概率论题解1000例

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One Thousand Exercises in Probability

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前　言

本书包括了概率论和随机过程中的 1000 多道练习题及其解答。它是牛津大学 2001 年出版的教程《概率论和随机过程》（以下简称 PRP）的题解手册，就这本书本身而言也是一部完全独立的习题集，不仅可以作为深入学习之用，也可以作为 PRP 教程的补充和进一步理解。本书是对早期《概率论及其题解》的扩展，新增加了 400 多道练习题。因为书中许多习题包括好几问，所以总共的问题超过 3000 道。

尽管本书具有很强的独立性，但作为 PRP 的伴随习题册，大量的书本功课难免，对 PRP 相关章节的参考都会有提示。在问题中多次出现“显然”，尽管运用这样的字样，但并不意味这些问题毫无难度。

本书的目录和 PRP 的完全一样，章节、习题编号也和 PRP 的相对应，偶尔有些参考资料也是 PRP 中的例题和方程。书中内容广泛，从概率论和随机变量基本理论开始，紧接着通过有关马尔科夫链和收敛、平稳和遍历理论、更新、排队、鞅和扩散这些章节，引进了期权定价。总体而言，练习是在检验个别理论的掌握情况，而问题综合性强，不会显得太专。这些问题都比较标准，大多数是初级难度或者中级难度。由于有些中级难度的练习题实际上已经相当棘手，加上星号或者标示以示区别。如果这本书作为自学教程，这些建议在初次阅读时将是不错的参考。

在此感谢无名老师的测试试卷和课后作业，一些教程都对本书的框架结构有莫大的影响。相反，也希望更多的读者从中受益。如果您发现错误，请尽可能的告知我们；如果您对一些问题有更好的解答，我们将很乐意在下一版中更新。

我们很感谢 Sarah Shea – Simonds 专家在本书 TEX 版本中做出的贡献，感谢 Andy Burbanks 在封面设计中的建议，这些都给作品增加了更大的影响力。

剑桥大学和牛津大学　　G. R. G.

2001 年 4 月　　D. R. S.

目　录

1

事件及其概率

1.2 集合事件

1. Let $\{A_i : i \in I\}$ be a collection of sets. Prove 'De Morgan's Laws'†:

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c, \qquad \left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c.$$

2. Let A and B belong to some σ-field $\mathcal{F}$. Show that $\mathcal{F}$ contains the sets $A \cap B$, $A \setminus B$, and $A \triangle B$.

3. A conventional knock-out tournament (such as that at Wimbledon) begins with 2^n competitors and has n rounds. There are no play-offs for the positions $2, 3, \ldots, 2^n - 1$, and the initial table of draws is specified. Give a concise description of the sample space of all possible outcomes.

4. Let $\mathcal{F}$ be a σ-field of subsets of Ω and suppose that $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-field of subsets of B.

5. Which of the following are identically true? For those that are not, say when they are true.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

(b) $A \cap (B \cap C) = (A \cap B) \cap C$;

(c) $(A \cup B) \cap C = A \cup (B \cap C)$;

(d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

1.3 概率

1. Let A and B be events with probabilities $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$. Show that $\frac{1}{12} \le \mathbb{P}(A \cap B) \le \frac{1}{3}$, and give examples to show that both extremes are possible. Find corresponding bounds for $\mathbb{P}(A \cup B)$.

2. A fair coin is tossed repeatedly. Show that, with probability one, a head turns up sooner or later. Show similarly that any given finite sequence of heads and tails occurs eventually with probability one. Explain the connection with Murphy's Law.

3. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with stars on. If the cups are placed randomly onto the saucers (one each), find the probability that no cup is upon a saucer of the same pattern.

†Augustus De Morgan is well known for having given the first clear statement of the principle of mathematical induction. He applauded probability theory with the words: "The tendency of our study is to substitute the satisfaction of mental exercise for the pernicious enjoyment of an immoral stimulus".

4. Let $A_1, A_2, \ldots, A_n$ be events where $n \geq 2$, and prove that

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n).$$

In each packet of Corn Flakes may be found a plastic bust of one of the last five Vice-Chancellors of Cambridge University, the probability that any given packet contains any specific Vice-Chancellor being $\frac{1}{5}$, independently of all other packets. Show that the probability that each of the last three Vice-Chancellors is obtained in a bulk purchase of six packets is $1 - 3(\frac{4}{5})^6 + 3(\frac{3}{5})^6 - (\frac{2}{5})^6$.

5. Let A_r, $r \geq 1$, be events such that $\mathbb{P}(A_r) = 1$ for all r. Show that $\mathbb{P}(\bigcap_{r=1}^{\infty} A_r) = 1$.

6. You are given that at least one of the events A_r, $1 \leq r \leq n$, is certain to occur, but certainly no more than two occur. If $\mathbb{P}(A_r) = p$, and $\mathbb{P}(A_r \cap A_s) = q$, $r \neq s$, show that $p \geq 1/n$ and $q \leq 2/n$.

7. You are given that at least one, but no more than three, of the events A_r, $1 \leq r \leq n$, occur, where $n \geq 3$. The probability of at least two occurring is $\frac{1}{2}$. If $\mathbb{P}(A_r) = p$, $\mathbb{P}(A_r \cap A_s) = q$, $r \neq s$, and $\mathbb{P}(A_r \cap A_s \cap A_t) = x$, $r < s < t$, show that $p \geq 3/(2n)$, and $q \leq 4/n$.

1.4 条件概率

1. Prove that $\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\mathbb{P}(A)/\mathbb{P}(B)$ whenever $\mathbb{P}(A)\mathbb{P}(B) \neq 0$. Show that, if $\mathbb{P}(A \mid B) > \mathbb{P}(A)$, then $\mathbb{P}(B \mid A) > \mathbb{P}(B)$.

2. For events $A_1, A_2, \ldots, A_n$ satisfying $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$, prove that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

3. A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?

He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head? He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head? He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?

He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?

4. What do you think of the following 'proof' by Lewis Carroll that an urn cannot contain two balls of the same colour? Suppose that the urn contains two balls, each of which is either black or white; thus, in the obvious notation, $\mathbb{P}(\text{BB}) = \mathbb{P}(\text{BW}) = \mathbb{P}(\text{WB}) = \mathbb{P}(\text{WW}) = \frac{1}{4}$. We add a black ball, so that $\mathbb{P}(\text{BBB}) = \mathbb{P}(\text{BBW}) = \mathbb{P}(\text{BWB}) = \mathbb{P}(\text{BWW}) = \frac{1}{4}$. Next we pick a ball at random; the chance that the ball is black is (using conditional probabilities) $1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{2}{3}$. However, if there is probability $\frac{2}{3}$ that a ball, chosen randomly from three, is black, then there must be two black and one white, which is to say that originally there was one black and one white ball in the urn.

5. **The Monty Hall problem: goats and cars.** (a) In a game show; you have to choose one of three doors. One conceals a new car, two conceal old goats. You choose, but your chosen door is not opened immediately. Instead the presenter opens another door, which reveals a goat. He offers you the opportunity to change your choice to the third door (unopened and so far unchosen). Let p be the (conditional) probability that the third door conceals the car. The presenter's protocol is:

(i) he is determined to show you a goat; with a choice of two, he picks one at random. Show $p = \frac{2}{3}$.

(ii) he is determined to show you a goat; with a choice of two goats (Bill and Nan, say) he shows you Bill with probability b. Show that, given you see Bill, the probability is $1/(1+b)$.

(iii) he opens a door chosen at random irrespective of what lies behind. Show $p = \frac{1}{2}$.

(b) Show that, for $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, there exists a protocol such that $p = \alpha$. Are you well advised to change your choice to the third door?

(c) In a variant of this question, the presenter is permitted to open the first door chosen, and to reward you with whatever lies behind. If he chooses to open another door, then this door invariably conceals a goat. Let p be the probability that the unopened door conceals the car, conditional on the presenter having chosen to open a second door. Devise protocols to yield the values $p = 0$, $p = 1$, and deduce that, for any $\alpha \in [0, 1]$, there exists a protocol with $p = \alpha$.

6. **The prosecutor's fallacy†.** Let G be the event that an accused is guilty, and T the event that some testimony is true. Some lawyers have argued on the assumption that $\mathbb{P}(G \mid T) = \mathbb{P}(T \mid G)$. Show that this holds if and only if $\mathbb{P}(G) = \mathbb{P}(T)$.

7. **Urns.** There are n urns of which the rth contains $r-1$ red balls and $n-r$ magenta balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:

(a) the second ball is magenta,

(b) the second ball is magenta, given that the first is magenta.

1.5 独立性

1. Let A and B be independent events; show that A^c, B are independent, and deduce that A^c, B^c are independent.

2. We roll a die n times. Let A_{ij} be the event that the ith and jth rolls produce the same number. Show that the events $\{A_{ij} : 1 \le i < j \le n\}$ are pairwise independent but not independent.

3. A fair coin is tossed repeatedly. Show that the following two statements are equivalent:

(a) the outcomes of different tosses are independent,

(b) for any given finite sequence of heads and tails, the chance of this sequence occurring in the first m tosses is 2^{-m}, where m is the length of the sequence.

4. Let $\Omega = \{1, 2, \dots, p\}$ where p is prime, $\mathcal{F}$ be the set of all subsets of Ω, and $\mathbb{P}(A) = |A|/p$ for all $A \in \mathcal{F}$. Show that, if A and B are independent events, then at least one of A and B is either $\varnothing$ or Ω.

5. Show that the conditional independence of A and B given C neither implies, nor is implied by, the independence of A and B. For which events C is it the case that, for all A and B, the events A and B are independent if and only if they are conditionally independent given C?

6. **Safe or sorry?** Some form of prophylaxis is said to be 90 per cent effective at prevention during one year's treatment. If the degrees of effectiveness in different years are independent, show that the treatment is more likely than not to fail within 7 years.

7. **Families.** Jane has three children, each of which is equally likely to be a boy or a girl independently of the others. Define the events:

$$A = \{\text{all the children are of the same sex}\},$$
$$B = \{\text{there is at most one boy}\},$$
$$C = \{\text{the family includes a boy and a girl}\}.$$

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†The prosecution made this error in the famous Dreyfus case of 1894.

(a) Show that A is independent of B, and that B is independent of C.

(b) Is A independent of C?

(c) Do these results hold if boys and girls are not equally likely?

(d) Do these results hold if Jane has four children?

8. **Galton's paradox.** You flip three fair coins. At least two are alike, and it is an evens chance that the third is a head or a tail. Therefore $\mathbb{P}(\text{all alike}) = \frac{1}{2}$. Do you agree?

9. Two fair dice are rolled. Show that the event that their sum is 7 is independent of the score shown by the first die.

1.7 旧题新问

1. There are two roads from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability p, independently of the others. Find the probability that there is an open road from A to B given that there is no open route from A to C.

If, in addition, there is a direct road from A to C, this road being blocked with probability p independently of the others, find the required conditional probability.

2. Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

3. A symmetric random walk takes place on the integers $0, 1, 2, \ldots, N$ with absorbing barriers at 0 and N, starting at k. Show that the probability that the walk is never absorbed is zero.

4. The so-called 'sure thing principle' asserts that if you prefer x to y given C, and also prefer x to y given C^c, then you surely prefer x to y. Agreed?

5. A pack contains m cards, labelled $1, 2, \ldots, m$. The cards are dealt out in a random order, one by one. Given that the label of the kth card dealt is the largest of the first k cards dealt, what is the probability that it is also the largest in the pack?

1.8 习题

1. A traditional fair die is thrown twice. What is the probability that:

(a) a six turns up exactly once?

(b) both numbers are odd?

(c) the sum of the scores is 4?

(d) the sum of the scores is divisible by 3?

2. A fair coin is thrown repeatedly. What is the probability that on the nth throw:

(a) a head appears for the first time?

(b) the numbers of heads and tails to date are equal?

(c) exactly two heads have appeared altogether to date?

(d) at least two heads have appeared to date?

3. Let $\mathcal{F}$ and $\mathcal{G}$ be σ-fields of subsets of Ω.

(a) Use elementary set operations to show that $\mathcal{F}$ is closed under countable intersections; that is, if $A_1, A_2, \ldots$ are in $\mathcal{F}$, then so is $\bigcap_i A_i$.

(b) Let $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ be the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{H}$ is a σ-field.

(c) Show that $\mathcal{F} \cup \mathcal{G}$, the collection of subsets of Ω lying in either $\mathcal{F}$ or $\mathcal{G}$, is not necessarily a σ-field.

4. Describe the underlying probability spaces for the following experiments:

(a) a biased coin is tossed three times;

(b) two balls are drawn without replacement from an urn which originally contained two ultramarine and two vermilion balls;

(c) a biased coin is tossed repeatedly until a head turns up.

5. Show that the probability that *exactly* one of the events A and B occurs is

$$\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).$$

6. Prove that $\mathbb{P}(A \cup B \cup C) = 1 - \mathbb{P}(A^c \mid B^c \cap C^c)\mathbb{P}(B^c \mid C^c)\mathbb{P}(C^c)$.

7. (a) If A is independent of itself, show that $\mathbb{P}(A)$ is 0 or 1.

(b) If $\mathbb{P}(A)$ is 0 or 1, show that A is independent of all events B.

8. Let $\mathcal{F}$ be a σ-field of subsets of Ω, and suppose $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfies: (i) $\mathbb{P}(\Omega) = 1$, and (ii) $\mathbb{P}$ is additive, in that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ whenever $A \cap B = \varnothing$. Show that $\mathbb{P}(\varnothing) = 0$.

9. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \in \mathcal{F}$ satisfies $\mathbb{P}(B) > 0$. Let $\mathbb{Q} : \mathcal{F} \to [0, 1]$ be defined by $\mathbb{Q}(A) = \mathbb{P}(A \mid B)$. Show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A \mid C) = \mathbb{P}(A \mid B \cap C)$; discuss.

10. Let $B_1, B_2, \dots$ be a partition of the sample space Ω, each B_i having positive probability, and show that

$$\mathbb{P}(A) = \sum_{j=1}^{\infty} \mathbb{P}(A \mid B_j)\mathbb{P}(B_j).$$

11. Prove **Boole's inequalities**:

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i), \qquad \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \ge 1 - \sum_{i=1}^{n} \mathbb{P}(A_i^c).$$

12. Prove that

$$\mathbb{P}\left(\bigcap_{1}^{n} A_i\right) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cup A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cup A_j \cup A_k) - \cdots - (-1)^n \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n).$$

13. Let $A_1, A_2, \dots, A_n$ be events, and let N_k be the event that exactly k of the A_i occur. Prove the result sometimes referred to as **Waring's theorem**:

$$\mathbb{P}(N_k) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{k} S_{k+i}, \text{ where } S_j = \sum_{i_1 < i_2 < \cdots < i_j} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}).$$

Use this result to find an expression for the probability that a purchase of six packets of Corn Flakes yields exactly three distinct busts (see Exercise (1.3.4)).

14. Prove **Bayes's formula**: if $A_1, A_2, \dots, A_n$ is a partition of Ω, each A_i having positive probability, then

$$\mathbb{P}(A_j \mid B) = \frac{\mathbb{P}(B \mid A_j)\mathbb{P}(A_j)}{\sum_1^n \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}.$$

15. A random number N of dice is thrown. Let A_i be the event that $N = i$, and assume that $\mathbb{P}(A_i) = 2^{-i}$, $i \geq 1$. The sum of the scores is S. Find the probability that:
(a) $N = 2$ given $S = 4$;
(b) $S = 4$ given N is even;
(c) $N = 2$, given that $S = 4$ and the first die showed 1;
(d) the largest number shown by any die is r, where S is unknown.

16. Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m, \quad C_n = \bigcap_{m=n}^{\infty} A_m.$$

Clearly $C_n \subseteq A_n \subseteq B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits

$$\lim B_n = B = \bigcap_n B_n = \bigcap_n \bigcup_{m \geq n} A_m, \qquad \lim C_n = C = \bigcup_n C_n = \bigcup_n \bigcap_{m \geq n} A_m.$$

The events B and C are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ respectively. Show that
(a) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$,
(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$.
We say that the sequence $\{A_n\}$ converges to a limit $A = \lim A_n$ if B and C are the same set A. Suppose that $A_n \to A$ and show that
(c) A is an event, in that $A \in \mathcal{F}$,
(d) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

17. In Problem (1.8.16) above, show that B and C are independent whenever B_n and C_n are independent for all n. Deduce that if this holds and furthermore $A_n \to A$, then $\mathbb{P}(A)$ equals either zero or one.

18. Show that the assumption that $\mathbb{P}$ is *countably* additive is equivalent to the assumption that $\mathbb{P}$ is continuous. That is to say, show that if a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfies $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$, and $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \varnothing$, then $\mathbb{P}$ is countably additive (in the sense of satisfying Definition (1.3.1b)) if and only if $\mathbb{P}$ is continuous (in the sense of Lemma (1.3.5)).

19. Anne, Betty, Chloë, and Daisy were all friends at school. Subsequently each of the $\binom{4}{2} = 6$ subpairs meet up; at each of the six meetings the pair involved quarrel with some fixed probability p, or become firm friends with probability $1 - p$. Quarrels take place independently of each other. In future, if any of the four hears a rumour, then she tells it to her firm friends only. If Anne hears a rumour, what is the probability that:
(a) Daisy hears it?
(b) Daisy hears it if Anne and Betty have quarrelled?
(c) Daisy hears it if Betty and Chloë have quarrelled?
(d) Daisy hears it if she has quarrelled with Anne?

20. A biased coin is tossed repeatedly. Each time there is a probability p of a head turning up. Let p_n be the probability that an even number of heads has occurred after n tosses (zero is an even number). Show that $p_0 = 1$ and that $p_n = p(1 - p_{n-1}) + (1 - p)p_{n-1}$ if $n \geq 1$. Solve this difference equation.

21. A biased coin is tossed repeatedly. Find the probability that there is a run of r heads in a row before there is a run of s tails, where r and s are positive integers.

22. A bowl contains twenty cherries, exactly fifteen of which have had their stones removed. A greedy pig eats five whole cherries, picked at random, without remarking on the presence or absence of stones. Subsequently, a cherry is picked randomly from the remaining fifteen.

(a) What is the probability that this cherry contains a stone?

(b) Given that this cherry contains a stone, what is the probability that the pig consumed at least one stone?

23. The '**ménages**' problem poses the following question. Some consider it to be desirable that men and women alternate when seated at a circular table. If n couples are seated randomly according to this rule, show that the probability that nobody sits next to his or her partner is

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

You may find it useful to show first that the number of ways of selecting k non-overlapping pairs of adjacent seats is $\binom{2n-k}{k}2n(2n-k)^{-1}$.

24. An urn contains b blue balls and r red balls. They are removed at random and not replaced. Show that the probability that the first red ball drawn is the $(k+1)$th ball drawn equals $\binom{r+b-k-1}{r-1}\Big/\binom{r+b}{b}$. Find the probability that the last ball drawn is red.

25. An urn contains a azure balls and c carmine balls, where $ac \neq 0$. Balls are removed at random and discarded until the first time that a ball (B, say) is removed having a different colour from its predecessor. The ball B is now replaced and the procedure restarted. This process continues until the last ball is drawn from the urn. Show that this last ball is equally likely to be azure or carmine.

26. Protocols. A pack of four cards contains one spade, one club, and the two red aces. You deal two cards faces downwards at random in front of a truthful friend. She inspects them and tells you that one of them is the ace of hearts. What is the chance that the other card is the ace of diamonds? Perhaps $\frac{1}{3}$?

Suppose that your friend's protocol was:

(a) with no red ace, say "no red ace",

(b) with the ace of hearts, say "ace of hearts",

(c) with the ace of diamonds but not the ace of hearts, say "ace of diamonds".

Show that the probability in question is $\frac{1}{3}$.

Devise a possible protocol for your friend such that the probability in question is zero.

27. Eddington's controversy. Four witnesses, A, B, C, and D, at a trial each speak the truth with probability $\frac{1}{3}$ independently of each other. In their testimonies, A claimed that B denied that C declared that D lied. What is the (conditional) probability that D told the truth? [This problem seems to have appeared first as a parody in a university magazine of the 'typical' Cambridge Philosophy Tripos question.]

28. The probabilistic method. 10 per cent of the surface of a sphere is coloured blue, the rest is red. Show that, irrespective of the manner in which the colours are distributed, it is possible to inscribe a cube in S with all its vertices red.

29. Repulsion. The event A is said to be repelled by the event B if $\mathbb{P}(A \mid B) < \mathbb{P}(A)$, and to be attracted by B if $\mathbb{P}(A \mid B) > \mathbb{P}(A)$. Show that if B attracts A, then A attracts B, and B^{c} repels A.

If A attracts B, and B attracts C, does A attract C?

30. Birthdays. If m students born on independent days in 1991 are attending a lecture, show that the probability that at least two of them share a birthday is $p = 1 - (365)!/\{(365-m)!\,365^m\}$. Show that $p > \frac{1}{2}$ when $m = 23$.

31. Lottery. You choose r of the first n positive integers, and a lottery chooses a random subset L of the same size. What is the probability that:

(a) L includes no consecutive integers?

(b) L includes exactly one pair of consecutive integers?

(c) the numbers in L are drawn in increasing order?

(d) your choice of numbers is the same as L?

(e) there are exactly k of your numbers matching members of L?

32. Bridge. During a game of bridge, you are dealt at random a hand of thirteen cards. With an obvious notation, show that $\mathbb{P}(4S, 3H, 3D, 3C) \simeq 0.026$ and $\mathbb{P}(4S, 4H, 3D, 2C) \simeq 0.018$. However if suits are not specified, so numbers denote the shape of your hand, show that $\mathbb{P}(4, 3, 3, 3) \simeq 0.11$ and $\mathbb{P}(4, 4, 3, 2) \simeq 0.22$.

33. Poker. During a game of poker, you are dealt a five-card hand at random. With the convention that aces may count high or low, show that:

$$\mathbb{P}(\text{1 pair}) \simeq 0.423, \quad \mathbb{P}(\text{2 pairs}) \simeq 0.0475, \quad \mathbb{P}(\text{3 of a kind}) \simeq 0.021,$$
$$\mathbb{P}(\text{straight}) \simeq 0.0039, \quad \mathbb{P}(\text{flush}) \simeq 0.0020, \quad \mathbb{P}(\text{full house}) \simeq 0.0014,$$
$$\mathbb{P}(\text{4 of a kind}) \simeq 0.00024, \quad \mathbb{P}(\text{straight flush}) \simeq 0.000015.$$

34. Poker dice. There are five dice each displaying 9, 10, J, Q, K, A. Show that, when rolled:

$$\mathbb{P}(\text{1 pair}) \simeq 0.46, \quad \mathbb{P}(\text{2 pairs}) \simeq 0.23, \quad \mathbb{P}(\text{3 of a kind}) \simeq 0.15,$$
$$\mathbb{P}(\text{no 2 alike}) \simeq 0.093, \quad \mathbb{P}(\text{full house}) \simeq 0.039, \quad \mathbb{P}(\text{4 of a kind}) \simeq 0.019,$$
$$\mathbb{P}(\text{5 of a kind}) \simeq 0.0008.$$

35. You are lost in the National Park of **Bandrika**†. Tourists comprise two-thirds of the visitors to the park, and give a correct answer to requests for directions with probability $\frac{3}{4}$. (Answers to repeated questions are independent, even if the question and the person are the same.) If you ask a Bandrikan for directions, the answer is always false.

(a) You ask a passer-by whether the exit from the Park is East or West. The answer is East. What is the probability this is correct?

(b) You ask the same person again, and receive the same reply. Show the probability that it is correct is $\frac{1}{2}$.

(c) You ask the same person again, and receive the same reply. What is the probability that it is correct?

(d) You ask for the fourth time, and receive the answer East. Show that the probability it is correct is $\frac{27}{70}$.

(e) Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is $\frac{9}{10}$.

36. Mr Bayes goes to Bandrika. Tom is in the same position as you were in the previous problem, but he has reason to believe that, with probability ϵ, East is the correct answer. Show that:

(a) whatever answer first received, Tom continues to believe that East is correct with probability ϵ,

(b) if the first two replies are the same (that is, either WW or EE), Tom continues to believe that East is correct with probability ϵ,

(c) after three like answers, Tom will calculate as follows, in the obvious notation:

$$\mathbb{P}(\text{East correct} \mid \text{EEE}) = \frac{9\epsilon}{11 - 2\epsilon}, \quad \mathbb{P}(\text{East correct} \mid \text{WWW}) = \frac{11\epsilon}{9 + 2\epsilon}.$$

Evaluate these when $\epsilon = \frac{9}{20}$.

†A fictional country made famous in the Hitchcock film 'The Lady Vanishes'.