CLASSICS IN MATHEMATICS

Joram Lindenstrauss · Lior Tzafriri

Classical Banach Spaces I and II

经典巴拿赫空间I和II

Springer

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Reprint of the 1977 Edition

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图书在版编目（CIP）数据

经典巴拿赫空间Ⅰ和Ⅱ：英文/（美）林登斯托斯（Lindenstrauss，J.）著．—影印本．—北京：世界图书出版公司北京公司，2010.2

书名原文：Classical Banach Spaces Ⅰ and Ⅱ

Springer 数学经典教材

ISBN 978-7-5100-0525-1

Ⅰ．①经… Ⅱ．①林… Ⅲ．①巴拿赫空间—教材—英文 Ⅳ．①O177.2

中国版本图书馆 CIP 数据核字（2010）第 010570 号

书　　名：	Classical Banach Spaces Ⅰ and Ⅱ
作　　者：	Joram Lindenstrauss & Lior Tzafriri
中 译 名：	经典巴拿赫空间Ⅰ和Ⅱ
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24 开
印　　张：	19
版　　次：	2010 年 01 月
版权登记：	图字：01-2009-1068
书　　号：	978-7-5100-0525-1/O · 741　　**定　　价：** 50.00 元

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AMS Subject Classification (1970): 46-02, 46 A45, 46Bxx, 46Jxx

ISBN 3-540-08072-4 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08072-4 Springer-Verlag New York Heidelberg Berlin

Library of Congress Cataloging in Publication Data. Lindenstrauss, Joram, 1936-. Classical Banach spaces. (Ergebnisse der Mathematik und ihrer Grenzgebiete; 92). Bibliography: v. H1, p. Includes index. CONTENTS: 1. Sequence spaces. 1. Banach spaces. 2. Sequence spaces. 3. Function spaces. I. Tzafriri, Lior, 1936- joint author. II. Title. III. Series QA322.2.L56. 1977- 515'.73. 77-23131.

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Classical Banach Spaces I

Sequence Spaces

Springer-Verlag
Berlin Heidelberg New York 1977

To Naomi and Marianne

Preface

The appearance of Banach's book [8] in 1932 signified the beginning of a systematic study of normed linear spaces, which have been the subject of continuous research ever since.

In the sixties, and especially in the last decade, the research activity in this area grew considerably. As a result, Banach space theory gained very much in depth as well as in scope. Most of its well known classical problems were solved, many interesting new directions were developed, and deep connections between Banach space theory and other areas of mathematics were established.

The purpose of this book is to present the main results and current research directions in the geometry of Banach spaces, with an emphasis on the study of the structure of the classical Banach spaces, that is $C(K)$ and $L_p(\mu)$ and related spaces. We did not attempt to write a comprehensive survey of Banach space theory, or even only of the theory of classical Banach spaces, since the amount of interesting results on the subject makes such a survey practically impossible.

A part of the subject matter of this book appeared in outline in our lecture notes [96]. In contrast to those notes, most of the results presented here are given with complete proofs. We therefore hope that it will be possible to use the present book both as a text book on Banach space theory and as a reference book for research workers in the area. It contains much material which was not discussed in [96], a large part of which being the result of very recent research work. An indication to the rapid recent progress in Banach space theory is the fact that most of the many problems stated in [96] have been solved by now.

In the present volume we also state some open problems. It is reasonable to expect that many of these will be solved in the not too far future. We feel, however, that most of the topics discussed here have reached a relatively final form, and that their presentation will not be radically affected by the solution of the open problems. Among the topics discussed in detail in this volume, the one which seems to us to be the least well understood and which might change the most in the future, is that of the approximation property.

We divided our book into four volumes. The present volume deals with sequence spaces. The notion of a Schauder basis plays a central role here. The classical spaces which are in the most natural way sequence spaces are c_0 and l_p, $1 \leqslant p \leqslant \infty$. Volumes II and III will deal with function spaces. In Volume II we shall present the general theory of Banach lattices with an emphasis on those notions concerning lattices which are related to $L_p(\mu)$ spaces. Volume III will be devoted to a study of the structure of the spaces $L_p(0, 1)$, $C(K)$ and general preduals of

$L_1(\mu)$ spaces. The division of the common Banach spaces into sequence and function spaces is made according to the usual practice. It should be remembered, however, that several spaces have natural representations both as sequence and function spaces. The best known example is the separable Hilbert space, which can be represented both as the sequence space l_2 and as the function space $L_2(0, 1)$. A less trivial example is the space l_p, $1 \leqslant p \leqslant \infty$, which is isomorphic to the function space $H_p(D)$ of the analytic functions on the disc $D=\{z;\ |z|<1\}$ with $\|f\|=(\iint|f(z)|^p\ dx\ dy)^{1/p}<\infty$ (cf. [88]). Also, the spaces $C(0, 1)$ and $L_p(0, 1)$, $1 \leqslant p<\infty$, have Schauder bases, and thus it is convenient sometimes to use their representations as sequence spaces.

In Volume IV we intend to present the local theory of Banach spaces. This theory deals with the structure of finite-dimensional Banach spaces and the relation between an infinite-dimensional Banach space and its finite-dimensional subspaces. A central part in this approach to Banach space theory is played by the evaluation of various parameters of finite-dimensional Banach spaces. The role of the classical finite-dimensional spaces, that is of the spaces l_p^n, $1 \leqslant p \leqslant \infty$, $n=1, 2, \ldots$ in the local theory of Banach spaces is even more central than the role of the classical spaces in the general theory of Banach sequence spaces and function spaces.

We sketch now briefly the contents of this volume. Chapter 1 contains a quite complete account of the main results on Schauder bases in general Banach spaces. Several notions related to Schauder bases—the various approximation properties, general biorthogonal systems and Schauder decompositions—as well as some examples are discussed in detail.

Chapter 2 is devoted to a study of the spaces c_0 and l_p, $1 \leqslant p<\infty$, and to some extent also of l_∞. Section *a* is devoted to an examination of the basic properties of these spaces, some of which are shown to characterize these spaces among general Banach spaces. The other sections of Chapter 2 are basically independent of each other and can thus be read in any order. In Sections *b* and *c* we discuss certain ideals of operators on general Banach spaces and show how they can be used in the study of the structure of the classical sequence spaces. Section *d* contains a structure theorem for "nice" subspaces of c_0 and l_p as well as examples of subspaces which are not "nice" (i.e. subspaces which fail to have the approximation property). This section contains also a discussion of general results related to the approximation property which complement the treatment of this property in Section *e* of Chapter 1. Section *f* contains an example of an infinite-dimensional Banach space which fails to have any of the classical sequence spaces as a subspace and also criteria for general Banach spaces to have subspaces isomorphic to c_0 and especially to l_1. The final section of Chapter 2 deals with the extension properties of c_0 and l_∞, the lifting property of l_1, and the closely related topic of the automorphisms of these spaces.

In Chapter 3 we discuss the special properties of symmetric bases and the relation between symmetric bases and general unconditional bases. A large part of this chapter is devoted to results and examples related to the possible characterizations of c_0 and l_p, $1 \leqslant p<\infty$, in the class of all spaces with a symmetric basis. The final chapter of this volume is devoted to a detailed study of the structure of some particular classes of spaces with symmetric bases, mainly Orlicz sequence spaces. The main emphasis is again on the relation between these spaces and the spaces c_0 and l_p. Several examples given there demonstrate how much more complicated the structure of general Orlicz sequence spaces is, as compared to that of l_p spaces. In section *d* it is shown that Orlicz sequence spaces enter naturally into the study of spaces like $l_p \oplus l_r$ with $p \neq r$. In Vol. III it will be shown that Orlicz sequence spaces arise naturally in the study of the structure of subspaces of $L_1(0, 1)$.

We assume that the reader is familiar with the basic results of real analysis and functional analysis which are usually covered in first year graduate courses in these subjects. An acquaintance with the main results in chapters I–VI of [33] will certainly suffice (much less is actually needed for being able to read this book).

The bibliography contains only those papers which are actually quoted in the text. We tried to indicate in the text the source of the main results which we present. The reference list is, however, far from being complete. Reference to papers where the basic results in Banach space theory were first proved can be found, for example, in [28] and [33]. Further references on bases may be found in [135]. References to further literature on Orlicz spaces may be found in [75].

The overlap between this book and existing books on related topics is very small.

We would like to acknowledge the contribution of G. Schechtman, who read the entire manuscript and made several very valuable suggestions, and of J. Arazy, who was very helpful in proofreading this volume and eliminated several mistakes. We also wish to thank Z. Altshuler and Y. Sternfeld for their help. We are grateful to Danit Sharon who expertly carried out the task of typing the manuscript of this book. We are also indebted to the U.S.-Israel Binational Science Foundation for partial support.

Finally, we would like to thank Springer-Verlag and especially Roberto Minio for their cooperation and help in all the stages of the preparation of this book.

Jerusalem Joram Lindenstrauss
January 1977 Lior Tzafriri

Standard Definitions, Notations and Conventions

For most of the results presented in this book it does not matter whether the field of scalars is real or complex. In the isometric theory there are some differences (usually minor) between real and complex spaces. As a rule we shall work with real scalars and, in a few places, we shall indicate the changes needed in the complex case. In a few instances e.g. where spaces of analytic functions are involved or where spectral theory is used we shall use complex scalars.

By $L_p(\mu)=L_p(\Omega, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$ we denote the Banach space of equivalence classes of measurable functions on (Ω, Σ, μ) whose p'th power is integrable (respectively, which are essentially bounded if $p=\infty$). The norm in $L_p(\mu)$ is defined by $\|f\|=(\int|f(\omega)|^p\, d\mu(\omega))^{1/p}$ (ess sup $|f(\omega)|$ if $p=\infty$). If (Ω, Σ, μ) is the usual Lebesgue measure space on [0, 1] we denote $L_p(\mu)$ by $L_p(0, 1)$. If (Γ, Σ, μ) is the discrete measure space on a set Γ, with $\mu(\{\gamma\})=1$ for every $\gamma \in \Gamma$, we denote $L_p(\mu)$ by $l_p(\Gamma)$. If Γ is the set of positive integers we denote $l_p(\Gamma)$ also by l_p while if $\Gamma=\{1, 2, \dots, n\}$, for some $n<\infty$, we denote $l_p(\Gamma)$ by l_p^n. The subspace of $l_\infty(\Gamma)$, of those functions which vanish at ∞, is denoted by $c_0(\Gamma)$ (if Γ is the set of positive integers we denote this space by c_0). The subspace of l_∞ consisting of convergent sequences is denoted by c. For a compact Hausdorff space K we denote by $C(K)$ the Banach space of all continuous scalar-valued functions on K with the supremum norm. If K is the unit interval [0, 1] in its usual topology we denote $C(K)$ by $C(0, 1)$.

In a Banach space X we denote the ball with center x and radius r, i.e. $\{y;\ \|y-x\| \leqslant r\}$, by $B_X(x, r)$. If the space X is clear from the context, we simply write $B(x, r)$. The unit ball $B_X(0, 1)$ of X is denoted also by B_X. For a sequence $\{x_n\}_{n=1}^\infty$ of elements of X we denote by span $\{x_n\}_{n=1}^\infty$ the algebraic linear span of $\{x_n\}_{n=1}^\infty$ i.e. the set of all finite linear combinations of $\{x_n\}_{n=1}^\infty$. The closure of span $\{x_n\}_{n=1}^\infty$ is denoted by $[x_n]_{n=1}^\infty$. A similar notation is used for the span of a set other than a sequence. For a set $A \subset X$ its norm closure is denoted by $\overline{A}$, e.g. $[x_n]_{n=1}^\infty=\overline{\text{span}\,\{x_n\}_{n=1}^\infty}$. The convex hull of a sequence $\{x_n\}_{n=1}^\infty$ is denoted by conv $\{x_n\}_{n=1}^\infty$; the closed convex hull by $\overline{\text{conv}\,\{x_n\}_{n=1}^\infty}$.

The term "operator" means a bounded linear operator unless specified otherwise. The space of all operators from X to Y with the usual operator norm is denoted by $L(X, Y)$. An operator $T \in L(X, Y)$ is called *compact* if $\overline{T B_X}$ is a norm compact subset of Y. The identity operator of a Banach space X is denoted by I_X (or simply by I if X is clear from the context). For an operator $T \in L(X, Y)$ the notation $T_{|Z}$ denotes the restriction of T to the subspace Z of X.

Two Banach spaces X and Y are called *isomorphic* (denoted by $X \approx Y$) if there exists an invertible operator from X onto Y. *The Banach-Mazur distance coefficient*

$d(X, Y)$ is defined by $\inf \|T\|\|T^{-1}\|$, the infimum being taken over all invertible operators from X onto Y (if X is not isomorphic to Y we put $d(X, Y)=\infty$). Notice that $d(X, Y)\geq 1$, for every X and Y, and that $d(X, Y)\, d(Y, Z)\geq d(X, Z)$, for every X, Y and Z. If there exists an invertible operator T from X onto Y so that $\|T\|=\|T^{-1}\|=1$ (i.e. $\|Tx\|=\|x\|$, for every $x\in X$) we say that *X is isometric to Y*. In this case $d(X, Y)=1$ (the converse is false in general; it is possible that $d(X, Y)=1$ but that the infimum in the definition of $d(X, Y)$ is not attained i.e. X is not isometric to Y). An operator $T\in L(X, Y)$ is said to be an isomorphism into Y if there is some constant $C>0$ so that $\|Tx\|\geq C\|x\|$ for every $x\in X$. In this case T^{-1} is a well defined element in $L(TX, X)$.

A closed linear subspace Y of a Banach space X is said to be a *complemented subspace* of X if there is a bounded linear projection from X onto Y, or what is the same, if there exists a closed linear subspace Z of X so that X is the direct sum of Y and Z, i.e. $X=Y\oplus Z$. We shall also use some direct sums of infinite sequences of Banach spaces. If $\{X_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces we define the direct sum of these spaces in the sense of l_p, $1\leqslant p<\infty$, namely $\left(\sum_{n=1}^{\infty}\oplus X_n\right)_p$, as the space of all sequences $x=(x_1, x_2, \ldots)$, with $x_n\in X_n$ for all n, for which $\|x\|=\left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{1/p}<\infty$. Similarly, $\left(\sum_{n=1}^{\infty}\oplus X_n\right)_0$ denotes the direct sum of $\{X_n\}_{n=1}^{\infty}$ in the sense of c_0 i.e. the space of all sequences $x=(x_1, x_2, \ldots)$, with $x_n\in X_n$ for all n, for which $\lim_n \|x_n\|=0$. The norm in this direct sum is taken as $\|x\|=\max_n \|x_n\|$. We shall occasionally use also other types of infinite direct sums. These will be defined in the proper places in the text.

Besides the norm (or strong) topology of a Banach space X we often use some other topologies. If Y is a subspace of the dual X^* of X then the Y-topology of X is the weakest topology making all the elements of Y continuous. A basis for the Y topology is obtained by taking all the sets of the form $V(x, \varepsilon, A)=\{u;\ |x^*(u)-x^*(x)|<\varepsilon,\ x^*\in A\}$, where $x\in X$, $\varepsilon>0$ and A is a finite subset of Y. If $Y=X^*$ the Y topology is called the weak topology (w topology). If $X=Z^*$ and we take as Y the canonical image of Z in $Z^{**}=X^*$ we obtain the w^* topology induced by Z (if Z is clear from the context we simply talk of the w^* topology). Convergence of sequences in the w topology (resp. w^* topology) is denoted by $x_n\xrightarrow{w} x$ or $w\lim_n x_n=x$ (resp. $x_n\xrightarrow{w^*}x$ or $w^*\lim_n x_n=x$). An operator $T\in L(X, Y)$ is said to be *w compact* if $\overline{T B_X}$ is a compact set in Y, in its w topology (i.e. a w compact set in Y).

Whenever we consider a Banach space X as a subspace of its second dual X^{**} we assume that it is embedded canonically. For a subset $A\subset X$ we denote by $A^{\perp}$ the subspace $\{x^*;\ x^*(x)=0,\ x\in A\}$ of X^*. For a subset $A\subset X^*$ we denote by $A^{\top}$ the subspace $\{x;\ x^*(x)=0,\ x^*\in A\}$ of X. For every subset $A\subset X$ we have $A^{\perp\top}\supset A$ and equality holds if and only if A is a closed linear subspace.

Besides subspaces of Banach spaces we shall also study quotient spaces. An operator $T: X\to Y$ is called a *quotient map* if $\overline{T B_X}=B_Y$. A Banach space Y is isomorphic to a quotient space of a space X if and only if there exists an operator T from X onto Y. If such a T exists then $Y\approx X/\ker T$, where $\ker T=\{x;\ Tx=0\}$, and Y^* is isomorphic to the subspace $(\ker T)^\perp$ of X^*. Similarly, if Z is a subspace of X then Z^* is isometric to the quotient space $X/(Z^\perp)$.

Among the general notations used in this book we want to single out the following. For a positive number S we denote by $[S]$ the largest integer $\leqslant S$. For a set A we denote by $\bar{\bar{A}}$ the cardinality of A. If A and B are sets we put $A \sim B = \{x, x \in A, x \notin B\}$.

Table of Contents

1. Schauder Bases

a. Existence of Bases and Examples

The aim of this volume is to describe some results concerning sequence spaces, i.e. those Banach spaces which can be presented in some natural manner as spaces of sequences. In general, such a representation is achieved by introducing in the space a sort of "coordinate system". There are, obviously, many different ways of giving a precise meaning to the terms "Banach sequence spaces" and "coordinate systems". The best known and most useful approach is by using the notion of a Schauder basis.

Definition 1.a.1. A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^\infty$ so that $x = \sum_{n=1}^\infty a_n x_n$. A sequence $\{x_n\}_{n=1}^\infty$ which is a Schauder basis of its closed linear span is called a *basic sequence.*

In this book we shall not consider any type of bases in infinite-dimensional Banach spaces besides Schauder bases. We shall therefore often omit the word Schauder. In addition to Schauder bases we shall only encounter algebraic bases in finite-dimensional spaces. This should not cause any confusion. As a matter of fact, quantitative notions concerning Schauder bases (like the basis constant defined below) have a meaning and will be used also in the context of algebraic bases in finite dimensional spaces.

Evidently, a space X with a Schauder basis $\{x_n\}_{n=1}^\infty$ can be considered as a sequence space by identifying each $x = \sum_{n=1}^\infty a_n x_n$ with the unique sequence of coefficients $(a_1, a_2, a_3, \ldots)$. It is important to note that for describing a Schauder basis one has to define the basis vectors not only as a set but as an ordered sequence.

Let $(X, \|\ \|)$ be a Banach space with a basis $\{x_n\}_{n=1}^\infty$. For every $x = \sum_{n=1}^\infty a_n x_n$ in X the expression $|||x||| = \sup_n \left\| \sum_{i=1}^n a_i x_i \right\|$ is finite. Evidently, $|||\cdot|||$ is a norm on X and $\|x\| \leqslant |||x|||$ for every $x \in X$. A simple argument shows that X is complete also with respect to $|||\cdot|||$ and thus, by the open mapping theorem, the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. These remarks prove the following proposition [8].

Proposition 1.a.2. *Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^\infty$. Then the* *projections* $P_n: X \to X$, *defined by* $P_n\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{n} a_i x_i$, *are bounded linear operators and* $\sup_n \|P_n\| < \infty$.

The projections $\{P_n\}_{n=1}^{\infty}$ are called the natural projections associated to $\{x_n\}_{n=1}^{\infty}$; the number $\sup_n \|P_n\|$ is called the *basis constant* of $\{x_n\}_{n=1}^{\infty}$. A basis whose basis constant is 1 is called a *monotone basis*. In other words, a basis is monotone if, for every choice of scalars $\{a_n\}_{n=1}^{\infty}$, the sequence of numbers $\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\|\right\}_{n=1}^{\infty}$ is non-decreasing. Every Schauder basis $\{x_n\}_{n=1}^{\infty}$ is monotone with respect to the norm $|||x||| = \sup_n \|P_n x\|$ which was already used above. Indeed,

$$|||P_n x||| = \sup_m \|P_m P_n x\| = \sup_{1 \leqslant m \leqslant n} \|P_m x\| \leqslant |||x||| .$$

Thus, given any Schauder basis $\{x_n\}_{n=1}^{\infty}$ of X, we can pass to an equivalent norm in X for which the given basis is monotone.

There is a simple and useful criterion for checking whether a given sequence is a Schauder basis.

Proposition 1.a.3. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of vectors in* X. *Then* $\{x_n\}_{n=1}^{\infty}$ *is a Schauder basis of* X *if and only if the following three conditions hold.*

(i) $x_n \neq 0$ *for all* n.

(ii) *There is a constant* K *so that, for every choice of scalars* $\{a_i\}_{i=1}^{\infty}$ *and integers* $n < m$, *we have*

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leqslant K \left\|\sum_{i=1}^{m} a_i x_i\right\| .$$

(iii) *The closed linear span of* $\{x_n\}_{n=1}^{\infty}$ *is all of* X.

The proof is easy. The necessity of (i) and (iii) is clear from the definition, while that of (ii) follows from 1.a.2. Conversely, if (i) and (ii) hold then $\sum_{n=1}^{\infty} a_n x_n = 0$ implies that $a_n = 0$ for all n. This proves the uniqueness of the expansion in terms of $\{x_n\}_{n=1}^{\infty}$. In order to prove that every $x \in X$ has such an expansion it is enough, in view of (iii), to show that the space of all elements of the form $\sum_{n=1}^{\infty} a_n x_n$ is a closed linear space. This latter fact can be easily proved by using (ii). □

Obviously conditions (i) and (ii) of 1.a.3, by themselves, form a necessary and sufficient condition for a sequence $\{x_n\}_{n=1}^{\infty}$ to be a basic sequence. It is also worthwhile to observe that in case we can take $K = 1$ it is enough to verify (ii) for $m = n + 1$.

A basis $\{x_n\}_{n=1}^{\infty}$ is called *normalized* if $\|x_n\| = 1$ for all n. Clearly, whenever $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis of X, the sequence $\{x_n / \|x_n\|\}_{n=1}^{\infty}$ is a normalized basis in X.

Before proceeding with the general discussion we present some examples of bases. The unit vectors $e_n=(0, 0, 0,\dots, \overset{n}{1}, 0,\dots)$ form a monotone and normalized basis in each of the spaces c_0 and l_p, $1\leqslant p<\infty$. An example of a basis in the space c, of convergent sequences of scalars, is given by

$$x_1=(1, 1, 1,\dots) \quad \text{and, for } n>1,\ x_n=e_{n-1}.$$

The expansion of $x=(a_1, a_2,\dots)\in c$ with respect to this basis is

$$x=(\lim_n a_n)x_1+(a_1-\lim_n a_n)x_2+(a_2-\lim_n a_n)x_3+\cdots.$$

An important example of a Schauder basis is the Haar system in $L_p(0, 1)$, $1\leqslant p<\infty$.

Definition 1.a.4. The sequence of functions $\{\chi_n(t)\}_{n=1}^\infty$ defined by $\chi_1(t)\equiv 1$ and, for $k=0, 1, 2,\dots, l=1, 2,\dots, 2^k$,

$$\chi_{2^k+l}(t)=\begin{cases} 1 & \text{if} \quad t\in[(2l-2)2^{-k-1}, (2l-1)2^{-k-1}] \\ -1 & \text{if} \quad t\in((2l-1)2^{-k-1}, 2l\cdot 2^{-k-1}] \\ 0 & \text{otherwise} \end{cases}$$

is called the *Haar system*.

The Haar system is (in its given order) a monotone (but obviously not normalized) basis of $L_p(0, 1)$ for every $1\leqslant p<\infty$. Indeed, since the linear span of the Haar system contains all the characteristic functions of dyadic intervals (i.e. intervals of the form $[l\cdot 2^{-k}, (l+1)\cdot 2^{-k}]$), it is clear that (iii) of 1.a.3 holds. We have only to verify that (ii) holds with $K=1$. Let $\{a_i\}_{i=1}^\infty$ be any sequence of scalars, let n be an integer and let $f(t)=\sum_{i=1}^{n} a_i\chi_i(t)$ and $g(t)=\sum_{i=1}^{n+1} a_i\chi_i(t)$. The only difference between f and g is that on some dyadic interval I where f has the constant value b, say, g has the value $b+a_{n+1}$ on the first half of I and $b-a_{n+1}$ on the second half. Since, for every $p\geqslant 1$, $|b+a_{n+1}|^p+|b-a_{n+1}|^p\geqslant 2|b|^p$ we get that $\|f\|\leqslant\|g\|$.

By integrating the Haar system or more precisely by putting

$$\varphi_1(t)\equiv 1; \qquad \varphi_n(t)=\int_0^t \chi_{n-1}(u)\,du, \quad n>1$$

we obtain another famous and important basis. The sequence $\{\varphi_n\}_{n=1}^\infty$ is called the *Schauder system*. The Schauder system is a monotone basis of $C(0, 1)$. Indeed, the linear span of the $\{\varphi_n\}_{n=1}^\infty$ consists exactly of the continuous piecewise linear functions on $[0, 1]$ whose nodes are dyadic points. This shows that (iii) of 1.a.3 is satisfied. Since, for every integer n, the interval on which the function $\varphi_{n+1}(t)$ is different from 0 is such that on it all the functions $\{\varphi_i(t)\}_{i=1}^n$ are linear it follows immediately that (ii) of 1.a.3 holds with $K=1$.

Schauder bases have been constructed in many other important Banach spaces appearing in analysis. Of particular interest in this direction are the results of Z. Ciesielski and J. Domsta [18] and S. Schonefeld [132] who proved the existence of a basis in $C^k(I^n)$ (=the space of all real functions $f(t_1, t_2,\ldots, t_n)$, $t_i \in [0, 1]$ which are k times continuously differentiable, with the obvious norm) and the result of S. V. Botschkariev [13] who proved the existence of a basis in the disc algebra A (=the space consisting of all the functions $f(z)$ which are analytic on $|z|<1$ and continuous on $|z|\leqslant 1$, with the sup norm). In these papers an important role is played by the *Franklin system*. The Franklin system consists of the sequence $\{f_n(t)\}_{n=1}^{\infty}$ of functions on [0, 1] which are obtained from the Schauder system $\{\varphi_n\}_{n=1}^{\infty}$ by applying the Gram–Schmidt orthogonalization procedure (with respect to the Lesbegue measure on [0, 1]). The Franklin system is (by definition) an orthonomal sequence which turns out to be also a Schauder basis of $C(0, 1)$. For a detailed study of the Franklin system we refer to the above mentioned papers as well as to [17].

The fact that in the common spaces there exists a Schauder basis led Banach to pose the question whether every separable Banach space has a basis. This problem (known as the basis problem) remained open for a long time and was solved in the negative by P. Enflo [37]. We shall present later on in this book (in Section 2.d) a variant of Enflo's solution.

The question whether every infinite-dimensional Banach space contains a basic sequence has, however, a positive answer. This simple fact was known already to Banach.

Theorem 1.a.5. *Every infinite dimensional Banach space contains a basic sequence.*

The proof, due to S. Mazur, is based on the following lemma.

Lemma 1.a.6. *Let X be an infinite dimensional Banach space. Let $B\subset X$ be a finite-dimensional subspace and let $\varepsilon>0$. Then there is an $x\in X$ with $||x||=1$ so that $||y||\leqslant(1+\varepsilon)||y+\lambda x||$ for every $y\in B$ and every scalar λ.*

Proof of 1.a.6. We may clearly assume that $\varepsilon<1$. Let $\{y_i\}_{i=1}^m$ be elements of norm 1 in B such that for every $y\in B$ with $||y||=1$ there is an i for which $||y-y_i||<\varepsilon/2$. Let $\{y_i^*\}_{i=1}^m$ be elements of norm 1 in X^* so that $y_i^*(y_i)=1$ for all i, and let $x\in X$ with $||x||=1$ and $y_i^*(x)=0$ for all i. This x has the desired property. Indeed, let $y\in Y$ with $||y||=1$, let i be such that $||y-y_i||\leqslant\varepsilon/2$ and let λ be a scalar. Then

$$||y+\lambda x||\geqslant||y_i+\lambda x||-\varepsilon/2\geqslant y_i^*(y_i+\lambda x)-\varepsilon/2=1-\varepsilon/2\geqslant||y||/(1+\varepsilon). \quad \square$$

Proof of 1.a.5. Let ε be any positive number and let $\{\varepsilon_n\}_{n=1}^{\infty}$ be positive numbers such that $\prod_{n=1}^{\infty}(1+\varepsilon_n)\leqslant 1+\varepsilon$. Let x_1 be any element in X with norm 1. By 1.a.6 we can construct inductively a sequence of unit vectors $\{x_n\}_{n=2}^{\infty}$ so that for every $n\geqslant 1$

$$||y||\leqslant(1+\varepsilon_n)||y+\lambda x_{n+1}|| \quad \text{for all } y\in \operatorname{span}\{x_1,\ldots, x_n\} \text{ and every scalar } \lambda.$$