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With a Foreword by Robert Fefferman

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A. ZYGMUND

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CHAPTER X

TRIGONOMETRIC INTERPOLATION

1. General remarks

In this chapter trigonometric polynomials will be systematically referred to simply as *polynomials*. We shall refer to ordinary polynomials, when we have occasion to speak of them, as *power polynomials*.

A polynomial
$$T(x)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)=\sum_{k=-n}^{n}c_k e^{ikx} \tag{1·1}$$
of order n has $2n+1$ coefficients, so that one would, in principle, expect that $2n+1$ constraints would be sufficient to determine T. A purely cosine polynomial of order n has $n+1$ coefficients; a sine one, n coefficients.

Let us fix $2n+1$ points
$$x_0,\ x_1,\ \ldots,\ x_{2n}$$
on the x-axis, distinct modulo 2π. (In what follows we shall speak simply of *distinct* points.) If desirable, we can always assume that these points are situated in any fixed interval of length 2π.

(1·2) THEOREM. *Given $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$ and arbitrary numbers $y_0, y_1, \ldots, y_{2n}$, real or complex, there is always a unique polynomial* (1·1) *such that*
$$T(x_k)=y_k \quad (k=0,\ 1,\ \ldots,\ 2n). \tag{1·3}$$

If we treat (1·3) as a system of linear equations in the c_k, the determinant of the system is
$$e^{-in(x_0+x_1+\cdots+x_{2n})}\begin{vmatrix}1 & e^{ix_0} & \cdots & e^{2nix_0}\\ 1 & e^{ix_1} & \cdots & e^{2nix_1}\\ \cdots & \cdots & \cdots & \cdots\\ 1 & e^{ix_{2n}} & \cdots & e^{2nix_{2n}}\end{vmatrix}=e^{-in(x_0+x_1+\cdots+x_{2n})}\prod_{\mu>\nu}(e^{ix_\mu}-e^{ix_\nu}),$$
and this is different from 0.

The polynomial $T(x)$ just defined is called the (*trigonometric*) *interpolating polynomial* corresponding to the points (abscissae) x_k and the values (ordinates) y_k. The points $x_0, x_1, \ldots, x_{2n}$ are often called the *fundamental*, or *nodal*, points of interpolation.

Let $t_j(x)$ be the polynomial of order n which takes the value 1 when $x=x_j$ and the value 0 at the remaining points x_k. Then
$$T(x)=\sum_{j=0}^{2n}y_j t_j(x), \tag{1·4}$$
since the right-hand side is a polynomial of order n taking the value y_k for $x=x_k$, for all k. The polynomials $t_j(x)$, $j=0, 1, \ldots, 2n$, are called the *fundamental polynomials* corresponding to the fundamental points $x_0, x_1, \ldots, x_{2n}$.

Clearly
$$t_j(x)=\prod_{k\neq j}2\sin\tfrac{1}{2}(x-x_k)\Big/\prod_{k\neq j}2\sin\tfrac{1}{2}(x_j-x_k). \tag{1·5}$$

For the expression on the right is equal to 1 when $x=x_j$ and to 0 at the remaining x_k; and it is a polynomial of order n, since the numerator consists of $2n$ factors each of the form $\alpha e^{\frac{1}{2}ix}+\beta e^{-\frac{1}{2}ix}$.

It is also easy to see that if

$$\Delta(x)=\prod_{k=0}^{2n} 2\sin\tfrac{1}{2}(x-x_k),$$

then

$$t_j(x)=\Delta(x)/\{2\Delta'(x_j)\sin\tfrac{1}{2}(x-x_j)\}. \tag{1·6}$$

By the *number of roots* of a polynomial $T(x)$ we shall mean the sum of the multiplicities of its distinct real roots (distinct mod 2π, that is). We have now:

(1·7) THEOREM. *The number of roots of any $T(x)\not\equiv 0$ of order n does not exceed $2n$.*

From (1·1) we see that

$$e^{-inx}\,T(x)=P(z), \tag{1·8}$$

where $z=e^{ix}$ and $P(z)$ is a power polynomial of degree $2n$ in z. If $x=\xi$ is a root of order k of $T(x)$, that is, if

$$T(\xi)=T'(\xi)=\ldots=T^{(k-1)}(\xi)=0, \quad T^{(k)}(\xi)\neq 0,$$

successive differentiation of (1·8) with respect to x shows that $\zeta=e^{i\xi}$ is a root of order k for $P(z)$, and conversely. Hence if the number of roots of $T(x)$ exceeds $2n$, the number of roots of $P(z)$, multiplicity being taken into account, also exceeds $2n$. Thus $P(z)\equiv 0$, that is, $T(x)\equiv 0$, contrary to the hypothesis.

As a corollary, we obtain that *if two polynomials $S(x)$ and $T(x)$ of order n vanish at the same $2n$ points $\xi_1, \xi_2, \ldots, \xi_{2n}$ of the interval $0\leqslant x<2\pi$, then one of S and T is a multiple of the other.* (If k of the points ξ coincide, we mean that S and T have roots of multiplicity at least k there.) For suppose that $S\not\equiv 0$ (otherwise, $S=0.T$), and let $C=T(\xi)/S(\xi)$, where ξ is distinct from $\xi_1, \xi_2, \ldots, \xi_{2n}$ and is such that $S(\xi)\neq 0$. The polynomial $T(x)-CS(x)$ of order n vanishes not only at the points $\xi_1, \xi_2, \ldots, \xi_{2n}$ but also at ξ. Hence $T-CS\equiv 0$, $T\equiv CS$.

In particular, if T vanishes at the roots of $\cos(nx+\alpha)$, then $T\equiv C\cos(nx+\alpha)$.

(1·9) THEOREM. *If a cosine polynomial $C(x)$ of order n vanishes at $n+1$ points $\xi_0<\xi_1<\ldots<\xi_n$ in $0\leqslant x\leqslant\pi$, then $C(x)\equiv 0$.*

If $\xi_0>0$ or $\xi_n<\pi$, $C(x)$ vanishes at $2n+1$ points and so is identically zero. For $C(x)$ is even, and if, for example, $\xi_0>0$, $C(x)$ vanishes at $\pm\xi_0, \pm\xi_1, \ldots, \pm\xi_{n-1}, \xi_n$ (ξ_n and $-\xi_n$ are not distinct if $\xi_n=\pi$). If simultaneously $\xi_0=0$ and $\xi_n=\pi$, then $C(x)$, being even, must have at least double roots at $x=0, \pi$, so that the number of roots of $C(x)$ is at least $4+2(n-1)=2n+2$, and again $C(x)\equiv 0$.

(1·10) THEOREM. *If a sine polynomial $S(x)$ of order n vanishes at n points $\xi_1<\xi_2<\ldots<\xi_n$ interior to $(0,\pi)$, then $S(x)\equiv 0$.*

It is enough to observe that $S(x)$ vanishes at $2n+2$ distinct points $0, \pm\xi_1, \ldots, \pm\xi_n, \pi$.

It is sometimes important to interpolate by means of purely cosine or purely sine polynomials.

(1·11) THEOREM. *Given any $n+1$ distinct points $\xi_0, \xi_1, \ldots, \xi_n$ in $0\leqslant x\leqslant\pi$, and any numbers $\eta_0, \eta_1, \ldots, \eta_n$, there is a unique cosine polynomial $C(x)$ of order n such that $C(\xi_k)=\eta_k$ for all k.*

Observe that

$$\tau_j(x)=\prod_{k\neq j}(\cos x-\cos\xi_k)/\prod_{k\neq j}(\cos\xi_j-\cos\xi_k)$$

is a cosine polynomial of order n which is equal to 1 when $x=\xi_j$ and vanishes at the remaining ξ_k, so that

$$C(x)=\sum_{j=0}^{n}\eta_j\tau_j(x)$$

is a cosine polynomial having the required properties. Its uniqueness is a consequence of (1·9).

If the points ξ_j are all in the interior of $(0,\pi)$, the roots of $\cos x-\cos\xi_j$ are all simple, and so

$$\tau_j(x)=\frac{\delta(x)\sin\xi_j}{\delta'(\xi_j)(\cos x-\cos\xi_j)},$$

where $$\delta(x)=\prod_k(\cos x-\cos\xi_k).$$

The case in which $C(x)$ is of order $n-1$ and

$$\xi_0^{(n-1)}=\frac{\pi}{2n},\quad \xi_1^{(n-1)}=\frac{3\pi}{2n},\quad \ldots,\quad \xi_{n-1}^{(n-1)}=\frac{(2n-1)\pi}{2n}$$

is particularly interesting. Here $\delta(x)$ has the same roots as $\cos nx$, so that

$$\delta(x)=C\cos nx,$$

and it is easy to verify that now

$$C(x)=\frac{\cos nx}{n}\sum_{j=0}^{n-1}\frac{(-1)^j\sin\xi_j}{\cos\xi_j-\cos x}\eta_j\quad\left(\xi_j=(2j+1)\frac{\pi}{2n}\right). \tag{1·12}$$

(1·13) Theorem. *Given any distinct points $\xi_1,\xi_2,\ldots,\xi_n$ interior to $(0,\pi)$ and any n numbers $\eta_1,\eta_2,\ldots,\eta_n$, there is a unique sine polynomial $S(x)$ of order n such that $S(\xi_k)=\eta_k$ for all k.*

It is enough to set

$$S(x)=\sum_{j=1}^{n}\eta_j\sigma_j(x),$$

where $$\sigma_j(x)=\frac{\sin x\prod_{k\neq j}(\cos x-\cos\xi_k)}{\sin\xi_j\prod_{k\neq j}(\cos\xi_j-\cos\xi_k)}.$$

Clearly σ_j is a sine polynomial of order n which is equal to 1 when $x=\xi_j$ and vanishes at the remaining ξ_k.

Return to the general formula (1·4). Given any function $f(x)$ of period 2π, the interpolating polynomial which coincides with $f(x)$ at the points x_k (and so also at the points congruent to x_k mod 2π) is equal to

$$\sum_{j=0}^{2n}f(x_j)t_j(x). \tag{1·14}$$

Suppose now that for each n we have a system

$$x_0^{(n)},\quad x_1^{(n)},\quad \ldots,\quad x_{2n}^{(n)} \tag{1·15}$$

of $2n+1$ fundamental points. It is natural to ask for conditions under which the sum (1·14) will tend to $f(x)$ as $n\to\infty$. This problem of the representation of functions by interpolating polynomials has something in common with the problem of the representation of functions by their Fourier series. It is natural to expect that the geometric structure of the fundamental sets (1·15) is of great importance here. Little is known about the behaviour of the interpolating polynomials for the general system (1·15), and in what follows we shall be concerned almost exclusively with the case of *equidistant* nodal points. By this we mean that

$$x_j^{(n)} = x_0^{(n)} + \frac{2\pi j}{2n+1} \quad (j=0, 1, \ldots, 2n). \tag{1·16}$$

Thus the points $\exp(ix_j^{(n)})$, $j=0, 1, \ldots, 2n$, are equally spaced over the circumference of the unit circle. This case has been particularly well investigated, and is the most important in applications. Moreover, the analogy with Fourier series is here particularly striking.

If no confusion arises, we shall write x_j for $x_j^{(n)}$.

The polynomial coinciding with the periodic function $f(x)$ at the points (1·16) will be denoted by $I_n(x, f)$ or by $I_n[f]$, or simply by $I_n(x)$, and will be called the *n-th interpolating polynomial* of f.

Consider the Dirichlet kernel

$$D_n(u) = \tfrac{1}{2} + \sum_{k=1}^{n} \cos ku = \frac{\sin(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}.$$

It is a polynomial of order n vanishing at the points $2\pi j/(2n+1)$, $j=1, 2, \ldots, 2n$, and equal to $n+\frac{1}{2}$ for $u=0$. Thus the polynomial $D_n(x-x_j)/(n+\frac{1}{2})$, which is equal to 1 when $x=x_j$ and to 0 at the remaining points x_k, is a fundamental polynomial for the system (1·16) and, by (1·14),

$$I_n(x, f) = \frac{2}{2n+1} \sum_{j=0}^{2n} f(x_j)\, D_n(x-x_j). \tag{1·17}$$

This expression can be written as a Stieltjes integral. Let ξ_0 be any real number, and for every positive integral p let $\omega_p(x)$, $-\infty < x < +\infty$, be any step function which has jumps $2\pi/p$ at the points

$$\xi_\nu = \xi_0 + 2\nu\pi/p \quad (\nu = 0, \pm 1, \pm 2, \ldots), \tag{1·18}$$

is constant in the interior of each interval $(\xi_\nu, \xi_{\nu+1})$ and has regular discontinuities at the ξ_ν. The function $\omega_p(x)$ is determined uniquely, except for an irrelevant additive constant, by the suffix p and by the position of any point ξ_ν; so no misunderstanding will occur if we denote the function simply by $\omega_p(x)$. If the set (1·18) contains a point ξ, or a point set E, we shall say that the function $\omega_p(x)$ is *associated* with ξ, or with E.

The formula (1·17) can now be written

$$I_n(x, f) = \frac{1}{\pi} \int_0^{2\pi} f(t)\, D_n(x-t)\, d\omega_{2n+1}(t), \tag{1·19}$$

where ω_{2n+1} is associated with the points (1·16). If $S(x)$ is a polynomial of order n, then $I_n(x, S) = S(x)$, since both sides are equal at the points (1·16). Thus

$$S(x) = \frac{1}{\pi} \int_0^{2\pi} S(t)\, D_n(x-t)\, d\omega_{2n+1}(t). \tag{1·20}$$

If $g(x)$ is periodic, then $\int_\alpha^{\alpha+2\pi} g\, d\omega_p$ is independent of α. In particular, the integral in (1·19) may be taken over any interval of length 2π.

If $f(x)$ is continuous, the integral in (1·19) certainly exists as a Riemann-Stieltjes integral. If f is discontinuous at some of the points (1·16), the integral does not exist in the Riemann-Stieltjes sense. We might here use the more general Lebesgue-Stieltjes definition, but it is much simpler to treat the integral in (1·19) merely as a different notation for the sum in (1·17), and we shall always do so. The advantage of the integral notation is that it brings to light the formal similarity between the nth interpolating polynomial of f and the nth partial sum

$$S_n(x;f)=\frac{1}{\pi}\int_0^{2\pi}f(t)\,D_n(x-t)\,dt \tag{1·21}$$

of $S[f]$. If we add a suitable constant to $\omega_{2n+1}(t)$, it will tend uniformly to t as n tends to infinity, and this might suggest that the behaviour of $I_n(x,f)$ as $n\to\infty$ should be similar to that of $S_n(x;f)$. We shall see later that within certain limits this is actually the case, though the parallelism does not go so far as might be expected from the formal resemblance of the integrals in (1·19) and (1·21).

In this chapter, unless otherwise stated, we shall consider only functions integrable in the classical Riemann sense and of period 2π. In particular, our functions will be bounded. The most interesting special case, and that in which the most important problems arise, is that of continuous functions. Usually the extension of results from continuous to R-integrable functions (if possible at all) does not require essentially new ideas; but R-integrability is as natural for the theory of interpolation as L-integrability is for the theory of Fourier series. That L-integrability is not of much use for interpolation is clear from the fact that the $I_n(x,f)$ are defined by the values of f at a denumerable set of points. By modifying f there, we can change the behaviour of the $I_n[f]$, while $S[f]$ remains unchanged.

The polynomial $\tilde{I}_n(x,f)$ conjugate to $I_n(x,f)$ is obtained from (1·17) by replacing each $D_n(x-x_j)$ by the conjugate Dirichlet kernel $\tilde{D}_n(x-x_j)$, where

$$\tilde{D}_n(x)=\sum_{k=1}^{n}\sin kx=\frac{\cos\frac{1}{2}x-\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}.$$

Thus

$$\tilde{I}_n(x,f)=\frac{1}{\pi}\int_0^{2\pi}f(t)\,\tilde{D}_n(x-t)\,d\omega_{2n+1}(t). \tag{1·22}$$

In particular, for any polynomial $S(x)$ of order n,

$$\tilde{S}(x)=\frac{1}{\pi}\int_0^{2\pi}S(t)\,\tilde{D}_n(x-t)\,d\omega_{2n+1}(t). \tag{1·23}$$

Trigonometric interpolation is analogous to interpolation by means of power polynomials. Given any $n+1$ distinct points $\zeta_0, \zeta_1, \ldots, \zeta_n$ of the complex plane, and any numbers $\eta_0, \eta_1, \ldots, \eta_n$, there is always a uniquely determined (interpolating) polynomial $P(\zeta)=c_0+c_1\zeta+\ldots+c_n\zeta^n$ of degree n satisfying

$$P(\zeta_k)=\eta_k\quad(k=0,1,\ldots,n). \tag{1·24}$$

The uniqueness follows from the fact that the difference of two such polynomials would be a polynomial of degree n having at least $n+1$ zeros, and so would vanish identically. If we set

$$\left.\begin{aligned}w(\zeta)&=(\zeta-\zeta_0)(\zeta-\zeta_1)\ldots(\zeta-\zeta_n),\\ l_j(\zeta)&=w(\zeta)/w'(\zeta_j)(\zeta-\zeta_j),\end{aligned}\right\} \tag{1·25}$$

then $l_j(\zeta)$ is a polynomial of degree n equal to 1 at ζ_j and vanishing at the remaining ζ_k. Thus, if $F(\zeta)$ is any function such that $F(\zeta_j)=\eta_j$ for all j, we get the classical Lagrange interpolating formula

$$P(\zeta)=\sum_{j=0}^{n}\eta_j l_j(\zeta)=\sum_{j=0}^{n}\eta_j\frac{w(\zeta)}{w'(\zeta_j)(\zeta-\zeta_j)}$$
$$=\sum_{j=0}^{n}F(\zeta_j)\frac{w(\zeta)}{w'(\zeta_j)(\zeta-\zeta_j)}.$$

Assume now that all the points ζ_j are real and situated in the interval $-1\leqslant\zeta\leqslant 1$, and consider the (standard) mapping

$$\zeta=\cos x \tag{1·26}$$

of the interval $-1\leqslant\zeta\leqslant 1$ on to the interval $0\leqslant x\leqslant\pi$. It transforms any function $F(\zeta)$, defined in $-1\leqslant\zeta\leqslant 1$, into $F(\cos x)=f(x)$, say, and the points $\zeta_0,\zeta_1,\ldots,\zeta_n$ into points $x_0,x_1,\ldots,x_n$. The power polynomial $P_n(\zeta)$ coinciding with $F(\zeta)$ at the points ζ_k becomes $P_n(\cos x)$, a purely cosine polynomial coinciding with $f(x)$ at the points $x_0,x_1,\ldots,x_n$.

Conversely, we suppose that $f(x)$ is any function defined for $0\leqslant x\leqslant\pi$, that $0\leqslant x_0<x_1<\ldots<x_n\leqslant\pi$, and that $C_n(x)$ is the cosine polynomial of order n coinciding with f at the points $x_0,x_1,\ldots,x_n$. We observe that $\cos kx$ is a power polynomial of degree k in $\cos x$. (This is obvious for $k=0,1$, and for general k it follows by induction from the formula $\cos kx+\cos(k-2)x=2\cos x\cos(k-1)x$.) Thus the transformation (1·26), which carries the function $f(x)$ into an $F(\zeta)$ defined in $-1\leqslant\zeta\leqslant 1$, also carries $C_n(x)$ into a power polynomial $P_n(\zeta)$ coinciding with F at the points $\zeta_j=\cos x_j$.

The problem of interpolating by means of power polynomials $P_n(\zeta)$ on the interval $-1\leqslant\zeta\leqslant 1$ is thus equivalent to that of interpolating by means of cosine polynomials on $0\leqslant x\leqslant\pi$. The case of the so-called *Tchebyshev abscissae*

$$\zeta_0^{(n-1)}=\cos\frac{\pi}{2n},\quad \zeta_1^{(n-1)}=\cos\frac{3\pi}{2n},\quad\ldots,\quad \zeta_{n-1}^{(n-1)}=\cos\frac{(2n-1)\pi}{2n}$$

is equivalent to cosine interpolation with equidistant fundamental points $\pi/2n$, $3\pi/2n$, ..., $(2n-1)\pi/2n$.

2. Interpolating polynomials as Fourier series

Write

$$I_n(x,f)=\tfrac{1}{2}a_0^{(n)}+\sum_{\nu=1}^{n}(a_\nu^{(n)}\cos\nu x+b_\nu^{(n)}\sin\nu x)$$
$$=\sum_{\nu=-n}^{+n}c_\nu^{(n)}e^{i\nu x}.$$

If we replace $D_n(u)$ in (1·19) by $\frac{1}{2}+\cos u+\ldots+\cos nu$ and compare the terms on both sides, we get

$$a_\nu^{(n)}=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos\nu t\,d\omega_{2n+1}(t),\quad b_\nu^{(n)}=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin\nu t\,d\omega_{2n+1}(t) \tag{2·1}$$

for $\nu=0,1,2,\ldots,n$. Similarly we have

$$c_\nu^{(n)}=\frac{1}{2\pi}\int_0^{2\pi}f(t)e^{-i\nu t}d\omega_{2n+1}(t) \tag{2·2}$$

for $|\nu|\leqslant n$. The numbers $a_\nu^{(n)}$, $b_\nu^{(n)}$ will be called the *Fourier-Lagrange coefficients* of f (corresponding to the fundamental points (1·16)). The $c_\nu^{(n)}$ are the *complex* Fourier-

Lagrange coefficients of f. Where no ambiguity arises we shall write a_ν, b_ν, c_ν for $a_\nu^{(n)}$, $b_\nu^{(n)}$, $c_\nu^{(n)}$. For a fixed ν, the integral defining $a_\nu^{(n)}$ is an approximate Riemann sum for the integral

$$\frac{1}{\pi}\int_0^{2\pi} f(t)\cos\nu t\,dt,$$

and similarly for $b_\nu^{(n)}$, $c_\nu^{(n)}$. Thus *as $n\to\infty$ the ν-th Fourier-Lagrange coefficient of f tends to the ν-th Fourier coefficient of f.*

We recall a definition from Chapter I, § 3. Let $\phi_1(x), \phi_2(x), \ldots$ be defined in an interval (a,b), and let $w(x)$ be a non-decreasing function in (a,b). We say that the system of functions ϕ_ν is orthogonal over (a,b) with respect to the weight dw if

$$\int_a^b \phi_j(x)\,\overline{\phi}_k(x)\,dw(x) = \begin{cases} 0 & \text{for } j\neq k, \\ \lambda_k>0 & \text{for } j=k. \end{cases}$$

Given any function $f(x)$ defined in (a,b), we call the numbers

$$c_j = \frac{1}{\lambda_j}\int_a^b f(x)\,\overline{\phi}_j(x)\,dw(x)$$

the *Fourier coefficients of f*, and the series

$$c_1\phi_1 + c_2\phi_2 + \ldots$$

the *Fourier series of f*, all with respect to the system $\{\phi_\nu\}$ and the weight dw. The system is called *complete* if the vanishing of all the c_j implies that f vanishes almost everywhere with respect to dw; that is, that the variation of $w(x)$ over the set of points at which f does not vanish is 0.

Return to (2·1). Taking for $f(x)$ one of the functions

$$\tfrac{1}{2},\quad \cos x,\quad \sin x,\quad \ldots,\quad \cos nx,\quad \sin nx, \tag{2·3}$$

we immediately deduce that *this system is orthogonal over* $(0,2\pi)$ (or any interval of length 2π) *with respect to the weight $d\omega_{2n+1}$*. The numbers λ here are equal to $\frac{1}{2}\pi, \pi, \pi, \ldots, \pi$. The formulae (2·1) imply that *$I_n(x,f)$ is the Fourier series of f with respect to the system* (2·3) *and the weight $d\omega_{2n+1}$*.

Similarly, we show that the system $e^{i\nu x}$, $\nu=0, \pm1, \ldots, \pm n$, is orthogonal over $(0,2\pi)$ with respect to $d\omega_{2n+1}$, and $I_n(x,f)$ is the Fourier series of f with respect to this system.

If for a given f the numbers $a_0, a_1, b_1, \ldots, a_n, b_n$ are all 0, then $I_n(x,f)\equiv 0$. This means that $f=0$ at the discontinuities of ω_{2n+1} (since $I_n(x,f)=f$ there), that is, that the total variation of ω_{2n+1} over the set where f does not vanish is 0. We may therefore say that *the system* (2·3) *is complete with respect to $d\omega_{2n+1}$*.

The orthogonality of the system (2·3) with respect to $d\omega_{2n+1}$ can also be proved directly; we have only to observe that

$$\int_0^{2\pi}\cos kx\,d\omega_N(x) + i\int_0^{2\pi}\sin kx\,d\omega_N(x) = \int_0^{2\pi} e^{ikx}\,d\omega_N(x), \tag{2·4}$$

and that the last integral is 0 if k is any integer not divisible by N (see Chapter II, (1·3)). Under this hypothesis both integrals on the left must vanish, and from this we easily infer that *the system* (2·3) *is orthogonal over any interval of length 2π with respect to $d\omega_m$, where m is any integer* (odd or even) *greater than $2n$.*

It follows from (2·4) that if $T(x)$ is any polynomial of order less than N, then

$$(2\pi)^{-1}\int_0^{2\pi} T\,d\omega_N$$

is equal to the constant term of $T(x)$. Thus

$$\int_0^{2\pi} T(x)\,d\omega_N(x) = \int_0^{2\pi} T(x)\,dx \tag{2·5}$$

for any polynomial T of order strictly less than N. In particular,

(2·6) THEOREM. *If $S(x)=\Sigma\gamma_\nu e^{i\nu x}$ and $T(x)=\Sigma\gamma'_\nu e^{i\nu x}$ are polynomials of order $n<\frac{1}{2}N$, we have the Parseval formulae*

$$\frac{1}{2\pi}\int_0^{2\pi} S(x)\,T(x)\,d\omega_N = \sum_{\nu=-n}^{n}\gamma_\nu\gamma'_{-\nu}, \tag{2·7}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |S(x)|^2\,d\omega_N = \sum_{\nu=-n}^{+n}|\gamma_\nu|^2. \tag{2·8}$$

The case $N=2n+1$ is particularly important.

If a system of functions $\{\phi_n(x)\}$ is orthogonal in (a,b) with respect to the weight $dw(x)$, and if S_k is a linear combination of the functions $\phi_1, \phi_2, \ldots, \phi_k$ with arbitrary constant coefficients, then the quadratic approximation

$$\int_a^b |f(x)-S_k(x)|^2\,dw(x)$$

of f by S_k is a minimum for fixed k if S_k is the kth partial sum of the Fourier series of f with respect to the system $\{\phi_n\}$ and weight $dw(x)$ (Chapter I, § 7). This and the fact that $I_n(x,f)$ is a Fourier series give significance to the kth partial sum of $I_n(x,f)$,

$$I_{n,k}(x,f) = \tfrac{1}{2}a_0^{(n)} + \sum_{\nu=1}^{k}(a_\nu^{(n)}\cos\nu x + b_\nu^{(n)}\cos\nu x)$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)\,D_k(t-x)\,d\omega_{2n+1}(t) \quad (k=0,1,\ldots,n), \tag{2·9}$$

and we have the following theorem:

(2·10) THEOREM. *The polynomial $I_{n,k}[f]$ minimizes the integral*

$$\int_0^{2\pi} |f(x)-S(x)|^2\,d\omega_{2n+1}(x)$$

among polynomials S of order k.

Hence $I_{n,k}[f]$ is the unique solution of the following problem: *among all polynomials $S(x)$ of order $k\leqslant n$ find the one which would approximate best—in the sense of least squares —to the function f at the points $x_0, x_1, \ldots, x_{2n}$.* For $k<n$ we cannot in general expect that the minimizing S would coincide with f at those points.

3. The case of an even number of fundamental points

In principle, any $2n+1$ conditions will suffice to determine a polynomial $T(x)$ of order n. In the previous section we assigned the value of T at $2n+1$ pre-assigned points. Now we shall choose a set of $2n$ equidistant points depending on T, more precisely on the phase of the highest term of T; and we show that T is uniquely determined by the conditions of having given values at these $2n$ points.

For write T in the form

$$T(x) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos \nu x + b_\nu \sin \nu x) + \rho \cos (nx + \alpha) \tag{3·1}$$

(ρ not necessarily positive), and consider the function ω_{2n} associated with the roots of $\sin (nx + \alpha)$. Since the product of any two of the functions

$$\tfrac{1}{2}, \quad \cos x, \quad \sin x, \quad \ldots, \quad \cos (n-1) x, \quad \sin (n-1) x, \quad \cos (nx + \alpha) \tag{3·2}$$

is a polynomial of order less than $2n$, it follows from (2·5) and the ordinary orthogonality of the system of functions (3·2) that this system is orthogonal with respect to $d\omega_{2n}$ over any interval of length 2π. (This holds for *any* ω_{2n}.) Thus the coefficients a_ν, b_ν, ρ of T can be determined in the usual Fourier fashion. However, while

$$\int_0^{2\pi} (\tfrac{1}{2})^2 \, d\omega_{2n} = \tfrac{1}{2}\pi,$$

and
$$\int_0^{2\pi} \cos^2 \nu t \, d\omega_{2n} = \int_0^{2\pi} \sin^2 \nu t \, d\omega_{2n} = \pi$$

for $\nu = 1, 2, \ldots, n-1$ (by (2·5)), we have

$$\int_0^{2\pi} \cos^2 (nt + \alpha) \, d\omega_{2n}(t) = \int_0^{2\pi} 1 \,.\, d\omega_{2n}(t) = 2\pi,$$

by virtue of the hypothesis on ω_{2n}. Thus

$$T(x) = \frac{1}{\pi} \int_0^{2\pi} T(t) \left\{ \tfrac{1}{2} + \sum_{\nu=1}^{n-1} \cos \nu (t - x) + \tfrac{1}{2} \cos (nt + \alpha) \cos (nx + \alpha) \right\} d\omega_{2n}(t).$$

To the last term in curly brackets we may add $\tfrac{1}{2} \sin (nt + \alpha) \sin (nx + \alpha)$, which is 0 at the discontinuities of ω_{2n}. The expression in brackets then becomes $D_n^*(t - x)$, where

$$D_n^*(u) = \tfrac{1}{2} + \sum_{\nu=1}^{n-1} \cos \nu u + \tfrac{1}{2} \cos nu = \frac{\sin nu}{2 \tan \tfrac{1}{2} u}$$

is the modified Dirichlet kernel (Chapter II, § 5), and we obtain the following result:

(3·3) Theorem. *For any polynomial* (3·1) *we have*

$$T(x) = \frac{1}{\pi} \int_0^{2\pi} T(t) \, D_n^*(t - x) \, d\omega_{2n}(t), \tag{3·4}$$

provided ω_{2n} *is associated with the roots of* $\sin (nt + \alpha)$.

The right-hand side here depends solely on the values of T at these roots.

Let now α be any real number. *Any* polynomial $S(x)$ of order n can be written

$$S(x) = T(x) + \sigma \sin (nx + \alpha),$$

where T is of the form (3·1). Thus in (3·4) we may replace $T(x)$ by $S(x) - \sigma \sin (nx + \alpha)$ and $T(t)$ by $S(t)$, since $\sin (nt + \alpha)$ vanishes at the discontinuities of $\omega_{2n}(t)$. This gives

$$S(x) = \sigma \sin (nx + \alpha) + \frac{1}{\pi} \int_0^{2\pi} S(t) \, D_n^*(t - x) \, d\omega_{2n}(t). \tag{3·5}$$

In particular, let $\phi_{2n}(t)$ and $\psi_{2n}(t)$ be the functions ω_{2n} associated respectively with the zeros of $\cos nt$ and $\sin nt$. Let $u_1, u_2, \ldots, u_{2n}$ and $v_1, v_2, \ldots, v_{2n}$ be the discontinuities of ϕ_{2n} and ψ_{2n}. We may suppose that

$$u_k = (2k-1)\pi/2n, \quad v_k = k\pi/n \quad (k = 1, 2, \ldots, 2n).$$

Then,

(3·6) THEOREM. *For every polynomial*

$$S(x) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

we have

$$S(x) = a_n \cos nx + \frac{1}{\pi}\int_0^{2\pi} S(t)\, D_n^*(t-x)\, d\phi_{2n}(t), \tag{3·7}$$

$$S(x) = b_n \sin nx + \frac{1}{\pi}\int_0^{2\pi} S(t)\, D_n^*(t-x)\, d\psi_{2n}(t). \tag{3·8}$$

These formulae are particularly useful for obtaining expressions for the polynomials $\tilde{S}$, S', $\tilde{S}'$, which, unlike S, contain $2n$ coefficients only. For example, differentiating (3·7), where the integral is actually a finite sum, we get

$$S'(x) = -na_n \sin nx + \frac{1}{\pi}\int_0^{2\pi} S(t) \left\{\frac{n\cos n(x-t)}{2\tan\frac{1}{2}(x-t)} - \frac{\sin n(x-t)}{4\sin^2\frac{1}{2}(x-t)}\right\} d\phi_{2n}(t). \tag{3·9}$$

In this put $x = 0$ and recall that ϕ_{2n} is associated with the zeros of $\cos nt$; then we obtain

$$S'(0) = \frac{1}{n}\sum_{k=1}^{2n} S(u_k)\frac{(-1)^{k+1}}{(2\sin\frac{1}{2}u_k)^2}, \tag{3·10}$$

and applying the result to the polynomial $S(\theta + x)$ we have

$$S'(\theta) = \frac{1}{n}\sum_{k=1}^{2n} S(\theta + u_k)\frac{(-1)^{k+1}}{(2\sin\frac{1}{2}u_k)^2} \quad \left(u_k = \frac{(k-\frac{1}{2})\pi}{n}\right). \tag{3·11}$$

This formula for the derivative of a trigonometric polynomial has interesting applications. If we write

$$\alpha_k = n^{-1}(2\sin\tfrac{1}{2}u_k)^{-2},$$

it gives

$$|S'(\theta)| \leqslant \sum_{k=1}^{2n} \alpha_k |S(\theta + u_k)|. \tag{3·12}$$

Now $\alpha_1 + \alpha_2 + \ldots + \alpha_{2n} = n$, as we may easily verify by taking $S(x) = \sin nx$ in (3·10). Hence, if $|S(x)| \leqslant M$ for all x, we have

$$|S'(\theta)| \leqslant M(\alpha_1 + \alpha_2 + \ldots + \alpha_{2n}) = Mn.$$

More precisely, we have $|S'(\theta)| < Mn$, unless $S(\theta + u_k)$ is alternately $\pm M$ for $k = 1, 2, \ldots, 2n$, that is, unless $S(\theta + x)$ coincides either with $M \sin nx$ or $-M\sin nx$ at the points u_k. To fix our ideas, consider the first case and let

$$\Delta(x) = S(\theta + x) - M\sin nx.$$

Then $\Delta(x)$ has roots $u_1, u_2, \ldots, u_{2n}$; but since $|S| \leqslant M$, both $S(\theta + x)$ and $M\sin nx$ attain their maxima and minima simultaneously at the points u_k and the roots must be at least double. It follows that $\Delta(x)$ has at least $4n > 2n+1$ roots. Hence

$$S(\theta + x) \equiv M \sin nx, \quad S(x) \equiv M\sin n(x - \theta),$$

and we get the following theorem:

(3·13) Theorem of S. Bernstein. *If a polynomial $S(x)$ of order n satisfies $|S(x)| \leqslant M$ for all x, then $|S'(x)| \leqslant nM$, with equality if and only if S is of the form $M\cos(nx+\alpha)$.*

Let now $\chi(u)$ be non-decreasing, non-negative and convex in $u \geqslant 0$. Dividing (3·12) by n and applying Jensen's inequality (Chapter I, (10·1)) we get

$$\chi(n^{-1}|S'(\theta)|) \leqslant \chi(n^{-1}\Sigma\alpha_k|S(\theta+u_k)|) \leqslant n^{-1}\Sigma\alpha_k\chi(|S(\theta+u_k)|), \tag{3·14}$$

and integrating over $0 \leqslant \theta \leqslant 2\pi$ we have

$$\int_0^{2\pi}\chi(n^{-1}|S'(\theta)|)\,d\theta \leqslant n^{-1}\Sigma\alpha_k\int_0^{2\pi}\chi(|S(\theta+u_k)|)\,d\theta$$
$$= n^{-1}\Sigma\alpha_k\int_0^{2\pi}\chi(|S(\theta)|)\,d\theta,$$

that is
$$\int_0^{2\pi}\chi(n^{-1}|S'(\theta)|)\,d\theta \leqslant \int_0^{2\pi}\chi(|S(\theta)|)\,d\theta. \tag{3·15}$$

Suppose now that χ is strictly increasing and that we have equality in (3·15). The two members here are the integrals of the extreme terms in (3·14), so that these terms must therefore be equal for all θ. Since χ is strictly increasing, this is only possible if we have equality for all θ in (3·12). The latter condition implies that for every θ the numbers $S(\theta+u_k)$ are of alternating sign. This in turn implies that the distance between two consecutive zeros of S never exceeds π/n. If $S \not\equiv 0$, none of these distances can be less than π/n, for otherwise S would have more than $2n$ zeros. Thus either $S \equiv 0$ or S has $2n$ equidistant zeros. In either case (see p. 2) $S = M\cos(nx+\xi)$. Hence

(3·16) Theorem. *For every function $\chi(u)$ non-negative, non-decreasing and convex in $u \geqslant 0$, we have*

$$\int_0^{2\pi}\chi(n^{-1}|S'(\theta)|)\,d\theta \leqslant \int_0^{2\pi}\chi(|S(\theta)|)\,d\theta. \tag{3·17}$$

If χ is strictly increasing, equality occurs if and only if $S = M\cos(nx+\xi)$. In particular,

$$\left(\int_0^{2\pi}|S'|^p\,d\theta\right)^{1/p} \leqslant n\left(\int_0^{2\pi}|S|^p\,d\theta\right)^{1/p} \quad \textit{for} \quad p \geqslant 1. \tag{3·18}$$

When $p \to \infty$, the last inequality reduces to $\max|S'| \leqslant n\max|S|$.

If $|S| \leqslant M$, then not only $|S'| \leqslant nM$ but also $|\tilde{S}'| \leqslant nM$. This is a corollary of the following result:

(3·19) Theorem. *If a polynomial $S(x)$ of order n satisfies $|S| \leqslant M$, then*

$$\{S'^2(x) + \tilde{S}'^2(x)\}^{\frac{1}{2}} \leqslant nM, \tag{3·20}$$

the sign of equality holding if and only if $S = M\cos(nx+\xi)$.

Since the integral in (3·7) is a linear combination of the $2n$ expressions $D_n^*(x-u_k)$, the conjugate polynomial $\tilde{S}(x)$ is given by

$$\tilde{S}(x) = a_n\sin nx + \frac{1}{\pi}\int_0^{2\pi}S(t)\,\tilde{D}_n^*(x-t)\,d\phi_{2n}(t), \tag{3·21}$$

where
$$\tilde{D}_n^*(u) = \sum_{k=1}^{n-1}\sin ku + \tfrac{1}{2}\sin nu = (1-\cos nu)\,\tfrac{1}{2}\cot\tfrac{1}{2}u$$

is the modified conjugate kernel of Dirichlet (Chapter II, (5·2)).