Mathematics Monograph Series 24

Global Superconvergence of Finite Elements for Elliptic Equations and Its Applications

Zi-Cai Li   Hung-Tsai Huang   Ningning Yan

（椭圆方程有限元方法的整体超收敛及其应用）

SCIENCE PRESS
Beijing

Mathematics Monograph Series 24

Global Superconvergence of Finite Elements for Elliptic Equations and Its Applications

(椭圆方程有限元方法的整体超收敛及其应用)

Zi-Cai Li   Hung-Tsai Huang   Ningning Yan

SCIENCE PRESS
Beijing

Responsible Editor: Yuzhuo Chen

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-033479-4

This book is dedicated to

Qun Lin and Michal Křížek

Preface

The *superconvergence* was first studied by Douglas and Dupont[55] in 1972. However, Oganesjan and Ruchovec[191] had already proved in 1969 that the linear interpolation on uniform triangular meshes is supercloseness to the finite element solution of second order elliptic problems. Since then many papers on superconvergence have been published. In summary, there are three types of superconvergence: *locally pointwise*, *average* and *global*. There exist many reports on superconvergence at special points (i.e., *locally pointwise*), see Hannukainen et al.[78], MacKinnon and Carey[181, 182], Nakao[189], Pehlivanov et al.[195], Wheeler and Whiteman[211] and Wahlbin[207, 208]. In Křížek and Neittaanmäki[103], superconvergence on average (or majority) of the nodal derivative solutions was introduced for Poisson's equation. The *global superconvergence* over the entire solution domain was proposed in Křížek and Neittaanmäki[104], Lin and Yan[161] and Lin et al.[163], and the plentiful bibliography on superconvergence was listed in Křížek and Neittaanmäki[105].

In Wahlbin[207, 208] a systematic analysis of locally pointwise superconvergence was given to general Galerkin finite elements, based on local error estimates and the Green functions. The analysis seems to be very complicated, and to suit only for the second order elliptic equations. Wahlbin's book[208] did not cover analysis for the fourth order elliptic problems, parabolic equations and systems of partial differential equations. The a posteriori processing is important not only to global superconvergence, see Lin and Zhu[170], but also to locally pointwise superconvergence, see Zienkiewicz and Zhu[243, 244], Babuška and Strouboulis[11], and Zhang[234, 235, 236]. For derivatives, the average technique, the integration recovery technique and the derivative patch interpolating recovery technique were proposed in Zhang[230] and Zhang et al.[231, 232, 233], for FEM to reach superconvergence and even ultraconvergence of numerical derivatives at mesh nodes. For 3D problems, more results on locally pointwise superconvergence were provided in Křížek[102]. Numerically, one superconvergence phenomenon for 3D elastic problems was observed by Xie [215] in 1975.

In general, the finite element methods (FEMs) are constructed in any partition finite space. In this book we discuss the rectangular elements, in particular, Adini's elements and p-order bi-Lagrange elements, which are constructed in rectangular finite space, see Lin and Yan[161]. The special rectangular partitions enable the finite element solutions more efficient to raise the convergence rates, although the rectangular elements are less flexible than the triangular elements. The convergence rate of

$\|u_h - u_{\rm I}\|_1$ is one or two orders higher than the optimal convergence rates. For post-processings we construct higher order solutions. Post-processing has the advantage that it can yield superconvergence result anywhere in the domain and even up to the boundaries, see [104, 161, 163]. However, post-processing techniques might be a little expensive because some of the processing has to be done over the whole domain. In this book we give the a posteriori interpolant formulas of Adini's elements and bi-quadratic elements to obtain the global superconvergence. The rectangular elements have the disadvantage that the solution domains are confined to those which can be decomposed of finite rectangles. This is a limitation for practical application due to less flexibility. In philosophy, *specialization rewards efficiency.* Rectangular elements can yield better superconvergence than triangular elements do, see [161].

The global superconvergence does not display much significance for the triangular FEMs, because only linear, quadratic and some cubic elements have been done, based on the analysis for $\|u_h - u_{\rm I}\|_{1,S}$ in [161]. In contrast, for locally pointwise superconvergence, all high order triangular elements can be analyzed systematically by the techniques of Wahlbin[208]. A remedy for this drawback of global superconvergence is to combine rectangular elements with triangular elements if necessary as shown in Li[114]. An important application of superconvergence is for a posteriori estimates of the numerical solutions obtained, and then for refining the partition to achieve a required accuracy, see Babuška and Strouboulis[11]. An example of global superconvergence is given in Li and Yan [146]. The global superconvergence in this book is confined to 2D problems only. For 3D problems, the global superconvergence of tetrahedral quadratic finite elements is explored in Brandts and Křížek[24], to reach $O(h^3)$ in H^1-norm.

An intrinsic feature of global superconvergence is the requirement of highly smooth solutions. In the analysis of this book, we need the solutions $u \in H^3(S)$ for the bilinear FEM and the FDM, $u \in H^5(S)$ for Adini's elements, and $u \in H^6(S)$ for the bi-quadratic elements. A discussion on $u \in H^3(S)$ is given in Křížek and Neittaanmäki[104]. It is well known that the solution of Laplace's equation on a concave domain satisfies $u \in H^{1+r}(S)$, where $0 < r < 1$. Even by the simplest linear elements, the poor solution with the H^1-norm error $O(h^r)$ can be obtained. Hence, many techniques have been developed to pursue the optimal convergence rate $O(h)$, see Li [114]. The reason for the global superconvergence rates to raise the convergence rates lies in that the bilinear elements (i.e., low order FEMs) will play a role of bi-quadratic elements (i.e., high order FEMs), by an a posteriori interpolant of bi-quadratic polynomials (high order interpolants) from the obtained solutions. Hence, the smoothness of bi-quadratic elements is requested for the bilinear elements in global superconvergence, see Chapter 1. All high order FEMs share the same drawback requirement of high regularity of the solution, and so does the global superconvergence. This book can be, indeed, regarded as a deep analysis in the rectangular elements, e.g., bilinear, bi-quadratic and Adini's elements for Poisson's equation, and bi-cubic Hermite elements for biharmonic equations. Although the requirement of $u \in H^6(S)$ is un-realistic, the surprisingly results of bi-quadratic elements may reach the superconvergence $O(h^5)$ in H^1-norm, and $O(h^{4-\delta})$, $0 < \delta \ll 1$, in the infinite norm, see Chapter 3. In general, the solution inside S is highly smooth, and the superconvergence is then valid wherein. When the inside subdomain is vast, a combination may be employed to retain the same superconvergence, see Chapters 1, 4, 8 and 10. Such a situation is just analogous to the case of the combined methods for singularity problems, see Li[114].

Several books on global superconvergence were written, e.g., Lin and Yan[161] with the updated version Yan[220], Chen and Huang[41], Chen[40], Lin and Zhu[170], Zhu and Lin[242], Yang[224], and Lin and Lin[154], to provide wide theoretical results of global superconvergence. Recently, new developments of global superconvergence in both theory and application are reported in [24, 29, 43, 106, 158, 159, 160, 162, 166, 167, 168, 169, 172, 173, 174, 175, 240, 241]. More references on the global superconvergence are provided in Yan[220], and Křížek and Neittaanmäki[103]. In this book some important literature may not be mentioned, for it we apologize in advance.

Our study on superconvergence has been greatly influenced by [154, 161], and this book is a further development of [154, 161]. Compared with [154, 161], this book has several different features.

(1) We report new discoveries of global superconvergence for Poisson's equation and biharmonic equation, eigenvalue problems and semilinear problems. The materials in this book are mainly summarized from our recent journal papers published in [45, 86, 88, 91, 92, 93, 115, 116, 117, 118, 119, 123, 125, 126, 127, 128, 129, 130, 131, 132, 133, 140, 141, 145, 146, 147, 221]. Only the most important results of global superconvergence in both theory and numerical verification are covered in this book.

(2) For the basic theory of global superconvergence, deeper and rather completed results are explored in Chapters 2, 3, 6, 9, etc., which are important to practical applications. Besides, the superconvergence of finite difference method is also provided in Chapter 5, where the discrete H^1-norms are used, which is similar to global superconvergence, and can also be viewed as the average superconvergence.

(3) The global superconvergence can be applied to singularity problems by the combined method in Li[114], and reported in Chapter 1 and Chapter 4. The coupling techniques in [114] are employed in Chapters 1, 4, 7, 8 and 10. The periodical boundary conditions and the semilinear problems are also studied by global superconvergence in Chapter 10 and Chapter 11.

(4) Numerical examples of models and application problems are provided, to verify the superconvergence and to display its significance in computation.

(5) The description of all chapters except Chapter 12 is problem-oriented, and each chapter can be regarded as an independent study. In fact, each chapter is adapted from one to several published papers. In general, the proofs in the error bounds for global superconvergence are tedious, based on repeated integration by parts. Our efforts are paid to seek rather simple proofs which are easy for others to follow. Hence, reader may read one's interested chapters without reading the entire book. It is our hope that the techniques and results of global superconvergence in this book may be extended to more problems in application.

A brief chapter-by-chapter description is given as follows.

Chapter 1 is an introduction to discuss the bilinear elements combined with the Ritz-Galerkin method using the singular solutions, to solve the Poisson equation with singularities. The simplified hybrid techniques are employed, to couple two methods along their common boundary, and the global supercloseness $O(h^{2-\delta})$, $0 < \delta \ll 1$, is derived, where h is the maximal meshspacing of quasiuniform rectangles used. By means of an a posteriori interpolation, the global superconvergence $O(h^{2-\delta})$ can also be achieved. This chapter presents a basic approach, to expose the global superconvergence and its application. Since the rectangular elements are not flexible, they should be combined with other methods, e.g., the triangular elements for rather arbitrary domains, or the singular functions for singularity problems.

In Chapters 2–5, as a fundamental level, we consider the Poisson equation by Adini's elements and the bi-quadratic Lagrange elements in Chapters 2–4, and by the finite difference method in Chapter 5.

Chapter 2 discusses new error estimates of Adini's elements for the Neumann problems of Poisson's equation. A new technique is given for the Neumann boundary problem, and new a posteriori interpolant polynomials of Adini's elements are provided which are the important posterior steps of global superconvergence.

Chapter 3 achieves new error estimates of biquadratic Lagrange elements for Poisson's equation (i.e., $-\Delta u = f$). The a posteriori interpolant polynomials with order six of the *point-line-area* variables are found. Moreover, by using the a posteriori interpolant with order six, the global superconvergence of biquadratic Lagrange elements may achieve $O(h^5)$ in H^1-norm and $O(h^6)$ in L^2-norm high, under the case of $f_{xxyy} = 0$ and $h = k$, where h and k are the boundary lengths of uniform rectangles $\square_{ij}$. When $f_{xxyy} \neq 0$, the same high global superconvergences may be retained by the extrapolation technique. The numerical results verify supercloseness $O(h^4)$ and $O(h^5)$, global superconvergence $O(h^5)$ in H^1-norm and the high convergence rates $O(h^6|\ln h|) = O(h^{6-\epsilon})$, $0 < \epsilon \ll 1$ in the infinity norm.

Chapter 4 is a continued study of Chapter 1 for the simplified hybrid combinations of the Ritz-Galerkin method and finite element methods. In Chapter 1, a basic error theorem is derived for the solution, which leads to superconvergence rates on the entire solution domain; however, only bilinear elements are discussed. In this chapter, we intend to achieve high global superconvergence by using high order Lagrange FEMs and Adini's elements in the regular subdomain.

Chapter 5 explores the superconvergence of the average nodal derivates of the finite difference method (FDM). Consider the singular boundary, where the Poisson solution is expressed as $u = O(x^\sigma)$, $\frac{1}{2} < \sigma < 2$, as $x \to 0$. By using a local refinement of difference grids near the axis y, the superconvergence $\overline{\|u - u_h\|}_1 = O(h^2)$ can also be reached, where $\overline{\|\cdot\|}_1$ is the discrete H^1-norm, h is the maximal meshspacing of the difference grids, and u_h is the FDM solution. Since the norm $\overline{\|u - u_h\|}_1$ and $\|u_I - u_h\|_{1,S}$ are equivalent to each other (see Li[114]), where u_I is the bilinear interpolant of u, the superconvergence of average nodal derivatives in Chapter 5 may be classified into global superconvergence.

In Chapters 6–9, as an advanced level of global superconvergence, we discuss the biharmonic equations with different boundary conditions by the bi-cubic Hermite functions on rectangular elements. More mathematical arguments, which are basically of the integration computation in calculus, are needed due to complexity of the biharmonic equations. In Chapters 6–8, the basic error estimates, stability and application of blending surfaces are explored. In Chapter 9, the eigenvalues of Laplace's operator are explored by nonconforming elements in Lin and Lin [154] with the focus of lower bounds.

Chapter 6 develops the basic error estimates in detail. The useful techniques for integration evaluation are illustrated, and easily understood. In Lin and Yan [161], the error estimates were given only for the clamped boundary condition. In this chapter, other popular boundary conditions, such as the simply supported condition, the natural condition and the symmetric condition are also discussed. The new global superconvergence has been verified by the numerical experiments, to provide a complete knowledge of the bi-cubic Hermite elements.

Chapter 7 makes a stability analysis for the blending problems by the bi-cubic Hermite elements with the hybrid plus penalty coupling for periodical boundary condition. The bounds of the condition number, $\text{Cond}(\boldsymbol{A}) = \lambda_{\max}(\boldsymbol{A})/\lambda_{\min}(\boldsymbol{A})$, are derived, where $\boldsymbol{A}$ is the associated matrix, which is symmetric and positive definite, and $\lambda_{\max}(\boldsymbol{A})$ and $\lambda_{\min}(\boldsymbol{A})$ are the maximal and the minimal eigenvalues of matrix $\boldsymbol{A}$, respectively.

Chapter 8 applies the bi-cubic Hermite elements in Chapter 6 to the blending surfaces by using some coupling techniques in Li [114], to match the surfaces to the outside boundary, which may be complicated. The analysis of this chapter displays *that not only is the global superconvergence retained, but also a better stability can* be achieved. This chapter also demonstrates a flexibility of global superconvergence in practical applications.

Chapter 9 explores new expansions of eigenvalues for $-\Delta u = \lambda \rho u$ in S with the Dirichlet boundary condition $u = 0$ on ∂S, by the bilinear element(denoted Q_1) and three nonconforming elements: the rotated bilinear element (denoted Q_1^{rot}), the extension of Q_1^{rot} (denoted EQ_1^{rot}) and Wilson's element. The expansions indicate that Q_1 and Q_1^{rot} provide upper bounds of the eigenvalues, and that EQ_1^{rot} and Wilson's elements provide lower bounds of the eigenvalues. By the extrapolation, the $O(h^4)$ convergence rate can be obtained, where h is the boundary length of uniform rectangles. Since upper bounds of eigenvalues are always obtained from conforming elements, low bounds of eigenvalues from EQ_1^{rot} and Wilson's element are interesting and important in application.

Chapter 10 and Chapter 11 apply the basic superconvergence to important application problems in physics and engineering. Chapter 10 applies the Adini's elements for nonlinear Schrödinger equations (NLS) defined in a square box with periodic boundary conditions. First, Adini's elements are applied to Laplace's eigenvalue problems in the unit square with periodical boundary conditions, and the leading eigenvalues are obtained from the Rayleigh quotient. The coupling techniques are developed to copy with periodical boundary conditions, and superconvergence is also explored for leading eigenvalues. The optimal convergence $O(h^6)$ of leading eigenvalues is obtained for quasi-uniform elements[10, 204]. When the uniform rectangular elements are used, the superconvergence $O(h^{6+p})$ with $p = 1$ or $p = 2$ of leading eigenvalues is proved, where h is the boundary length of Adini's elements.

Chapter 11 studies finite element approximations for positive solutions of semilinear elliptic eigenvalue problems with folds, to explore the superconvergence of finite element methods (FEMs). To apply the superconvergence of FEM for Poisson's equation in Chapters 2, 3 and 5 to parameter-dependent problems with folds, this chapter provides the framework of error analysis, accompanied with the proof of the strong monotonicity of the nonlinear problem. It is worthy pointing out that the superconvergence of nonlinear problems in this chapter is different from that in Chen and Huang[41]. A continuation algorithm is described to trace solution curves of semilinear elliptic eigenvalue problems, where Adini's elements are employed to discretize the PDEs.

The last Chapter 12 states the basic ideas and proof techniques of global superconvergence with bilinear elements, and summaries important results of integral identity analysis and global superconvergence for general equations by different elements. The description of Chapters 1−11 is problem-oriented; the summary of this chapter is method-oriented. Hence, the techniques and results of global superconvergence may be extended to more problems of wide applications.

Zi-Cai Li
Hung-Tsai Huang
Ningning Yan

March 2011

Acknowledgements

This monograph displays the collaborated research between Chinese Academy of Sciences, Beijing and Sun Yat-sen University, Kaohsiung for global superconvergence of numerical methods for elliptic equations. In April 1997, Professor Q. Lin visited Sun Yat-sen University, Kaohsiung for one month, and introduced his new book[161]. Since then Z.C. Li has studied global superconvergence. Professor N. Yan visited Sun Yat-sen University, Kaohsiung from March 2000 for nearly five months. In June 2001, Professor A. Zhou also visited Sun Yat-sen University, Kaohsiung, and a co-authored paper was published in [93] which is adapted in Chapter 3 in this book. From September 1999 to June 2003, H.T. Huang was a Ph.D. student under supervision by Z.C. Li, and was engaged to study the global superconvergence[85]. Later, Z.C. Li and H.T. Huang visited Q. Lin and N. Yan in Beijing several times.

We are sincerely appreciated to Professors Q. Lin and M. Křížek for their encouragements and supports during our study of global superconvergence. We are also grateful to Professors Q. Lin, M. Křížek, T. Yamamoto, A. Zhou, H. Xie, S. Zhang, C. S. Chien, T.T. Lu, Y. Wei, X.P. Xie, T. Zhang and Q. Fang for their collaboration or valuable suggestions on the research works related to this book.

We are obliged to Y.Z. Chen and the editorial staff at Science Press, Beijing for their support and assistance in publishing this book. We also wish to thank Professors H.D. Yang, H.K. Xu and N.C. Wong at Sun Yat-sen University, Taiwan, for their support during the writing process.

This book and the related works were supported in part by the Science Council of Taiwan, the National Basic Research Program of China under the Grant 2010CB731505 and the National Natural Science Foundation of China under Grant 11171337.

Contents

Chapter 1

Basic Approaches

To solve the elliptic boundary value problems with singularities, the simplified hybrid combinations of the Ritz-Galerkin and finite element methods (simply written as RGM-FEM) are explored, and the *global* superconvergence rates are derived on the entire solution domain, based on a posteriori interpolation techniques of global superconvergence (see Chapter 12), which only cost a little more computation. Let the solution domain be divided by Γ_0 into two subdomains S_1 and S_2: $S = S_1 \cup S_2 \cup \Gamma_0$ and $S_1 \cap S_2 = \varnothing$. Suppose that S_1 can be partitioned into quasiuniform rectangles: $S_1 = \sum_{ij} \Box_{ij}$, a singular point occurs at ∂S_2, and the singular functions are chosen in S_2. Then for bilinear elements, it is proven that the simplified hybrid combinations of RGM-FEM can provide the global superconvergence rate $O(h^{2-\delta})$, $0 < \delta \ll 1$ for solution gradients over the entire subdomains S_1 and S_2, where h is the maximal boundary length of $\Box_{ij}$. Note that numerical stability of the simplified hybrid combinations of RGM-FEM is also optimal[121]. This chapter presents the important results for the general case of Poisson-problems on a polygonal domain S, where the error estimates for the Sobolev norm $\| \cdot \|_1$ are given in *a much more general sense* than known before, cf. [52, 103, 104, 113, 182, 189, 195, 207, 211]. The materials of this chapter are adapted from [117].

1.1 Introduction

In this chapter, the *global* superconvergence rates of solution gradients on the entire solution domain are established by combinations of the Ritz-Galerkin and finite element methods (RGM-FEM). There exist many reports on superconvergence, such as Křížek and Neittaanmäki[103, 104], MacKinnon and Carey[182], Nakao[189], Pekhlivanov et al.[195], Wahlbin[207], and Wheeler and Whiteman[211]. Most of them deal with superconvergence at specific points. In this chapter, the global superconvergence rates are obtained over the entire region, based on a posteriori interpolation of the numerical solutions proposed by Lin and his colleagues[149, 161, 165, 220], see Chapter 12.

A recent study on superconvergence in combinations is reported in [113]. where