3^na^-K_2^nK_4^n\}$$
$$U_6^n = \{aK_1^nK_4^na^-K_2^nK_3^n\} \quad U_7^n = \{aK_1^na^-K_3^nK_2^nK_4^n\}$$
$$U_8^n = \{K_1^nK_2^naK_3^na^-bK_4^nb^-\} \quad U_9^n = \{K_1^nK_3^naK_2^na^-bK_4^nb^-\}$$
$$U_{10}^n = \{K_1^nK_4^naK_2^na^-bK_3^nb^-\} \quad U_{11}^n = \{K_1^naK_2^na^-bK_3^nb^-cK_4^nc^-\}$$

where $K_1^n = a_{h_1}a_{h_2}a_{h_3}\cdots a_{h_r}$, $K_2^n = a_{h_{r+1}}a_{h_{r+2}}a_{h_{r+3}}\cdots a_{h_n}$, $K_3^n = a_{l_1}^-a_{l_2}^-a_{l_3}^-\cdots a_{l_s}^-$, $K_4^n = a_{l_{s+1}}^-a_{l_{s+2}}^-a_{l_{s+3}}^-\cdots a_{l_n}^-$, $n \ge h_1 > h_2 > h_3 > \cdots > h_r \ge 1$, $1 \le h_{r+1} < h_{r+2} < h_{r+3} < \cdots < h_n \le n$, $1 \le l_1 < l_2 < l_3 < \cdots < l_s \le n$, $n \ge l_{s+1} > l_{s+2} > l_{s+3} > \cdots > l_n \ge 1$ and $1 \le r, s \le n$, $h_p \ne h_q$, $l_p \ne l_q$ for $p \ne q$.

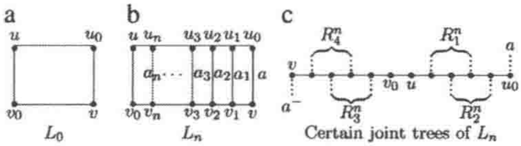

Fig. 2. L_0, L_n and certain joint trees of L_n.

Theorem 1 *(Theorem 3.1 of [7]). Let $g_i(GL_n)$ denote the number of distinct embeddings with genus i in GL_n and let $g_{i_j}(n)$ denote the number of surfaces with genus i in S_j^n. $g_i(G_n)$ is a linear combination of $g_{m_j}(n)$'s for $1 \le j \le 11, 0 \le m \le i$ and $n \ge 1$.* □

Theorem 2. *Let $g_i(GC_n)$ denote the number of distinct embeddings with genus i in GC_n and let $\mu_{i_j}(n)$ denote the number of surfaces with genus i in U_j^n. $g_i(GC_n)$ is a linear combination of $\mu_{m_j}(n)$'s for $1 \le j \le 11, 0 \le m \le i$ and $n \ge 1$.*

Proof. This conclusion holds on using arguments similar to those in the proof of Theorem 1. □

Theorem 3. *Suppose that $g_{i_j}(n)$ and $\mu_{i_j}(n)$ denote the number of surfaces for the surface sets S_j^n and U_j^n with genus i for $n \ge 1$, $1 \le j \le 11$ and $i \ge 0$ respectively. Let $f_{S_j^0}(x) = f_{U_j^0}(x) = 0$. Then,*

$$g_{i_1}(n) = \mu_{i_2}(n), \quad g_{i_2}(n) = \mu_{i_1}(n), \quad g_{i_3}(n) = \mu_{i_3}(n), \quad g_{i_4}(n) = \mu_{i_4}(n),$$
$$g_{i_5}(n) = \mu_{i_6}(n), \quad g_{i_6}(n) = \mu_{i_5}(n), \quad g_{i_7}(n) = \mu_{i_7}(n), \quad g_{i_8}(n) = \mu_{i_8}(n),$$
$$g_{i_9}(n) = \mu_{i_{10}}(n), \quad g_{i_{10}}(n) = \mu_{i_9}(n), \quad g_{i_{11}}(n) = \mu_{i_{11}}(n).$$

Proof. Let a_l denote distinct letters for $l \ge 1$ and let

$$R_1^n = a_{k_1}a_{k_2}a_{k_3}\cdots a_{k_r}, \quad R_2^n = a_{k_{r+1}}a_{k_{r+2}}a_{k_{r+3}}\cdots a_{k_n},$$
$$R_3^n = a_{t_1}^- a_{t_2}^- a_{t_3}^- \cdots a_{t_s}^- \quad \text{and} \quad R_4^n = a_{t_{s+1}}^- a_{t_{s+2}}^- a_{t_{s+3}}^- \cdots a_{t_n}^-$$

where $n \ge k_1 > k_2 > k_3 > \cdots > k_r \ge 1$, $1 \le k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_n \le n$, $n \ge t_1 > t_2 > t_3 > \cdots > t_s \ge 1$, $1 \le t_{s+1} < t_{s+2} < t_{s+3} < \cdots < t_n \le n$ and $1 \le r, s \le n$, $k_p \ne k_q$, $t_p \ne t_q$ for $p \ne q$. The corresponding cross surface sets U_j^n are obtained by letting

$$K_1^n = a_{k_1}a_{k_2}a_{k_3}\cdots a_{k_r}, \quad K_2^n = a_{k_{r+1}}a_{k_{r+2}}a_{k_{r+3}}\cdots a_{k_n},$$
$$K_3^n = a_{t_{s+1}}^- a_{t_{s+2}}^- a_{t_{s+3}}^- \cdots a_{t_n}^- \quad \text{and} \quad K_4^n = a_{t_1}^- a_{t_2}^- a_{t_3}^- \cdots a_{t_s}^-.$$

Let ψ be a map defined on $\bigcup_{j=1}^{11} S_j^n$ such that R_1^n, R_2^n, R_3^n and R_4^n correspond to K_1^n, K_2^n, K_4^n and K_3^n.

For any surface $R_1^nR_2^nR_3^nR_4^n \in S_1^n$, we have $\psi(R_1^nR_2^nR_3^nR_4^n) = K_1^nK_2^nK_4^nK_3^n$. For any surface $K_1^nK_2^nK_4^nK_3^n$, $\psi^{-1}(K_1^nK_2^nK_4^nK_3^n) = R_1^nR_2^nR_3^nR_4^n$. Then, ψ is a bijection from S_j^1 to U_j^2.

Since $\gamma(R_1^nR_2^nR_3^nR_4^n) = \gamma(K_1^nK_2^nK_4^nK_3^n)$, $g_{i_1}(n) = \mu_{i_2}(n)$.

The other equations can be verified by using a similar map ψ as well as by applying Lemma 1. □

3. Applications

Let L_0 be the graph shown in Fig. 2 (a). The closed-end ladder L_n is formed by adding n parallel edges u_lv_l, denoted by a_l, in Fig. 2 (b). A spanning tree T_n of L_n is obtained by letting a and a_l be cotree edges for $1 \le l \le n$. Joint trees of L_n are obtained by splitting cotree edges. Let each vertex have a clockwise rotation in each joint tree. Thus, the associated embedding surfaces of L_n are $aR_1^nR_4^na^-R_3^nR_2^n$. Certain joint trees of L_n are shown in Fig. 2 (c).
By Op2 and Lemma 1

$$aR_1^nR_4^na^-R_3^nR_2^n \sim aR_1^nR_4^na^-R_2^nR_3^n.$$

Thus,

$$f_{L_n}(x) = f_{S_6^n}(x).$$

By Theorem 3

$$f_{L_n}(x) = f_{U_5^n}(x).$$

Then, by using Theorem 4 of [10] we have:

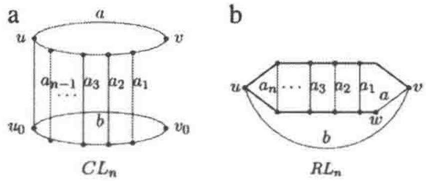

Fig. 3. CL_n and RL_n.

Corollary 1 ([1]). *Let $g_i(L_n)$ be the number of distinct embeddings for L_n and let $C_n(i) = \binom{n-2-i}{i}$. Then*

$$g_i(L_n) = \begin{cases} 2^{n+i-1}\dfrac{2n-3i+2}{n-i+1}C_{n+3}(i), & \text{if } 0 \le i \le \left[\dfrac{n+1}{2}\right] \text{ and } n \ge 1; \\ 0, & \text{otherwise.} \end{cases}$$

A subdivision of CL_n, still denoted by CL_n, is obtained by adding $n-1$ parallel edges a_l such that the ends of a_l are, respectively, on uv and u_0v_0 for positive integers n and l with $n \ge 2$ and $1 \le l \le n-1$ (Fig. 3(a)). A spanning tree of CL_n is obtained by letting a, b and a_l be cotree edges. Then CL_n has four types of joint trees for certain R_1^{n-1}, R_2^{n-1}, R_3^{n-1} and R_4^{n-1} according to distinct rotation pairs at ends of a and b. Accordingly, it has four embedding surfaces $R_1^{n-1}aR_2^{n-1}a^-bR_3^{n-1}b^-R_4^{n-1}$, $a^-R_1^{n-1}aR_2^{n-1}bR_3^{n-1}b^-R_4^{n-1}$, $R_1^{n-1}aR_2^{n-1}a^-R_3^{n-1}b^-R_4^{n-1}b$ and $a^-R_1^{n-1}aR_2^{n-1}R_3^{n-1}b^-R_4^{n-1}b$. Thus,

$$g_i(CL_n) = 2g_{i_9}(n-1) + 2g_{i_{10}}(n-1) = 2\mu_{i_9}(n-1) + 2\mu_{i_{10}}(n-1).$$

By using Theorem 4 of [10], we obtain

Corollary 2 ([5]). *Let $g_i(CL_n)$ be the number of distinct embeddings with genus i for CL_n and let $g_0(CL_1) = 4$ where $n \ge 1$ and $i \ge 0$. Let $C_n(i) = \binom{n-2-i}{i}$ and $D_n(i) = \frac{n}{i}2^i$. Then,*

$$g_i(CL_n) = \begin{cases} 4, & \text{if } i = 0 \text{ and } n = 2; \\ 12, & \text{if } i = 1 \text{ and } n = 2; \\ 2^n + 8n + 6, & \text{if } i = 1 \text{ and } n = 3, 4; \\ 2^n + 8n - 2, & \text{if } i = 1 \text{ and } n \ge 5; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i), & \text{if } 2 \le i < \dfrac{n}{2} - 1 \text{ and } n \ge 3; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i) + 2^{n-1}, & \text{if } i = \dfrac{n}{2} - 1 \text{ and } n \ge 5; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i) + 2^n, & \text{if } \dfrac{n}{2} - 1 < i \le \dfrac{n-1}{2} \text{ and } n \ge 4; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{\frac{3n}{2}+1} - 3 \cdot 2^{n-1}, & \text{if } \dfrac{n-1}{2} < i \le \dfrac{n}{2} \text{ and } n \ge 3; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1), & \text{if } \dfrac{n}{2} < i \le \dfrac{n+1}{2} \text{ and } n \ge 3; \\ 0, & \text{otherwise.} \end{cases}$$

RL_n is obtained by adding parallel edges a_l such that the ends of a_l are on uv and uw each for a positive integer n and $1 \le l \le n$ (Fig. 3(b)). Let a, b and a_l be cotree edges. A spanning tree of RL_n is obtained. RL_n has four types of embedding surfaces: $R_1^n baR_2^n b^- R_3^n a^- R_4^n$, $b^- R_1^n baR_2^n R_3^n a^- R_4^n$, $R_1^n abR_2^n b^- R_3^n a^- R_4^n$ and $b^- R_1^n abR_2^n R_3^n a^- R_4^n$.

$$g_i(RL_n) = 2g_{(i-1)_3}(n) + 2g_{i_{10}}(n) = 2\mu_{(i-1)_3}(n) + 2\mu_{i_9}(n).$$

The following conclusion can be obtained by using Theorem 4 of [10]:

Corollary 3 ([4]). *Let $g_i(RL_n)$ denote the number of distinct embeddings for RL_n with genus i and let $C_n(i) = \binom{n-i}{i-2}\frac{n}{i-1}2^{i-1}$. Then*

$$\begin{aligned} g_i(RL_n) = {} & 2^n C_n(i) \quad (\text{if } 1 \le i \le (n+1)/2) \\ & -2^{2i-2}C_n(i) \quad (\text{if } 2 \le i \le (n+1)/2) \\ & +2^{2i}C_n(i+1) \quad (\text{if } 1 \le i \le (n-1)/2) \\ & +2^n \quad (\text{if } i = (n-1)/2) \\ & +2^{n-1} \quad (\text{if } i = n/2 \text{ or } i = (n/2) - 1) \end{aligned}$$

$$+2^n(2^{(n/2)+1}-2) \quad (\textit{if } i = n/2)$$
$$-2 \quad (\textit{if } i = 1)$$
$$+2 \quad (\textit{if } i = 0). \quad \square$$

Corollary 4 ([4]). *The embedding distribution by genus for ML_n equals that of CL_n, except that ML_n has two extra embeddings of genus 1 and two fewer embeddings of genus 0.* □

References

[1] M.L. Furst, J.L. Gross, R. Statman, Genus distributions for two classes of graphs, J. Combin. Theory B 46 (1989) 22–36.
[2] Y.P. Liu, Embeddability in Graphs, Kluwer Academic Publisher, Dordrecht, Boston, London, 1995.
[3] Y.P. Liu, Advances in Combinatorial Maps, Northern Jiaotong University Press, Beijing, 2003 (in Chinese).
[4] L.A. McGeoch, Algorithms for two graph problems: Computing maximum-genus imbeddings and the two-server problem, Ph.D. Thesis, Computer Science Dept., Carnegie Mellon University, PA, 1987.
[5] E.H. Tesar, Genus distribution of Ringel ladders, Discrete Math. 216 (2000) 235–252.
[6] C. Thomassen, The graph genus problem is NP-complete, J. Algorithms 10 (1989) 568–576.
[7] L.X. Wan, Genus distribution of general ladders, Acta Math. Appl. Sin. (in press) (in Chinese).
[8] L.X. Wan, Y.P. Liu, Orientable embedding genus distribution for certain types of graphs, J. Combin. Theory, Ser. B 98 (2008) 19–32.
[9] L.X. Wan, Y.P. Liu, Orientable embedding distributions by genus for certain type of non-planar graphs (I), Ars Combin. 79 (2006) 97–105.
[10] L.X. Wan, Y.P. Liu, Orientable embedding distributions by genus for certain type of non-planar graphs (II), Ars Combin (in press).

On Three Types of Embedding of Pseudowheels on the Projective Plane*

Yan Yang, Yanpei Liu

Department of Mathematics, Beijing Jiaotong University,
Beijing 100044, *P.R.China*

Abstract A weak (or strong) embedding of a graph G is an embedding of G with no repeated edges (or vertices) on the boundary of each face of it. A pseudowheel is a wheel graph with multi-spokes. In this paper, we obtain the numbers (of equivalence classes) of three types of embedding, *i.e.*, general embedding, weak embedding and strong embedding, of pseudowheels on the projective plane.

Keywords pseudowheel; embedding; weak embedding; strong embedding

1. Introduction

A *surface* is a compact 2-dimensional manifold without boundary. It can be represented by a polygon of even edges in the plane whose edges are identified and directed clockwise or counterclockwise in pairs. Such polygonal representations of surfaces can be also written by words. For example, the sphere is written as $O_0 = aa^-$ where a^- is with the opposite direction of a on the boundary of the polygon. The projective plane, the torus and the Klein bottle are, respectively, aa, aba^-b^- and $aabb$. Suppose $A = a_1a_2\cdots a_t$, $t \geq 1$ is a word, then $A^- = a_t^- \cdots a_2^- a_1^-$ is called the *inverse* of A.

An *embedding* of a graph G into a surface S is a homeomorphism $h : G \to S$ of G into S such that every component of $S - h(G)$ is a 2-cell. In this paper, this type of embedding is also called general embedding. Two embeddings $h : G \to S$ and $g : G \to S$ of G into a surface S are said to be *equivalent* if there is an orientation-preserving homeomorphism $f : S \to S$ such that $f \circ h = g$. The connected components of $S - h(G)$ are called *faces* or *regions* of the embedding. A closed curve C on a surface S is called *contractible* if $S - C$ is disconnected and one of the regions of $S - C$ is homeomorphic to an open disc; otherwise it is called *noncontractible.*

A *weak embedding* (also called *edge-strong embedding* see [23]) of a graph G is an embedding of G such that there are no repeated edges (repeated vertices are allowed) on the boundary of each face. If the boundary of a face contains no repeated vertices and edges, then the boundary of the face is a circuit. A *strong embedding* (also called *closed* 2-*cell embedding* see [23] and *circular embedding* see [17]) of a graph G is an embedding of G with all the face boundaries being circuits. The *weak* (or *strong*) *embedding conjecture* states that every 2-connected graph has a weak (or strong) embedding on some surfaces. These conjectures have close relations to the circuit double cover conjecture (see [5],[12] for details).

*Supported by NNSFC under Grant No.60373030

E-mail address: yanyang0206@126.com (Y.Yang)

The *maximum orientable* (or *nonorientable*) *genus* of a graph G is the maximum integer k such that G can be embedded on the orientable (or nonorientable) surface of genus k, denoted by $\gamma_M(G)$ (or $\tilde{\gamma}_M(G)$). If the embedding is strong, the corresponding maximum orientable (or nonorientable) genus is called *strong maximum orientable* (or *nonorientable*) *genus*, denoted by $\gamma_{sM}(G)$ (or $\tilde{\gamma}_{sM}(G)$).

Given a graph, how many nonequivalent embeddings does it have on each orientable surface. This problem was inaugurated by Gross and Furst in [3]. Gross et al.[4] did it for bouquets of circles; Furst et al.[2] for closed-end ladders and cobblestone paths; Kwak and Lee [6] for dipoles; and Tesar [18] for Ringel ladders, etc. Chen et al. generalized this problem to nonorientable surface, and calculated the total genus distribution of necklace, closed-end ladders and cobblestone paths in [1]; Kwak and Shim [7] for bouquets of circles and dipoles. Liu [8] gave the number of embeddings of a graph on the sphere.

A natural question, one can posed is that, how many nonequivalent weak (or strong) embeddings of a graph does it have on each orientable (or nonorientable) surface. There are two main steps to solve this problem. First, one must find that on which orientable (or nonorientable) surfaces the graph can be weakly (or strongly) embedded. Second, enumerate the nonequivalent weak (or strong) embeddings of the graph on the surfaces it can be embedded.

For $n \geq 3$, the *wheel graph* of n spokes is the graph W_n obtained from a circuit C_n by adding a new vertex v_0 and joining it to all vertices of C_n. A wheel graph with multi-spokes is the *pseudowheel*. Suppose that the vertices of C_n are $v_1, \ldots, v_i, \ldots, v_n$ in clockwise. For $1 \leq i \leq n$, join v_0 to v_i with k_i multi-edges, then we can get the pseudowheel $W_n^{(k_1,\ldots,k_n)}$. Let $\bar{k} = (k_1, \ldots, k_n)$ be a n-dimensional vector, $W_n^{(k_1,\ldots,k_n)}$ can be denoted by $W_n^{\bar{k}}$ simply. We denote the k_i multi-edges joining v_0 to v_i by $a_i^1, \ldots, a_i^{k_i}$, the edge joining v_i to v_{i+1} in C_n by a_i, for $1 \leq i \leq n$, and denote the edge joining v_n to v_1 in C_n by a_0. Fig.1, Fig.2 illustrate wheel graph W_4 and pseudowheel $W_4^{(k_1,k_2,k_3,k_4)}$, respectively.

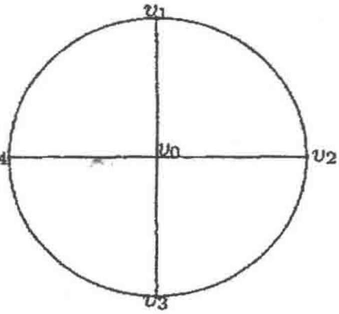

Fig.1. The graph W_4

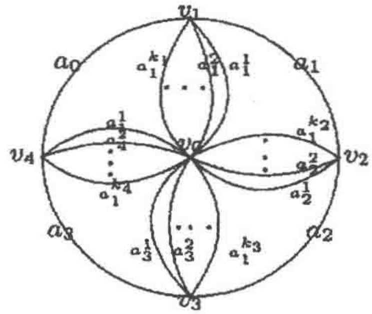

Fig.2. The graph $W_4^{(k_1,\ldots,k_4)}$

In the following, we will introduce the joint tree model of a graph embedding, established in [9] by Liu, based on his initial work in [10]. By using the joint trees method, Wan and Liu [21,22] calculated orientable embedding distributions for certain type of non-planar graphs.

Given a spanning tree T of a graph G, for $1 \leq i \leq \beta$, we split each cotree edge e_i into two semi-edges and label them by the same letter as a_i where β is the betti number of G. The resulting graph consisting of tree edges in T and 2β semi-edges is a tree. We denote this new tree by $\hat{T}$. Then indexing the 2β semi-edges of $\hat{T}$ by $+$(always omitted) or $-$, so that the indices of each pair of semi-edges labelled with same letter can be the same or distinct. A *rotation* at a vertex v, denoted by σ_v, is a cyclic permutation of edges incident with v. Let $\sigma_G = \prod_{v \in V(G)} \sigma_v$ be a rotation system of G.

The tree $\hat{T}$ with an index of each semi-edge and a rotation system of it is called a *joint tree* [9,11] of G. Denote the joint tree by $\hat{T}_\sigma^\delta$, in which $\delta = (\delta_1, \delta_2, \cdots, \delta_\beta)$ be a binary vector, δ_i can be 0 or 1 where $\delta_i = 0$ means that the two indices of a_i are the distinct; otherwise, the same. By reading these lettered semi-edges with indices of a $\hat{T}_\sigma^\delta$ in a fixed orientation (clockwise or counterclockwise), we can get an algebraic representation for a surface. It is a cyclic order of 2β letters with indices. Such a surface is called an *associated surface* [11] of G. If two associate surfaces of G have same cyclic order with the same δ in their algebraic representations, then we say that they are the same; otherwise, distinct.

From [11], there is a 1-to-1 correspondence between associate surfaces and embeddings of a graph, hence an embedding of a graph on a surface can be represented by an associate surface of it.

In Fig.3 we give two embeddings and their joint trees of pseudowheel $W_3^{(2,2,2)}$. From the joint trees we can get the two associate surfaces corresponding to embeddings (a) and (b), they are $a_2^1 a_2^2 a_1^2 a_1^1 a_0 a_0^- a_3^1 a_3^{1-} a_2^2 a_1^2 a_1^1 a_2^{1-}$ and $a_1^1 a_0 a_1^2 a_2^2 a_2^1 a_3^1 a_0 a_1^2$ $a_2^2 a_2^1 a_3^1 a_1^{1-}$, respectively.

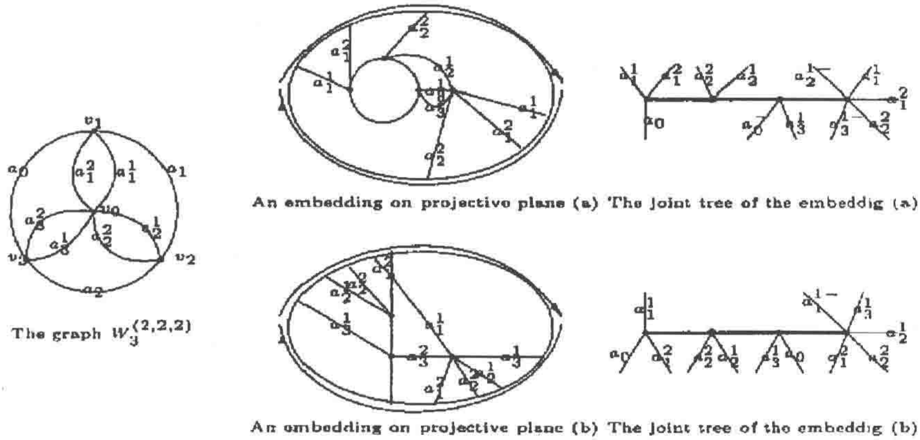

Fig.3. The graph $W_3^{(2,2,2)}$, embeddings and joint trees of it

From the definition of strong embedding, we have

Lemma 1.1 [20] *Suppose Ψ is a strong embedding of a graph G on some surfaces, $F(\Psi)$ is the set of faces in Ψ, then $|F(\Psi)| \geq \nabla(G)$ where $\nabla(G)$ denotes the maximum degree of the graph.*

From [13], we can get that

Lemma 1.2 [13] *Let G be a 3-connected planar graph. If G contains a triangle, then G has a strong embedding on the projective plane.*

Lemma 1.3 *For the pseudowheel $W_n^{\bar{k}}$, $\gamma_{sM}(W_n^{\bar{k}}) = 0$, $\tilde{\gamma}_{sM}(W_n^{\bar{k}}) = 1$ where $\bar{k} = (k_1, \ldots, k_n)$, $k_i \geq 1$, for $1 \leq i \leq n$.*

Proof The maximum degree of pseudowheel graph $W_n^{\bar{k}}$ is $k_1 + \cdots + k_n$. Suppose Ψ is a strong embedding of $W_n^{\bar{k}}$, $F(\Psi)$ is the set of faces in Ψ, by Lemma 1.1, we have that $|F(\Psi)| \geq k_1 + \cdots + k_n$. By Euler formula, it is impossible to have strong embeddings of $W_n^{\bar{k}}$ on surfaces other than the sphere and the projective plane. Because $W_n^{\bar{k}}$ is a 3-connected planer graph without loops, there exists a strong embedding on the sphere. And by Lemma 1.2, there exits a strong embedding on the projective plane too. So $\gamma_{sM}(W_n^{\bar{k}}) = 0$, $\tilde{\gamma}_{sM}(W_n^{\bar{k}}) = 1$. Thus the proof is complete. □

From the definition of weak embedding and Lemma 1.3, we have that the pseudowheel $W_n^{\bar{k}}$ also has weak embeddings on the sphere and the projective plane.

Lemma 1.4 [16] *The number of non-negative integral solutions of the equation $x_1 + x_2 + \cdots + x_r = n$ $(n \geq 0)$ is $\binom{r+n-1}{n}$.*

In this paper, we obtain the numbers (of equivalence classes) of three types of embedding, *i.e.*, general embedding, weak embedding and strong embedding, of pseudowheels on the projective plane, by using the joint tree method.

There are two main reasons why we do these research. Firstly, wheel graph have some special characters, which had been proved very useful in the research of 3-connected graphs [19]. Ren and Deng[15] investigate the flexibility of wheel graphs on the torus. Secondly, there is no result about the number of weak (or strong) embeddings of graphs on surfaces as we known up to now. The results here and the method we used will be helpful for the further research of this type of problem.

2. The number of general embeddings of pseudowheel $W_n^{\bar{k}}$ on the projective plane

Theorem 2.1 *The number of general embeddings of pseudowheel $W_n^{\bar{k}}$ on the projective plane is*

$$\left(\sum_{i=1}^{n} k_i^2 + 2\sum_{i=1}^{n-1}\sum_{\beta=1}^{n-i} k_i k_{i+\beta} - \sum_{i=1}^{n} k_i\right)\prod_{j=1}^{n} k_j! + \prod_{j=1}^{n}(k_j+1)!$$

Proof The circuit C_n of a pseudowheel $W_n^{\bar{k}}$ can be classified into two cases in the embeddings. Case 1: C_n is contractible; Case 2: C_n is noncontractible. According to [13], in Case 1 and Case 2, the embeddings of $W_n^{\bar{k}}$ on the projective plane have the structures as shown in Fig.4 and Fig.5, respectively. We choose $v_1 a_1 v_2 \cdots v_{n-1} a_{n-1} v_n a_n^{k_n} v_0$ as the spanning tree of $W_n^{\bar{k}}$ and we will discuss the two cases respectively.

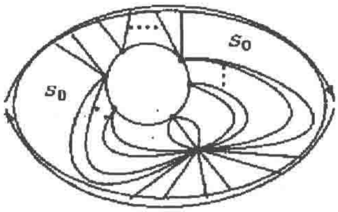

Fig.4. The embedding of $W_n^{\bar{k}}$ when C_n is contractible

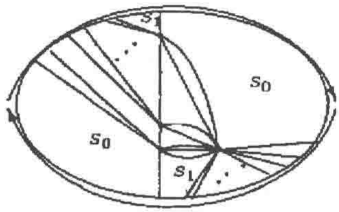

Fig.5. The embedding of $W_n^{\bar{k}}$ when C_n is noncontractible

Case 1 C_n is contractible.

Since C_n is contractible, it is facial. Let

$$E_i = \{a_i^1, \ldots, a_i^{k_i}\},\ 1 \le i \le n;$$

$$\mathcal{H} = \{E_i \mid \exists a_i^t \in E_i, st.\ \delta(a_i^t) = 1,\ 1 \le t \le k_i,\ 1 \le i \le n\};$$

and

$$A_j = a_j^{\varepsilon_1} \cdots a_j^{\varepsilon_{k_j}}, 1 \le j \le n-1,\ A_n = a_n^{\varepsilon_1} \cdots a_n^{\varepsilon_{k_n - 1}}.$$

The sequence $\varepsilon_1, \ldots, \varepsilon_{k_j}$ is a permutation of $\{1, 2, \ldots, k_j\}$. It is easy to get that there are $k_i!$ possible forms of $A_i, 1 \le i \le n-1$, according to the rotation at v_i. We classify the embeddings according to the number $|\mathcal{H}|$.

Subcase 1.1 $|\mathcal{H}| = 1$.

Subcase 1.1.1 $\nexists\ a_n^t \in E_n, st.\ \delta(a_n^t) = 1$, for $1 \le t \le k_n$.
In this case, according to the structure as shown in Fig.4 and the joint tree method, we can get that the joint trees of pseudowheel $W_n^{\bar{k}}$ on the projective plane have the form as (a) or (b) in Fig.6, where

$$A_i = A_i^1 \widetilde{A}_i A_i^2, |\widetilde{A}_i| \ge 1, 1 \le i \le n-1;\ A_n^1 A_n^2 = A_n;$$

$$B_1 = A_n^{1-} A_{n-1}^{-} \cdots A_{i+1}^{-} A_i^{2-} \widetilde{A}_i A_i^{1-} A_{i-1}^{-} \cdots A_1^{-} A_n^{2-};$$

$$B_2 = A_n^{2-} A_1^{-} \cdots A_{i-1}^{-} A_i^{1-} \widetilde{A}_i A_i^{2-} A_{i+1}^{-} \cdots A_{n-1}^{-} A_n^{1-}.$$

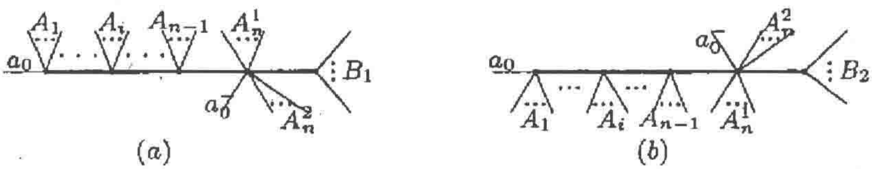

Fig.6. The joint trees

By reading the lettered semi-edges of joint trees (a) in clockwise in Fig.6, one can get the corresponding associate surfaces, they are of the forms

$$(A_1 \cdots A_{i-1} A_i^1 \widetilde{A}_i A_i^2 A_{i+1} \cdots A_{n-1} A_n^1 A_n^{1-} A_{n-1}^{-} \cdots A_{i+1}^{1-} A_i^{2-}$$