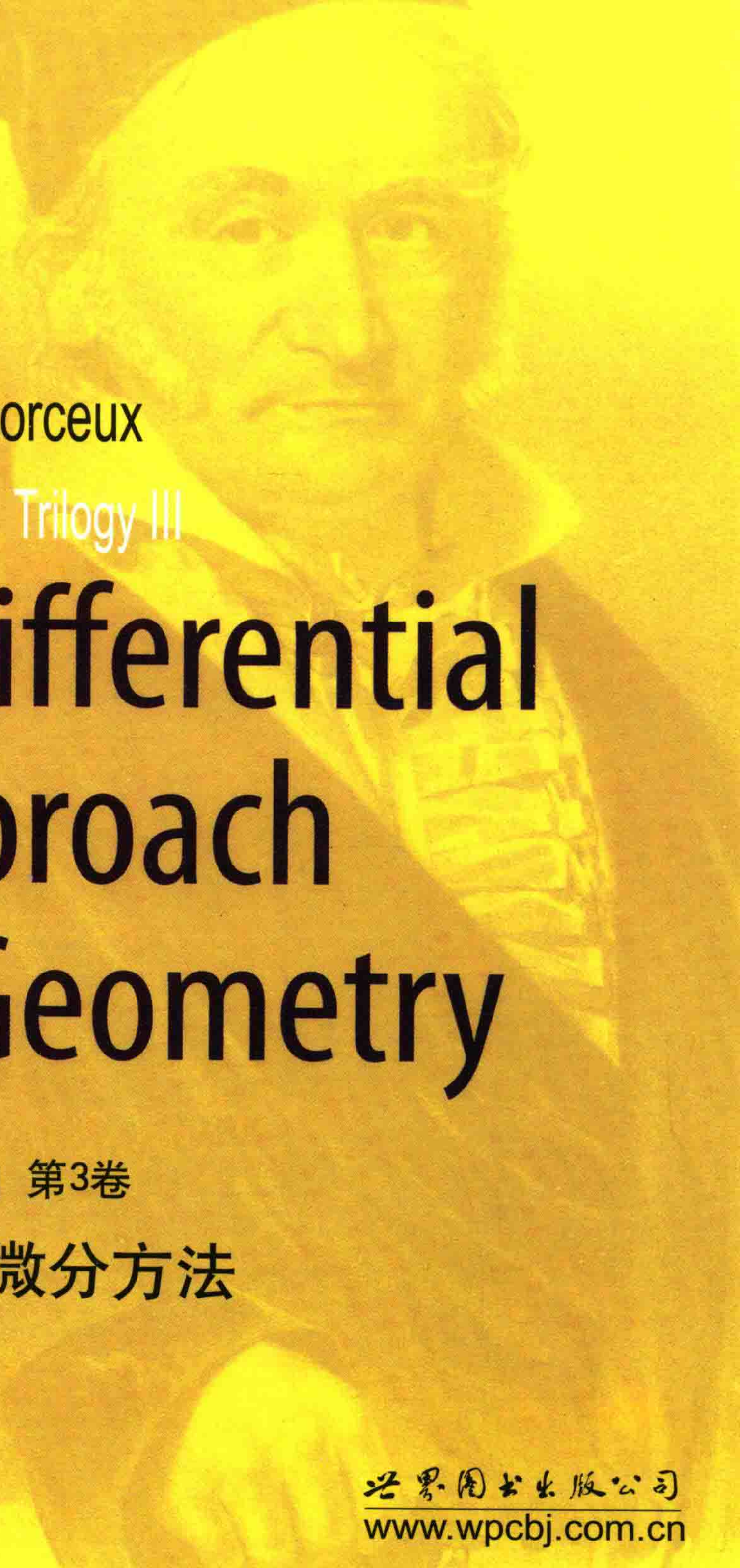

Francis Borceux

Geometric Trilogy III

# A Differential Approach to Geometry

几何三部曲 第3卷

# 几何的微分方法

Springer

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# A Differential Approach to Geometry

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by Francis Borceux

*To Océane, Anaïs, Magali, Lucas, Cyprien,*
*Théophile, Constance, Léonard, Georges, ...*
*and those still to come*

# Preface

The reader is invited to immerse himself in a "love story" which has been unfolding for 35 centuries: the love story between mathematicians and geometry. In addition to accompanying the reader up to the present state of the art, the purpose of this *Trilogy* is precisely to tell this story. The *Geometric Trilogy* will introduce the reader to the multiple complementary aspects of geometry, first paying tribute to the historical work on which it is based and then switching to a more contemporary treatment, making full use of modern logic, algebra and analysis. In this *Trilogy*, Geometry is definitely viewed as an autonomous discipline, never as a sub-product of algebra or analysis. The three volumes of the *Trilogy* have been written as three independent but complementary books, focusing respectively on the axiomatic, algebraic and differential approaches to geometry. They contain all the useful material for a wide range of possibly very different undergraduate geometry courses, depending on the choices made by the professor. They also provide the necessary geometrical background for researchers in other disciplines who need to master the geometric techniques.

In the 1630s Fermat and Descartes were already computing the tangents to some curves using arguments which today we would describe in terms of derivatives (see [4], *Trilogy II*). However, these arguments concerned algebraic curves, that is, curves whose equation is expressed by a polynomial, and the derivative of a polynomial is something that one can describe algebraically in terms of its coefficients and exponents, without having to handle limits. Some decades later, the development of differential calculus by Newton and Leibniz allowed these arguments to be formalized in terms of actual derivatives, for rather arbitrary curves. In the present book, we focus on this general setting of curves and surfaces described by functions which are no longer defined by polynomials, but are arbitrary functions having sufficiently well behaved properties with respect to differentiation.

We have deliberately chosen to restrict our attention to curves in the 2- and 3-dimensional real spaces and surfaces in the 3-dimensional real space. Although we occasionally give a hint on how to generalize several of our results to higher dimensions, our focus on lower dimensions provides the best possible intuition of the basic notions and techniques used today in advanced studies of differential geometry.

An important notion is the consideration of *parametric equations*, following an idea of *Euler* (see [4], *Trilogy II*). A closer look at such equations suggests that we should view a curve not as the set of points whose coordinates satisfy some equation(s), but as a continuous deformation of the real line in $\mathbb{R}^2$ or in $\mathbb{R}^3$, according to the case. When a parameter $t$ varies on the real line, the parametric equations describe successively all the points of the curve. Analogously, a surface can be seen as a continuous deformation of the real plane in the space $\mathbb{R}^3$. This is the notion of a *parametric representation*, which is the basic tool that we shall use in our study.

Our first chapter is essentially historical: its purpose is to explain where the ideas of differential geometry came from and why we choose this or that precise definition and not another possible one.

The formalized study of curves then begins with Chap. 2, where we restrict our attention to the simplest case: the plane curves. We pay special attention to basic notions like tangency, length and curvature, but we also prove very deep theorems, such as the *Hopf theorem* for simple closed curves. Working in the plane makes certainly things easier to grasp in a first approach. However, it is also a matter of fact that the study of plane curves offers many interesting aspects, such as envelopes, evolutes, involutes, which have beautiful applications. Many of these aspects do not generalize elegantly to higher dimensions.

Our Chap. 3 is a kind of parenthesis in our theory of differential geometry: we present a *museum* of some specimens of curves which have played an important historical role in the development of the theory.

Chapter 4 is then devoted to the study of curves in three dimensional space: the so-called *skew curves*. We focus our attention on the main aspects of the theory, namely, the study of the *curvature* and the *torsion* of skew curves and the famous *Frenet trihedron*.

Next we switch to surfaces in $\mathbb{R}^3$. In Chap. 5 we concentrate our attention on the *local properties* of surfaces, that is, properties "in a neighborhood of a given point of the surface", such as the tangent plane at that point or the various notions of curvature at that point: normal curvature, Gaussian curvature, and the information that we can get from these on the shape of the surface in a neighborhood of the point.

Chapter 6 then begins by repeating many of the arguments of Chap. 5, but using a different notation: the notation of Riemannian geometry. Our objective is to provide in this way a good intuitive approach to notions such as the *metric tensor*, the *Christoffel symbols*, the *Riemann tensor*, and so on. We provide evidence that these apparently very technical notions reduce, in the case of surfaces in $\mathbb{R}^3$, to very familiar notions studied in Chap. 5. We also devote special attention to the case of *geodesics* and establish the main properties (including the existence) of the systems of geodesic coordinates.

The last chapter of this book is devoted to some *global properties* of surfaces: properties for which one has to consider the full surface, not just what happens in a neighborhood of one of its points. We start with a basic study of the surfaces of revolution, the ruled and the developable surfaces and the surfaces with constant curvature. Next we switch to results and notions such as the *Gauss–Bonnet* theorem and the *Euler characteristic*, which represent some first bridges between the elementary theory of surfaces and more advanced topics.

Each chapter ends with a section of "problems" and another section of "exercises". Problems are generally statements not treated in this book, but of theoretical interest, while exercises are more intended to allow the reader to practice the techniques and notions studied in the book.

Of course reading this book assumes some familiarity with the basic notions of linear algebra and differential calculus, but these can be found in all undergraduate courses on these topics. An appendix on general topology introduces the few ingredients of that theory which are needed to properly follow our approach to Riemannian geometry and the global theory of surfaces. A second appendix states with full precision (but without proofs this time) some theorems on the existence of solutions of differential equations and partial differential equations, which are required in some advanced geometrical results.

A selective bibliography for the topics discussed in this book is provided. Certain items, not otherwise mentioned in the book, have been included for further reading.

The author thanks the numerous collaborators who helped him, through the years, to improve the quality of his geometry courses and thus of this book. Among them a special thanks to *Pascal Dupont*, who also gave useful hints for drawing some of the illustrations, realized with *Mathematica* and *Tikz*.

# The Geometric Trilogy

**III. A Differential Approach to Geometry**

# Contents

# Chapter 1
# The Genesis of Differential Methods

This first chapter is intentionally provocative, and useless! By *useless* (besides being at once provocative) we mean: this first chapter is not formally needed to follow the systematic treatment of the theory of curves and surfaces developed in the subsequent chapters.

So what is this chapter about? Usually, when you open a book on—let us say—the theory of curves in the real plane, you expect to find first "the" precise definition of a plane curve, followed by a careful study of the properties of such a notion. We all have an intuitive idea of what a plane curve is. Everybody knows that the straight line, the circle or the parabola *are* curves, but a single point or the empty set *are not* curves! Nevertheless, all these "figures" can be described by an equation $F(x, y) = 0$, with $F$ a polynomial: for example, $x^2 + y^2 = 0$ is an "equation of the origin" in $\mathbb{R}^2$ while $x^2 + y^2 = -1$ is "an equation" of the empty set. Thus a curve cannot simply be defined via an equation $F(x, y) = 0$, even when $F$ is a "very good" function! For example, consider the picture comprised of 7 hyperbolas, thus 14 branches. Is this *one* curve, or *seven* curves, or *fourteen* curves? After all, it is not so clear what a curve should be!

Starting at once with a precise definition of a curve would give the false impression that this is *the* definition of a curve. Instead it should be stressed that such a definition is *a possible* definition. Discussing the advantages and disadvantages of the various possible definitions, in order to make a sensible choice, is an important aspect of every mathematical approach.

There is also a second aspect that we want to stress. For *Euclid*, a straight line was *What has a length and no width and is well-balanced at each of its points* (see Definition 3.1.1 in [3], *Trilogy I*). The intuition behind such a sentence is clear, but such a "definition" assumes that before beginning to develop geometry, we know what a *length* is. Of course what we want to do concerning a *length* is then to find a formula to compute it, such as $2\pi R$ for a circle of radius $R$.

With more than two thousand years of further mathematical developments and experience, we now feel quite uneasy about such an approach. How can we establish a formula to compute the *length of a curve* if we did not define first what the *length of a curve* is?

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For many centuries—essentially up to the 17th century—mathematicians could hardly handle problems of length for curves other than the straight line and the circle. Differential calculus, with the full power of the theories of derivatives and integrals, opened the door to the study of arbitrary curves. However, in some sense, one was still taking the notion of length (or surface or volume) as something "which exists and that one wants to calculate".

Like many authors today, we adopt in the following chapters a completely different approach: the theory of integration is a well-established part of analysis and we use it to define a length. Analogously the theory of derivatives is a well-established part of analysis and we use it to define a tangent. And so on.

This first chapter is intended to be a "bridge" between the "historical" and the "contemporary" approaches. We present typical arguments developed in the past (and sometimes, still today) to master some geometrical notions (like length, or tangent), but we do that in particular to develop an intuition for the contemporary definitions of these notions. In this introductory chapter, we refer freely to [3] and [4], *Trilogy I* and *II*, when the historical arguments that we have in mind have been developed there.

Various arguments in this chapter can appear quite disconcerting. We often rely on our intuition, without trying to formalize the argument. We freely apply many results borrowed from a first calculus course, taking as a blanket assumption that when we apply a theorem, the necessary assumptions for its validity should be satisfied, even if we have not tried to determine the precise context in which this is the case! This is not a very rigorous attitude, however our point in this chapter is not to *prove* results, but to *guess* what possible "good" definitions should be.

## 1.1 The Static Approach to Curves

Originally, Greek geometry (see [3], *Trilogy I*) was essentially concerned with the study of two curves: the line and the circle.

> *The line is what has length and no width and is well-balanced around each of its points.*
>
> *The circle is the locus of those points of the plane which are at a fixed distance R from a fixed point O of the plane.*

Passing analogously to three dimensional space, using a circle in a plane and a point not belonging to the plane of the circle, you can then—using lines—construct the cone on this circle with vertex the given point. "Cutting" this cone by another plane then yields new curves that, according to the position of the "cutting plane", you call *ellipse*, *hyperbola* or *parabola*. This is the origin of the theory of curves.

It is common practice to describe a curve by giving its equation with respect to some basis. In this book, we are interested in the study of curves in the real plane $\mathbb{R}^2$. For example a circle of radius $R$ centered at the origin admits the equation (see Chap. 1 in [4], *Trilogy II*)

$$x^2 + y^2 = R^2$$