HIT

国外优秀数学著
原 版 系

Linear Algebra and Geometry

线性代数与几何

[荷] 凯珀（Kuiper, N. H.） 著

哈爾濱工業大學出版社
HARBIN INSTITUTE OF TECHNOLOGY PRESS

VECTORS IN THE PLANE AND IN SPACE

Chapter 1

In the ordinary plane (or in ordinary space) we take a fixed point O, that we call the *origin*. We consider arrows in the plane. An arrow can be characterized by its initial point and its endpoint. The word arrow in therefore used as synonym for "ordered pair of points", the first point of a pair being the initial point, the second the endpoint of the arrow. An arrow with O as initial point will be called a *vector*[①].

A special vector is that with O as initial as well as endpoint; it is special since it cannot be drawn as an ordinary arrow. It is called the zero vector and designated by O.

There is a one-to-one relationship between points of the plane and vectors; indeed, to each point there corresponds one vector with that point as endpoint, and conversely, to any vector there belongs an endpoint.

To any arrow $\overrightarrow{BC}$ (cf. Fig. 1.1) there corresponds exactly one vector (the arrow with O as initial point) which can be obtained from $\overrightarrow{BC}$ by displacing it parallel to itself so that O becomes the initial point. The vector $\overrightarrow{OA}$ so obtained may thus be represented by the arrow $\overrightarrow{BC}$. The vector $\overrightarrow{OA}$ has many such representatives, of course. Sometimes we shall say "the vector $\overrightarrow{BC}$" and mean "the vector represented by $\overrightarrow{BC}$".

The set of all arrows in the plane (in space) with initial point O is called a 2-*dimensional vector space* (a 3-dimensional vector space). Furthermore the following algebraic conventions will be observed:

---

① Another (equivalent) definition which, however, is slightly less realistic to the beginner, is: a vector is a complete set (i.e. a set that cannot be extended) of arrow, each of which can be obtained from any other one by translation (a parallel displacement).

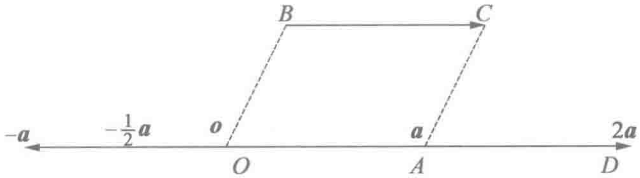

Fig. 1. 1

A_1. *Addition.* To any pair of vectors $\boldsymbol{a} = \overrightarrow{OA}, \boldsymbol{b} = \overrightarrow{OB}$ there corresponds a vector $\boldsymbol{c} = \overrightarrow{OC}$ in the following way: the arrow $\overrightarrow{BC}$ must be a representative of $\boldsymbol{a}$, or equivalently, the arrow $\overrightarrow{AC}$ must be a representative of $\boldsymbol{b}$, or: C is the fourth vertex of a parallelogram with sides OA and OB (cf. Fig. 1. 2. The last formulation, known as the parallelogram-construction, is not unambiguous if O, A and B are collinear). This vector $\boldsymbol{c}$ is called the *sum* of $\boldsymbol{a}$ and $\boldsymbol{b}$, and is denoted by $\boldsymbol{a} + \boldsymbol{b}$.

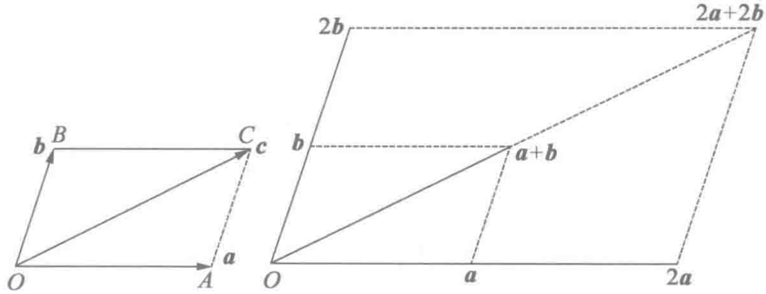

Fig. 1. 2

A_2. *Multiplication by a real number* λ. To any pair consisting of a vector $\boldsymbol{a} = \overrightarrow{OA}$ and a real number λ there corresponds a vector $\boldsymbol{d} = \overrightarrow{OD}$, called *the product of* λ and $\boldsymbol{a}$, denoted by $\lambda\boldsymbol{a}$, in the following way: O, A and D lie on a line and

$$\text{length of } \overrightarrow{OD} = |\lambda| \cdot \text{length of } \overrightarrow{OA}$$

whilst A and D will be on the same or on opposite sides of O depending on whether λ is positive or negative. $|\lambda|$ means the absolute value of λ, i. e. the non-negative on of the number λ and $-\lambda$; for example $|5| = 5$, $|-7| = 7$, $|0| = 0$ (cf. Fig. 1. 1 with $\lambda = 2$). The concept of multiplication considered here is the *geometrical multiplication* known from plane geometry.

Combining addition and multiplication we obtain expressions in vectors and numbers like in ordinary algebra.

These expressions satisfy a number of identities known from ordinary algebra. E. g. whatever the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and the number λ may be, the vector $\lambda(\boldsymbol{a} + \boldsymbol{b})$

will always be equal to the vector $\lambda \boldsymbol{a} + \lambda \boldsymbol{b}$. This will be clear from the geometrical multiplication of the parallelogram on $\boldsymbol{a}$ and $\boldsymbol{b}$ by the factor λ.

We mention some identities which are not so very important when considered as theorems (they are almost trivial), but which will be chosen as starting point (axioms) in the following chapters. From the moment on theorems will no longer be deduced from any knowledge of plane or solid geometry, but from the axioms only. Nevertheless ordinary geometry will play an important part, simply since many theorems originate in it. Also many theorems and their proofs will be much better understood and menorized if interpreted in terms of ordinary geometry.

Important rules, true for arbitrary choice of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and numbers λ, μ, are the following:

A_3	$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$	associative law
A_4	$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$	commutative law
A_5	$1\boldsymbol{a}=\boldsymbol{a}$	
A_6	$\lambda(\mu\boldsymbol{a})=(\lambda\mu)\boldsymbol{a}$	
A_7	$(\lambda+\mu)\boldsymbol{a}=\lambda\boldsymbol{a}+\mu\boldsymbol{a}$	distributive law
A_8	$\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\boldsymbol{a}+\lambda\boldsymbol{b}$	

Problem(1.1). Take vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in space, and numbers λ, μ. Construct geometrically the above expressions, and check that the relations hold.

Chapter 2

SUBSET, PRODUCT SET, RELATION AND MAPPING

In this chapter we introduce some general notions and symbols that will be used later on.

If A is a set of elements, and a is one of them, we shall write $a \in A$. If any element of A is also an element of the set B, then A will be called a *subset* of B, denoted by $A \subset B$, or, which means the same, $B \supset A$. If moreover A is different from B, the A is called a *proper* subset of B. The set of all elements belonging to the set A as well as to B, is called the *intersection* of A and B, denoted by $A \cap B$. Fig. 2. 1 shows point sets A and B and their intersection. Note, that $A \cap B$ is a subset of A: $(A \cap B) \subset A$, and of B.

The set of all elements belonging to A or to B or the both is called the *union* of A and B, denoted by $A \cup B$.

The set of all pairs (a, b) with $a \in A$ and $b \in B$, is called the *product set* of A and B, denoted by $A \times B$. Hence $A \times A$ is the set of all ordered pairs of elements of A. Also any pair of the form (a, a) belongs to it; but if $a \neq a'$ the pair (a, a') is to be distinguished from (a', a). $A \times A \times A$ is the set of all triples of elements of A.

A subset R of $A \times B$ is called a *relation* between the elements of A and those of B. If both A and B are the sets of real numbers, then the set of pairs (a, b) with $a \in A$, $b \in B$ such that $a < b$, is a relation, viz, the relation "less than". In Fig. 2. 2 the relation $a < b$ has been shaded. Similarly we have the relations $a = 2b$ and $a^2 + b^2 = 1$ and $b \leqslant \sin a$. We will say that the relation R between a and b holds if $(a, b) \in R$.

The relation consisting of the set of all pairs (a, a) in $A \times A$ is called the *identity*. A relation $R \subset A \times A$ is called *symmetric* if $(a, b) \in R$ implies that $(b, a) \in R$, for every $a, b \in A$. The relation is called *transitive* if from $(a,b) \in R$ and $(b,c) \in R$ it follows that $(a,c) \in R$, for every $a,b,c \in A$. A relation containing the identity is called *reflexive*.

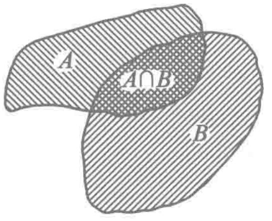

Fig. 2.1

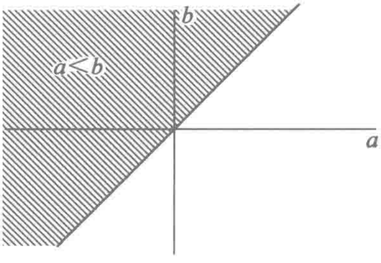

Fig. 2.2

A reflexive, symmetric, transitive relation is called an *equivalence*. We will usually denote equivalence between a and b by $a \sim b$, and we say: "a is equivalent to b". From the definition it follows that for an equivalence we have: $a \sim a$ (reflexive);

If $a \sim b$, then $b \sim a$ (symmetric);

If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitive).

The relation "similarity" between plane figures is an instance of an equivalence. The identity is also an equivalence. The relation "less than" for real number is transitive, but it is not an equivalence.

If an equivalence R in $A \times A$ is given, the elements of A can be divided into classes, such that two elements belong to the same class if and only if they are equivalent. These equivalence classes can be considered as elements of a new set, denoted by A/R, which is called the *quotient* of A by the equivalence R. An element of a class will sometimes be called a *representative* of that class.

For a geometrical example, consider the set A of all points in a vertical plane. Two points a and b will be called equivalent, $a \sim b$ or $(a,b) \in R$, if they are at the same height. It can easily be seen that this relation is an equivalence. The equivalence classes are the horizontal lines.

As another application of the notion of equivalence we mention the construction of the retional number from the integers. Let A be the set of pairs of integers (p,q) with $q \neq 0$. Two elements (p,q) and (r,s) will be equivalent if $ps - qv = 0$. Again it is easy to see that this relation also is an equivalence. The equivalence classes are the rational numbers. The class of which (p,q) is a representative, i. e. the class to which it belongs, is denoted by p/q, and also by r/s if $ps - rq = 0$.

If in a subset $R \subset A \times B$ any element $a \in A$ occurs exactly once as leading element in a pair, then the relation is called a *mapping of A into B*. The element $b \in B$ occurring with $a \in A$ in a pair, is called the *image* of a and may be denoted by $b =$

$f(a)$, or, still shorter, by fa. The mapping is then denoted by $f:A \to B$. We shall also write $f:a \to b$ when $b = f(a)$.

The mapping f will be called a *function* if B is a set of numbers, or more generally a set of elements of a field F (for definition of a field cf. p. 68).

By the image of a subset V of A under f is meant the set of all images fa of elements $a \in V$. It is denoted by fV. Briefly, fA is called the image of the mapping f.

In Fig. 2.3, A is a set of 6 vertical lines, B a set of 8 horizontal lines. Any element of the product space $A \times B$ will be a pair consisting of a vertical and a horizontal line, and can be represented by the point of intersection. A relation is given by marking a number of such points. The relation given in the figure is a mapping of A into B. This is similar to the ordinary graph of a function.

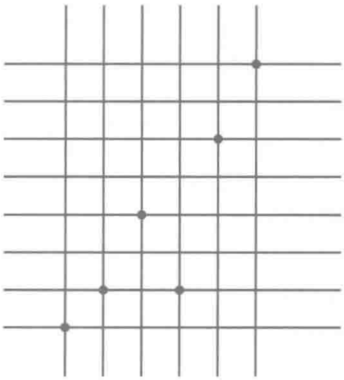

Fig. 2.3

The relation is called a *mapping of A onto B* if the image coincides with B. The set of all $a \in A$ having b as image is designated by $f^{-1}(b)$ and is called a *level set* of the mapping f. The level set of the mapping which associates to points on the earth the temperature, the atmospherical pressure or the altitude at hose points, are called isotherms, isobars and isohypses respectively. Sometimes one of the points of the image space plays a special part, and is denoted by 0 (zero). In that case the level set $f^{-1}(0)$ is called the *kernel* of the mapping.

If $c \geqslant B$ and $f(a) = c$ for all $a \in A$ then f is called *the constant function c*. For example, *the function zero*.

If in the relation $R \subset A \times B$ any element $a \in A$ occurs exactly once as leading element in a pair, and any element $b \in B$ at most (exactly) once as second element, then the relation is called a one-to-one mapping into (onto).

The relations $R \subset A \times B$ and $S \subset B \times C$ determine a relation $SR \subset A \times C$ called the *composite relation*, consisting of all pairs $(a,c) \in A \times C$ for which there exists $ab \in B$ *such that* $(a,b) \in R$ and $(b,c) \in S$. We shall use this only in the case that the relations are mappings, say $f:A \to B$ and $g:B \to C$. The composite relation is then called the *product mapping*. The product mapping associates to any element $a \in A$ the element $gfa = g(f(a)) \in C$. It is denoted by $gf:A \to C$.

If the one-to-one mappings $f:A \to B$ and $g:B \to A$ are such that the product mapping $fg:A \to A$ is the identity, then g is called the inverse of f. In that case $fg:B \to B$ is the identity in B, and consequently f is also the inverse of g. The inverse of f is denoted by $f^{-1}:B \to A$.

As an example we mention: the product of the mappings (in this case functions of real variables) $x \to x+3$, $x \to x^2$, $x \to \sin x$ is the composite function $x \to \sin(x+3)^2$.

THE n-DIMENSIONAL VECTOR SPACE V^n

Chapter 3

We shall now present a precise formulation of our starting point by giving a definition of vector space. This definition will be such that the geometrical vector spaces of chapter 1 can be considered as (important) instances. The definition will not, however, be based on any geometrical experience of the reader. Also in proving theorems we shall only use the definition and properties already established. Hence, all properties that the vector space will eventually have, it will have by virtue of our definitions only.

In the definition we shall make use of *scalars*. The reader may think of these as real numbers, or also as complex numbers; more generally, scalars will be elements of an arbitrary field F of characteristic unequal to 2, unless in particular cases we explicitly state the contrary[①]. (A field is said to have characteristic unequal to 2 if any non-zero element differs from its opposite). The field of real numbers will be denoted by $(F=)R$, the complex number field by $(F=)C$.

Definition: A vector space (or linear space) over the field F is a get V of elements $\boldsymbol{a}, \boldsymbol{b}, \cdots$, called *vectors*, having the following properties $A_1, \cdots, A_8$.

A_1. There is a mapping of $V \times V$ into V which is called *addition* of vectors. The image of the pair $(\boldsymbol{a}, \boldsymbol{b}) \in V \times V$ is called the *sum* of $\boldsymbol{a}$ and $\boldsymbol{b}$, and it is denoted by $\boldsymbol{a}+\boldsymbol{b}$.

A_2. There is a mapping of $F \times V$ into V which is called *multiplication* of vectors by scalars. The image of the pair $(\lambda, \boldsymbol{a}) \in F \times V$ is called the *product*, and is denoted by $\lambda \boldsymbol{a}$.

---

① A further generalization to arbitrary characteristic and non-commutative fields is possible, but requires too many precautions for our purposes.

For arbitrary $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in V$ and $\lambda,\mu, \in F$:

A_3	$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$	associative law
A_4	$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$	commutative law
A_5	$1\boldsymbol{a}=\boldsymbol{a}$	
A_6	$\lambda(\mu\boldsymbol{a})=(\lambda\mu)\boldsymbol{a}$	
A_7	$(\lambda+\mu)\boldsymbol{a}=\lambda+\mu\boldsymbol{a}$	distributive law
A_8	$\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\boldsymbol{a}+\lambda\boldsymbol{b}$	

From properties $A_3,\cdots,A_8$ it follows that to a certain extent the manipulations with scalars and vectors are the same as in ordinary algebra. For example, A_3 says that no ambiguity will arise if in a sum of three or more vectors the parentheses are omitted. A_4 implies that altering the order in a sum of vectors has no influence on the result. For example $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{c})+\boldsymbol{b}=(\boldsymbol{c}+\boldsymbol{a})+\boldsymbol{b}=\boldsymbol{c}+(\boldsymbol{a}+\boldsymbol{b})$, which without any harm may be written as $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ or $\boldsymbol{c}+\boldsymbol{a}+\boldsymbol{b}$.

Problem(3.1). Using $A_3,\cdots,A_8$ only, prove

$$\lambda\{(\mu+v)(\boldsymbol{c}+\boldsymbol{b})\}+(\lambda\mu)\boldsymbol{a}+\nu(\lambda\boldsymbol{a})=\lambda(\mu+v)(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$$

$\lambda,\mu,\nu\in F;\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in V.$

A set of generators of V will be any set of vectors $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_m$ in V such that for any vector $\boldsymbol{a}\in V$ there exist scalars $\lambda_1,\cdots,\lambda_m\in F$ satisfying

$$\boldsymbol{a}=\lambda_1\boldsymbol{a}_1+\cdots+\lambda_m\boldsymbol{a}_m$$

In this case we shall say that the vectors $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_m$ *generate* or *span* the space V.

If none of the proper subsets of the set $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_m$ is also a set of generators of V, then the set $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_m$ is called a *basic* of V.

If a finite set of generators which is not a basis, is given, then by the definition it should be possible to omit at least one of the vectors of the set and still have a set of generators. Repeating this, one finds a basis after a finite number of steps.

The properties $A_1,\cdots,A_8$ do not imply that there exists a finite set of generators. Consider e. g. the set of all polynomials in one variable with real coefficients, or the set of all continuous functions on an interval.

Since we wish to restrict ourselves in this book to a *finite set of* generators, we add:

A_9. V has a finite basis.

The smallest number occurring as the number of elements in a basis is called the *dimension* of V. If, however, V consists of exactly one vector, we shall say that the dimension is zero. A vector space of dimension n will be denoted by V^n.

The vectors in the ordinary plane with a fixed point O, as introduced in chapter

1, form a vector space over the real numbers. Since these vectors are not all multiples of one of them, the dimension is at least two. In Fig. 3. 1 we see that the dimension is exactly two: $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are vectors different from zero and situated on two intersecting lines through O. It is easy to find for any vector $\boldsymbol{b}$ a parallelogram having $\boldsymbol{b}$ as diagonal and two sides along the given lines. Then there are real numbers λ_1 and λ_2 such that

$$\boldsymbol{b} = \lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2$$

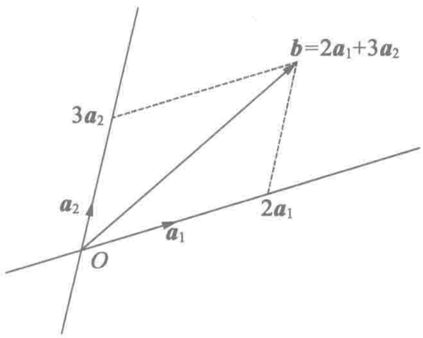

Fig. 3. 1

Problem(3. 2). The vectors in ordinary space with fixed point O form a vector space of dimension *three*. Prove this by means of solid geometry.

Problem(3. 3). In Fig. 3. 1 the vectors on the line on which $\boldsymbol{a}_1$ lies form a vector space of dimension *one*.

Remark: *If in a problem like* (3. 3) *just an assertion is given, then this assertion is to be proved.*

The *zero vector*. For any two vectors $\boldsymbol{a} = \lambda_1\boldsymbol{a}_1 + \cdots + \lambda_n\boldsymbol{a}_n$ and $\boldsymbol{b} = \mu_1\boldsymbol{a}_1 + \cdots + \mu_n\boldsymbol{b}_n$ in the space V with basis $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ we have

$$0 \times \boldsymbol{a} = 0 \times \boldsymbol{b} \text{ and } \boldsymbol{a} + 0 \times \boldsymbol{b} = \boldsymbol{a}$$

as the reader can readily verify from the axioms. The vector $0 \times \boldsymbol{a} = 0 \times \boldsymbol{b}$ is called the *zero vector*, denoted by $\boldsymbol{O}$. Clearly and $\boldsymbol{a} \in \boldsymbol{V}$ satisfies

$$\boldsymbol{a} + \boldsymbol{O} = \boldsymbol{a}, \text{and in addition } \lambda\boldsymbol{O} = \boldsymbol{O} \text{ for any scalar } \lambda$$

In view of the axioms vector $\boldsymbol{O}$ on its own is a vector space of dimension zero.

The *difference vector*. By the vector $-\boldsymbol{a}$ we will mean $(-1)\boldsymbol{a}$. The vector $\boldsymbol{x}$ satisfying $\boldsymbol{a} + \boldsymbol{x} = \boldsymbol{c}$ for given $\boldsymbol{a}, \boldsymbol{c} \in \boldsymbol{V}$ is called the *difference* of $\boldsymbol{c}$ and $\boldsymbol{a}$, and is denoted by $\boldsymbol{c} - \boldsymbol{a}$. From $\boldsymbol{a} + (-\boldsymbol{a}) = 1 \cdot \boldsymbol{a} + (-1) \cdot \boldsymbol{a} = 0 \cdot \boldsymbol{a} = \boldsymbol{O}$, it follows that $\boldsymbol{a} + \boldsymbol{c} + (-\boldsymbol{a}) = \boldsymbol{a} + (-\boldsymbol{a}) + \boldsymbol{c} = \boldsymbol{O} + \boldsymbol{c} = \boldsymbol{c}$, hence $\boldsymbol{c} + (-\boldsymbol{a})$ satisfies the equation for $\boldsymbol{x}$, i. e. there exists such an $\boldsymbol{x}$, and we have $\boldsymbol{c} - \boldsymbol{a} = \boldsymbol{c} + (-\boldsymbol{a})$; it is the vector to be added to $\boldsymbol{a}$ in order to obtain $\boldsymbol{c}$. In Fig. 1. 2 it is vector $\boldsymbol{b}$.

Problems. Using the definitions of vector space, subspace, basis, etc., prove the following assertions, and draw figures illustration the low-dimensional cases. V will always be a vector space.

Problem(3.4). If the pair $\boldsymbol{a}, \boldsymbol{b} \in V$ forms a basis of V, then so does the following pair of vectors ($\lambda \neq 0$)

$$\boldsymbol{a}, -\boldsymbol{b}; \lambda, \boldsymbol{b}; \boldsymbol{a}+\boldsymbol{b}, \boldsymbol{b}$$

Problem(3.5). If $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ form $\boldsymbol{a}$ basis of V, then so do

$$\boldsymbol{a}_1, \boldsymbol{a}_1+\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \cdots, \boldsymbol{a}_m$$

Problem(3.6). If $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$ are vectors in V then the set B of all vectors $\lambda_1\boldsymbol{a}_1 + \cdots + \lambda_k\boldsymbol{a}_k$ with arbitrary $\lambda_1, \cdots, \lambda_k \in F$ is also a vector space. It is called the *(sub)space generated by* $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$. The set of all vectors $\lambda\boldsymbol{a}$ with fixed $\boldsymbol{a} \in V$ and variable $\lambda \in F$ is a one-dimensional space and is denoted by Fa. If U and W are subsets of V, then the set of all $\boldsymbol{b}+\boldsymbol{c}$ with $\boldsymbol{b} \in U$ and $\boldsymbol{c} \in W$ is denoted by $U+W$ (cf. Fig. 3.2). Hence the subspace generated by $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$ can be represented by $F\boldsymbol{a}_1 + F\boldsymbol{a}_2 + \cdots + F\boldsymbol{a}_k$. Similarly λU may be defined.

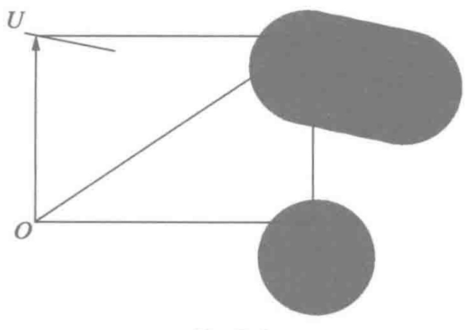

Fig. 3.2

Problem(3.7). If W is a set in the two-dimensional geometrical vector space of chapter 1 and $\boldsymbol{a}$ is a point (vector), prove that $\boldsymbol{a}+W$ is congruent to W.

Problem(3.8). If U and W and non-parallel line segments in a plane, prove that $U+W$ is a parallelogram together with its interior.

Problem(3.9). If U is the interior of a triangle prove that $\frac{1}{2}(U+U) = U$.

Chapter 4

THE PARAMETRIC REPRESENTATION OF A LINE

Fig. 4. 1 shows a so-called number axis. In connection with it we consider two sets:

a) the set of points on the line;

b) the set of real numbers.

The figure suggests a one-to-one mapping between the two sets, making possible the designation of points by numbers.

Fig. 4. 1

Similarly in the geometrical plane with fixed point O of chapter 1 we may consider two sets:

a) the set A of points in the plane;

b) the set V of vectors.

There is again a one-to-one mapping of either of them onto the other set. Fig. 4. 2 shows the two sets *apart*, and the mapping k is indicated by the arrow k.

In A our interest will concentrate on points, lines and other configurations, and later on motions. In V our attention will be directed to the vector algebra.

We may speak of A in terms of V. For example B is the point represented by vector $\boldsymbol{b}$ under the mapping k. It will simply be called "*the point b*". For the time being we shall work with a single fixed mapping $A \to V$; later on we shall consider various mappings $A \to V$ and we shall make a clearer distinction between A and V.

After this introduction we give the following preliminary definition of the n-dimensional affine space A^n: It is a set whose elements are called points, having a one-to-one mapping

$$k: A^n \to V^n$$

onto the n-dimensional vector space V^n.

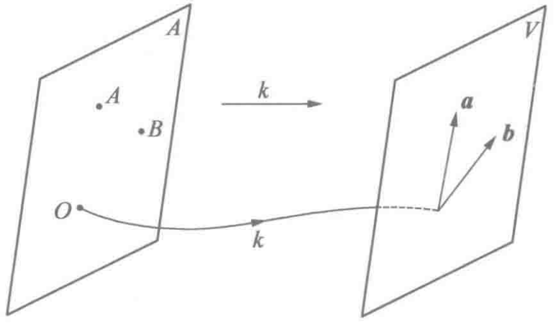

Fig. 4. 2

In the following we shall alternately consider A^n and V^n; which of the two is under consideration can be seen from the terminology: points and lines belong to A^n, vectors and vector subspaces belong to V^n. Moreover, vector spaces and vectors will be distinguished by Roman type.

Definition: The *line* through the points (represented by the vectors) $\boldsymbol{a}$ and $\boldsymbol{b}$ is the point set (cf. Fig. 4. 3)

$$\boldsymbol{a} + \mu(\boldsymbol{b} - \boldsymbol{a}), \mu \in F \tag{4.1}$$

or

$$\lambda \boldsymbol{a} + \mu \boldsymbol{b}, \lambda + \mu = 1; \lambda, \mu \in F$$

This definition of course refers to the affine space A; (4. 1) is called the *parametric representation* of the line. $\boldsymbol{b} - \boldsymbol{a}$ is called *the vector from point a to point b*. For $\mu = 0$ or 1 respectively we obtain the points a and b again. For $\mu = \frac{1}{2}$ we obtain the point $z = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$, the middle or centroid of a and b.

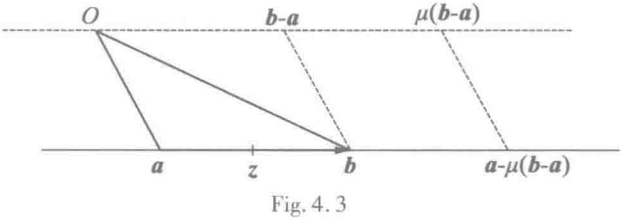

Fig. 4. 3

If real numbers and being used ($F = R$) then the definition just gives the ordinary line through a and b. The points with $0 < \mu < 1$ lie between a and b on the line through a and b. This can be seen from the figure, but it cannot be seen independently of that, since "between" is a yet undefined term. A suitable definition is therefore: A point (4. 1) on the line through a and b will be said to lie between a and b

if $0 < \mu < 1$.

Suppose for general F that "the line throught the different points a' and b'" contains the points a and $b(\neq a)$. We shall now prove that the line coincide with "the line through a and b".

This is expressed by

Theorem[4.1]: *There is exactly one line containing two given different points.*

Proof: Since a and b are on the line through a' and b' there exist scalars ν' and $\eta' \neq \nu' \in F$ such that

$$\boldsymbol{a} = \boldsymbol{a}' + \nu'(\boldsymbol{b}' - \boldsymbol{a}'), \boldsymbol{b} = \boldsymbol{a}' + \eta'(\boldsymbol{b}' - \boldsymbol{a}')$$

The line through a' and b' consists of the points

$$\boldsymbol{a}' + \mu'(\boldsymbol{b}' - \boldsymbol{a}'), \mu' \text{ variable in } F$$

The line through a and b consists of the points

$$\boldsymbol{a} + \mu(\boldsymbol{b} - \boldsymbol{a}) = \boldsymbol{a}' + \{\nu' + \mu(\eta - \nu')\}(\boldsymbol{b}' - \boldsymbol{a}'), \mu \in F$$

These two sets are identical: Take

$$\mu' = \nu' + \mu(\eta' - \nu'), \mu = (\mu' - \nu')(\eta' - \nu')^{-1}$$

If $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b}$ are vectors such that $\boldsymbol{b} = \lambda \boldsymbol{a}$, then the scalar λ is called *the ratio of* $\boldsymbol{b}$ *and* $\boldsymbol{a}$. If also $\boldsymbol{b} = \mu \boldsymbol{a}$, then $\lambda \boldsymbol{a} - \mu \boldsymbol{a} = (\lambda - \mu)\boldsymbol{a} = \boldsymbol{0}$. If $\lambda - \mu \neq 0$ then $(\lambda - \mu)^{-1}(\lambda - \mu)\boldsymbol{a} = \boldsymbol{a} = \boldsymbol{0}$, contrary to the assumption. Hence $\lambda = \mu$ and the ratio

$$\frac{\boldsymbol{b}}{\boldsymbol{a}} = \lambda \tag{4.2}$$

is uniquely determined.

N. B. For *arbitrary* vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the left-hand side of (4.2) usually has no meaning.

Two lines are said to be *parallel* if there exist points a and $b(\neq a)$ on one of them, points c and $d(\neq c)$ on the other, and a scalar λ such that

$$\boldsymbol{d} - \boldsymbol{c} = \lambda(\boldsymbol{b} - \boldsymbol{a}), \lambda = \frac{\boldsymbol{d} - \boldsymbol{c}}{\boldsymbol{b} - \boldsymbol{a}} \tag{4.3}$$

λ is called the *ratio of the line segments* "cd" and "ab".

Problem(4.1). If arbitrary points a' and $b' \neq a'$ are chosen on one of two parallel lines and points c' and $d'(\neq c')$ on the other one, then there exists a scalar λ' such that $(\boldsymbol{d}' - \boldsymbol{c}') = \lambda'(\boldsymbol{b}' - \boldsymbol{a}')$.

Problem(4.2). Given points a and b in the ordinary plane, determine a point c on ab such that the ratio of the line segments ab and ac equals (i) -1; (ii) $+4$; (iii) $+1$.

Problem(4.3). If two lines are parallel to the third line then they are parallel themselves.

Three points a, b, c are said to be *collinear* if they are on one line. Three line are said to be *concurrent* if they either pass through one point or are mutually parallel. (Fig. 4.4).

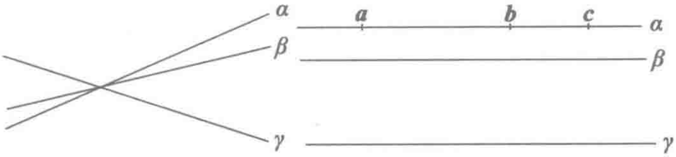

Fig. 4.4 Three concurrent lines

As an application of what precedes we shall prove the

Theorem: *The medians of a triangle pass through one point*[①].

Proof: Let $a, b, c (\in V)$ be the vertices of an arbitrarily given triangle. We first determine an expression for the median from a to the middle of b and c: $\frac{1}{2}(\boldsymbol{b}+\boldsymbol{c})$.

It is: $\boldsymbol{a}+\lambda\{\frac{1}{2}(\boldsymbol{b}+\boldsymbol{c})-\boldsymbol{a}\}, \lambda \in F$. The choice $\lambda=\frac{2}{3}$ yields the point

$$\frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) \tag{4.3}$$

On the medians from b and c we find in a similar way a point that can be obtained from (4.3) by interchanging the letters. However, (4.3) is invariant under interchange of letters; hence the point given by (4.3) lies on all medians and is then the centroid of the triangle.

Problem(4.4). The points (represented by vectors) a, b, c lie on a line if and only if there are $\alpha, \beta, \gamma \in F((\alpha, \beta, \gamma) \neq (0,0,0))$ such that

$$\alpha \boldsymbol{a}+\beta \boldsymbol{b}+\gamma \boldsymbol{c}=\boldsymbol{0} \text{ and } \alpha+\beta+\gamma=0$$

Problem(4.5). Give a definition of a parallelogram that can also be applied to a "quadrangle" having four collinear vertices. Let a, b, c, d be the consecutive vertices of a parallelogram in space. Express vector $\boldsymbol{d}$ in terms of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

Problem(4.6). The diagonals in a parallelogram bisect each other.

Problem(4.7). The midpoints of the sides of an arbitrary plane or non-plane quadrangle form a parallelogram.

Problem(4.8). Each of the sides ab and bc of a triangle is extended by itself to yield the points p and q resp. such that b is between a and p, c between b and q. Line pq intersects ac in r. Prove that c divides the segment ar into pieces with ration 3:1.

---

① In this theorem and the following problems we assume that the scalars are taken from a field with characteristic 0 such as the real or complex number fields.

Problem(4.9). The four lines through one of the points a, b, c, d and the centroid of the other three pass through one point, the centroid of a, b, c, d.

Problem(4.10). In a space of arbitrary dimension the centroid of k points $a_1, \cdots, a_k$ is by definition the point

$$z = \frac{1}{k}(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a}_k)$$

Prove that the line through the centroid of $a_1, \cdots, a_p$ and the centroid of $a_{p+1}, \cdots, a_{p+q}$ $(p + q = m)$, passes through the centroid of $a_1, \cdots, a_m$.

What consequences does this have for a plane quadrangle and for a tetrahedron? (Take $(p, q) = (1,3)$ and $(p, q) = (2,2)$).

Problem(4.11). The midpoints of the consecutive sides of the hexagon $a_1, a_2, \cdots, a_6$ are $b_1, b_2, \cdots, b_6$ respectively. Prove that the triangles $b_1b_3b_5$ and $b_2b_4b_6$ have the same centroid.

Problem(4.12). Give a definition of a parallelopiped. Let *abcdefgh* be a parallelopiped; let u be the centroid of *deb*, v that of *cfh*, w' that of *bdeg*, w'' that of *acfh*, w that of *abcdefgh*. Prove that $w = w' = w''$ and that a, u, v, w and g are collinear. Calculate the ratio of the line segments thus formed on ag. Hint: take $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{p}, \boldsymbol{d} = \boldsymbol{a} + \boldsymbol{q}, \boldsymbol{e} = \boldsymbol{a} + \boldsymbol{r}$ and express the vertices in terms of a, p, q and r.

Problem(4.13). A set G of points of a real affine space is called *convex* if it contains together with any two points a, b also the points on ab between a and b. The intersection of two convex sets is itself convex.

Problem(4.14). If $a_1, \cdots, a_m$ are arbitrary points in a real affine space then the set G of points

$$\boldsymbol{x} = \frac{\gamma_1 \boldsymbol{a}_1 + \cdots + \gamma_m \boldsymbol{a}_m}{\gamma_1 + \cdots + \gamma_m}$$

$\gamma_1, \cdots, \gamma_m$ real numbers is convex.

The same is true if $\gamma_1, \cdots, \gamma_m$ are required to be non-negative.

In this last case prove that we obtain the smallest convex set containing $a_1, \cdots, a_m$ (cf. Fig. 4.5). x is called a *weighted mean* of the vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ with "weights" $\gamma_1 \geqslant 0, \cdots, \gamma_m \geqslant 0$

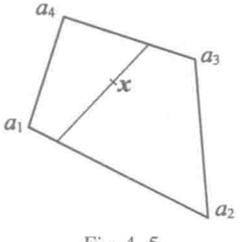

Fig. 4.5