PEARSON

高等学校数学双语教学推荐教材

微积分

（下册）

Calculus

威廉·布里格斯（William Briggs）
莱尔·科克伦（Lyle Cochran） 著
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北京

图书在版编目（CIP）数据

微积分（下册）：英文/（美）布里格斯等著. —北京：中国人民大学出版社，2012.6
高等学校数学双语教学推荐教材
ISBN 978-7-300-15940-9

Ⅰ.①微… Ⅱ.①布… Ⅲ.①微积分-双语教学-高等学校-教材-英文 Ⅳ.①O172

中国版本图书馆 CIP 数据核字（2012）第 119841 号

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出版发行	中国人民大学出版社		
社　　址	北京中关村大街 31 号	**邮政编码**	100080
电　　话	010－62511242（总编室）		010－62511398（质管部）
	010－82501766（邮购部）		010－62514148（门市部）
	010－62515195（发行公司）		010－62515275（盗版举报）
网　　址	http：//www.crup.com.cn		
	http：//www.ttrnet.com（人大教研网）		
经　　销	新华书店		
印　　刷	涿州市星河印刷有限公司		
规　　格	215 mm×275 mm　16 开本	**版　　次**	2012 年 6 月第 1 版
印　　张	34.25 插页 1	**印　　次**	2012 年 6 月第 1 次印刷
字　　数	968 000	**定　　价**	68.00 元

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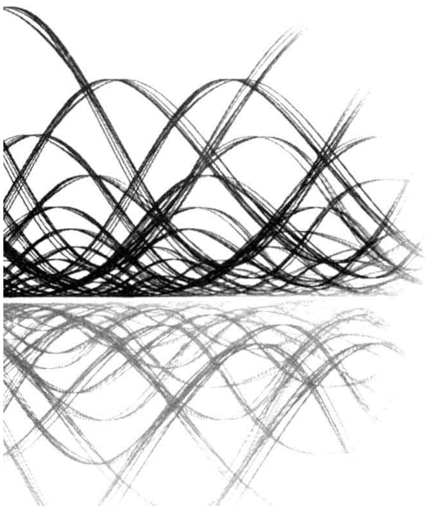

9

Sequences and Infinite Series

Chapter Preview This chapter covers topics that lie at the foundation of calculus—indeed, at the foundation of mathematics. The first task is to make a clear distinction between a *sequence* and an *infinite series*. A sequence is an ordered *list* of numbers, $a_1, a_2, \ldots$, while an infinite series is a *sum* of numbers, $a_1 + a_2 + \cdots$. The idea of convergence to a limit is important for both sequences and series, but convergence is analyzed differently in the two cases. To determine limits of sequences, we use the same tools used for limits at infinity of functions. Convergence of infinite series is a different matter, and we develop the required methods in this chapter. The study of infinite series begins with the ubiquitous *geometric series*; it has theoretical importance and it is used to answer many practical questions (When is your auto loan paid off? How much antibiotic do you have in your blood if you take three pills a day?). We then present several tests that are used to determine whether series with positive terms converge. Finally, alternating series, whose terms alternate in sign, are discussed in anticipation of power series in the next chapter.

9.1 An Overview

➤ Keeping with common practice, the terms *series* and *infinite series* are used interchangeably throughout this chapter.

To understand sequences and series, you must understand how they differ and how they are related. The purposes of this opening section are to introduce sequences and series in concrete terms and to illustrate their differences and their crucial relationships with each other.

Examples of Sequences

➤ The dots ($\ldots$) after the last number (called an *ellipsis*) mean that the list goes on indefinitely.

Consider the following *list* of numbers:

$$\{1, 4, 7, 10, 13, 16, \ldots\}$$

Each number in the list is obtained by adding 3 to the previous number. With this rule, we could extend the list indefinitely.

This list is an example of a **sequence**, where each number in the sequence is called a **term** of the sequence. We denote sequences in any of the following forms:

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} \qquad \{a_n\}_{n=1}^{\infty} \qquad \{a_n\}$$

The subscript n that appears in a_n is called an **index**, and it indicates the order of terms in the sequence. The choice of a starting index is arbitrary, but sequences usually begin with $n = 0$ or $n = 1$.

The sequence $\{1, 4, 7, 10, \ldots\}$ can be defined in two ways. First, we have the rule that each term of the sequence is 3 more than the previous term; that is, $a_2 = a_1 + 3$, $a_3 = a_2 + 3$, $a_4 = a_3 + 3$, and so forth. In general, we see that

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + 3, \qquad \text{for } n = 1, 2, 3, \ldots.$$

This way of defining a sequence is called a **recurrence relation** (or an **implicit formula**). It specifies the initial term of the sequence (in this case, $a_1 = 1$) and gives a general rule for computing the next term of the sequence from previous terms. For example, if you know a_{100}, the recurrence relation can be used to find a_{101}.

Suppose instead you want to find a_{147} directly without computing the first 146 terms of the sequence. The first four terms of the sequence can be written

$$a_1 = 1 + (3 \cdot 0), \qquad a_2 = 1 + (3 \cdot 1), \qquad a_3 = 1 + (3 \cdot 2), \qquad a_4 = 1 + (3 \cdot 3).$$

Observe the pattern: The nth term of the sequence is 1 plus 3 multiplied by $n - 1$, or

$$a_n = 1 + 3(n - 1) = 3n - 2, \qquad \text{for } n = 1, 2, 3, \ldots$$

QUICK CHECK 1 Find a_{10} for the sequence $\{1, 4, 7, 10, \ldots\}$ using the recurrence relation and then again using the explicit formula for the nth term.

With this **explicit formula**, the nth term of the sequence is determined directly from the value of n. For example, with $n = 147$,

$$\underbrace{a_{147}}_{n} = 3 \cdot \underbrace{147}_{n} - 2 = 439.$$

➤ When defined by an explicit formula $a_n = f(n)$, it is evident that sequences are functions. The domain is the set of positive, or nonnegative, integers, and one real number a_n is assigned to each integer in the domain.

DEFINITION Sequence

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

A sequence may be generated by a **recurrence relation** of the form $a_{n+1} = f(a_n)$, for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an **explicit formula** for the nth term in the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

EXAMPLE 1 Explicit formulas Use the explicit formula for $\{a_n\}_{n=1}^{\infty}$ to write the first four terms of each sequence. Sketch a graph of the sequence.

a. $a_n = \dfrac{1}{2^n}$ **b.** $a_n = \dfrac{(-1)^n n}{n^2 + 1}$

SOLUTION

a. Substituting $n = 1, 2, 3, 4, \ldots$ into the explicit formula $a_n = \dfrac{1}{2^n}$, we find that the terms of the sequence are

$$\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}.$$

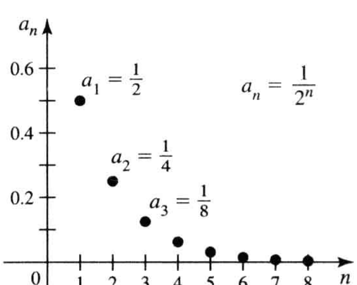

FIGURE 9.1

The graph of a sequence is like the graph of a function that is defined only on a set of integers. In this case, we plot the coordinate pairs (n, a_n) for $n = 1, 2, 3, \ldots$, resulting in a graph consisting of individual points. The graph of the sequence $a_n = \dfrac{1}{2^n}$ suggests that the terms of this sequence approach 0 as n increases (Figure 9.1).

b. Substituting $n = 1, 2, 3, 4, \ldots$ into the explicit formula, the terms of the sequence are

➤ The "switch" $(-1)^n$ is used frequently to alternate the signs of the terms of sequences and series.

$$\left\{\frac{(-1)^1(1)}{1^2 + 1}, \frac{(-1)^2 2}{2^2 + 1}, \frac{(-1)^3 3}{3^2 + 1}, \frac{(-1)^4 4}{4^2 + 1}, \ldots\right\} = \left\{-\frac{1}{2}, \frac{2}{5}, -\frac{3}{10}, \frac{4}{17}, \ldots\right\}.$$

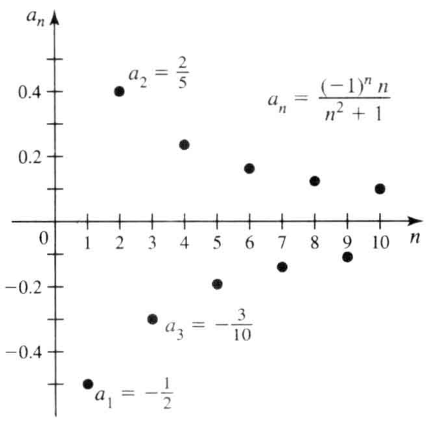

FIGURE 9.2

From the graph (Figure 9.2), we see that the terms of the sequence alternate in sign and appear to approach 0 as n increases. *Related Exercises 9–12* ◂

EXAMPLE 2 **Recurrence relations** Use the recurrence relation for $\{a_n\}_{n=1}^{\infty}$ to write the first four terms of the sequences

$$a_{n+1} = 2a_n + 1, a_1 = 1 \quad \text{and} \quad a_{n+1} = 2a_n + 1, a_1 = -1.$$

SOLUTION Notice that the recurrence relation is the same for the two sequences; only the first term differs. The first four terms of the two sequences are as follows.

n	a_n with $a_1 = 1$	a_n with $a_1 = -1$
1	$a_1 = 1$ (given)	$a_1 = -1$ (given)
2	$a_2 = 2a_1 + 1 = 2 \cdot 1 + 1 = 3$	$a_2 = 2a_1 + 1 = 2(-1) + 1 = -1$
3	$a_3 = 2a_2 + 1 = 2 \cdot 3 + 1 = 7$	$a_3 = 2a_2 + 1 = 2(-1) + 1 = -1$
4	$a_4 = 2a_3 + 1 = 2 \cdot 7 + 1 = 15$	$a_4 = 2a_3 + 1 = 2(-1) + 1 = -1$

We see that the terms of the first sequence increase without bound, while all terms of the second sequence are -1. Clearly, the initial term of the sequence has a lot to say about the behavior of the entire sequence. *Related Exercises 13–16* ◂

QUICK CHECK 2 Find an explicit formula for the sequence $\{1, 3, 7, 15, \dots\}$ (Example 2). ◂

EXAMPLE 3 **Working with sequences** Consider the following sequences.

a. $\{a_n\} = \{-2, 5, 12, 19, \dots\}$ **b.** $\{b_n\} = \{3, 6, 12, 24, 48, \dots\}$

(i) Find the next two terms of the sequence.
(ii) Find a recurrence relation that generates the sequence.
(iii) Find an explicit formula for the nth term of the sequence.

SOLUTION

a. (i) Each term is obtained by adding 7 to its predecessor. The next two terms are $19 + 7 = 26$ and $26 + 7 = 33$.

(ii) Because each term is seven more than its predecessor, the recurrence relation is

$$a_{n+1} = a_n + 7, a_0 = -2, \quad \text{for } n = 0, 1, 2, \dots$$

▸ In Example 3, we chose the starting index to be $n = 0$. Other choices are possible.

(iii) Notice that $a_0 = -2$, $a_1 = -2 + (1 \cdot 7)$, and $a_2 = -2 + (2 \cdot 7)$, so the explicit formula is

$$a_n = 7n - 2, \quad \text{for } n = 0, 1, 2, \dots.$$

b. (i) Each term is obtained by multiplying its predecessor by 2. The next two terms are $48 \cdot 2 = 96$ and $96 \cdot 2 = 192$.

(ii) Because each term is two times its predecessor, the recurrence relation is

$$a_{n+1} = 2a_n, a_0 = 3, \quad \text{for } n = 0, 1, 2, \dots$$

(iii) To obtain the explicit formula, note that $a_0 = 3$, $a_1 = 3(2^1)$, and $a_2 = 3(2^2)$. In general,

$$a_n = 3(2^n), \quad \text{for } n = 0, 1, 2, \dots.$$

Related Exercises 17–22 ◂

Limit of a Sequence

Perhaps the most important question about a sequence is this: If you go farther and farther out in the sequence, $a_{100}, \ldots, a_{10,000}, \ldots, a_{100,000}, \ldots$, how do the terms of the sequence behave? Do they approach a specific number, and if so, what is that number? Or do they grow in magnitude without bound? Or do they wander around with or without a pattern?

The long-term behavior of a sequence is described by its **limit**. The limit of a sequence is defined rigorously in the next section. For now, we work with an informal definition.

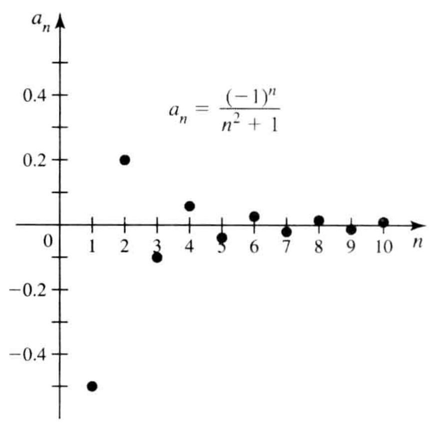

FIGURE 9.3

DEFINITION Limit of a Sequence

If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim_{n \to \infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

EXAMPLE 4 Limit of a sequence Write the first four terms of each sequence. If you believe the sequence converges, make a conjecture about its limit. If the sequence appears to diverge, explain why.

a. $\left\{\frac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty}$ Explicit formula

b. $\{\cos(n\pi)\}_{n=1}^{\infty}$ Explicit formula

c. $\{a_n\}_{n=1}^{\infty}$, where $a_{n+1} = -2a_n$, $a_1 = 1$ Recurrence relation

SOLUTION

a. Beginning with $n = 1$, the first four terms of the sequence are

$$\left\{\frac{(-1)^1}{1^2+1}, \frac{(-1)^2}{2^2+1}, \frac{(-1)^3}{3^2+1}, \frac{(-1)^4}{4^2+1}, \ldots\right\} = \left\{-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, \ldots\right\}.$$

The terms decrease in magnitude and approach zero with alternating signs. The limit appears to be 0 (Figure 9.3).

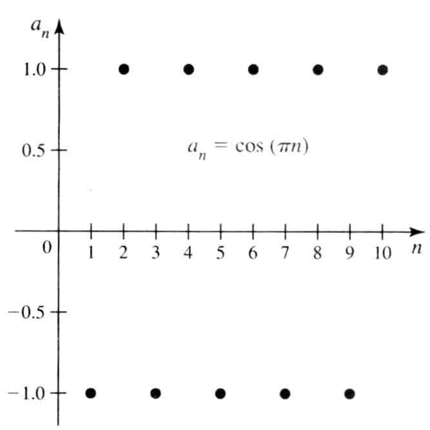

FIGURE 9.4

b. The first four terms of the sequence are

$$\{\cos \pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \ldots\} = \{-1, 1, -1, 1, \ldots\}.$$

In this case, the terms of the sequence alternate between -1 and $+1$, and never approach a single value. Thus, the sequence diverges (Figure 9.4).

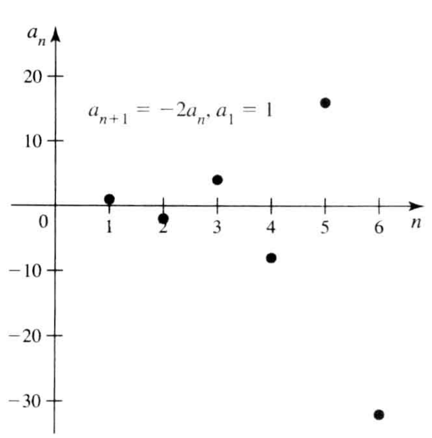

FIGURE 9.5

c. The first four terms of the sequence are

$$\{1, -2a_1, -2a_2, -2a_3, \ldots\} = \{1, -2, 4, -8, \ldots\}.$$

Because the magnitudes of the terms increase without bound, the sequence diverges (Figure 9.5).

Related Exercises 23–30 ◀

EXAMPLE 5 Limit of a sequence Enumerate and graph the terms of the following sequence and make a conjecture about its limit.

$$a_n = \frac{4n^3}{n^3+1}, \quad \text{for } n = 1, 2, 3, \ldots \quad \text{Explicit formula}$$

SOLUTION The first 14 terms of the sequence $\{a_n\}$ are tabulated in Table 9.1 and graphed in Figure 9.6. The terms appear to approach 4.

Table 9.1

n	a_n	n	a_n
1	2.000	8	3.992
2	3.556	9	3.995
3	3.857	10	3.996
4	3.938	11	3.997
5	3.968	12	3.998
6	3.982	13	3.998
7	3.988	14	3.999

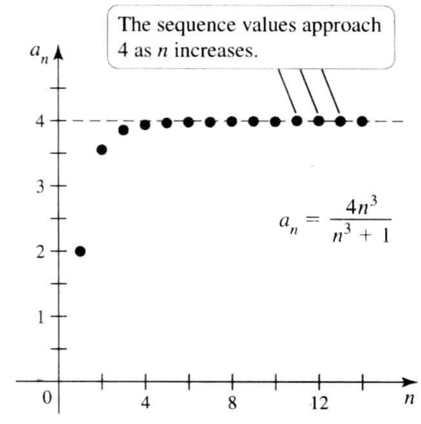

FIGURE 9.6

Related Exercises 31–44 ◄

EXAMPLE 6 **A bouncing ball** A basketball tossed straight up in the air reaches a high point and falls to the floor. Assume that each time the ball bounces on the floor it rebounds to 0.8 of its previous height. Let h_n be the high point after the nth bounce, with the initial height being $h_0 = 20$ ft.

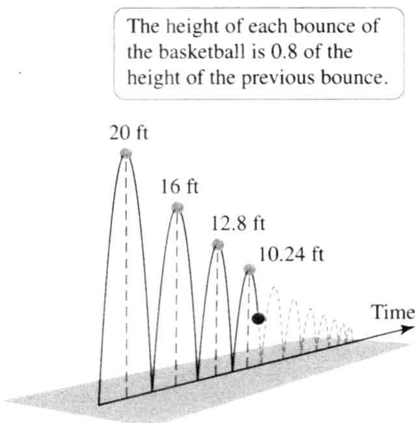

FIGURE 9.7

a. Find a recurrence relation and an explicit formula for the sequence $\{h_n\}$.

b. What is the high point after the 10th bounce? after the 20th bounce?

c. Speculate on the limit of the sequence $\{h_n\}$.

SOLUTION

a. We first write and graph the heights of the ball for several bounces using the rule that each height is 0.8 of the previous height (Figure 9.7). For example, we have

$$\begin{aligned} h_0 &= 20 \text{ ft} \\ h_1 &= 0.8\,h_0 = 16 \text{ ft} \\ h_2 &= 0.8\,h_1 = 0.8^2\,h_0 = 12.80 \text{ ft} \\ h_3 &= 0.8\,h_2 = 0.8^3\,h_0 = 10.24 \text{ ft} \\ h_4 &= 0.8\,h_3 = 0.8^4\,h_0 \approx 8.19 \text{ ft.} \end{aligned}$$

Each number in the list is 0.8 of the previous number. Therefore, the recurrence relation for the sequence of heights is

$$h_{n+1} = 0.8\,h_n, \qquad \text{for } n = 0, 1, 2, 3, \ldots, h_0 = 20 \text{ ft.}$$

To find an explicit formula for the nth term, note that

$$h_1 = h_0 \cdot 0.8, \qquad h_2 = h_0 \cdot 0.8^2, \qquad h_3 = h_0 \cdot 0.8^3, \qquad \text{and} \qquad h_4 = h_0 \cdot 0.8^4.$$

In general, we have

$$h_n = h_0 \cdot 0.8^n = 20 \cdot 0.8^n, \qquad \text{for } n = 0, 1, 2, 3, \ldots,$$

which is an explicit formula for the terms of the sequence.

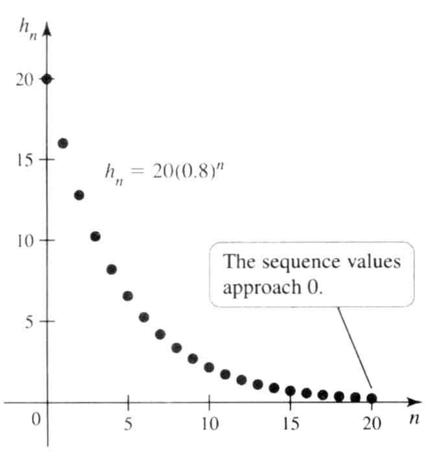

FIGURE 9.8

b. Using the explicit formula for the sequence, we see that after $n = 10$ bounces, the next height is

$$h_{10} = 20 \cdot 0.8^{10} \approx 2.15 \text{ ft.}$$

After $n = 20$ bounces, the next height is

$$h_{20} = 20 \cdot 0.8^{20} \approx 0.23 \text{ ft.}$$

c. The terms of the sequence (Figure 9.8) appear to decrease and approach 0. A reasonable conjecture is that $\lim_{n \to \infty} h_n = 0$.

Related Exercises 45–48 ◄

Infinite Series and the Sequence of Partial Sums

An infinite series can be viewed as a sum of an infinite set of numbers; it has the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k,$$

where the terms of the series, $a_1, a_2, \ldots$, are real numbers. *An infinite series is quite distinct from a sequence.* We first answer the question: How is it possible to sum an infinite set of numbers and produce a finite number? Here is an informative example.

Consider a unit square (sides of length 1) that is subdivided as shown in Figure 9.9. We let S_n be the area of the colored region in the nth figure of the progression. The area of the colored region in the first figure is

$$S_1 = 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

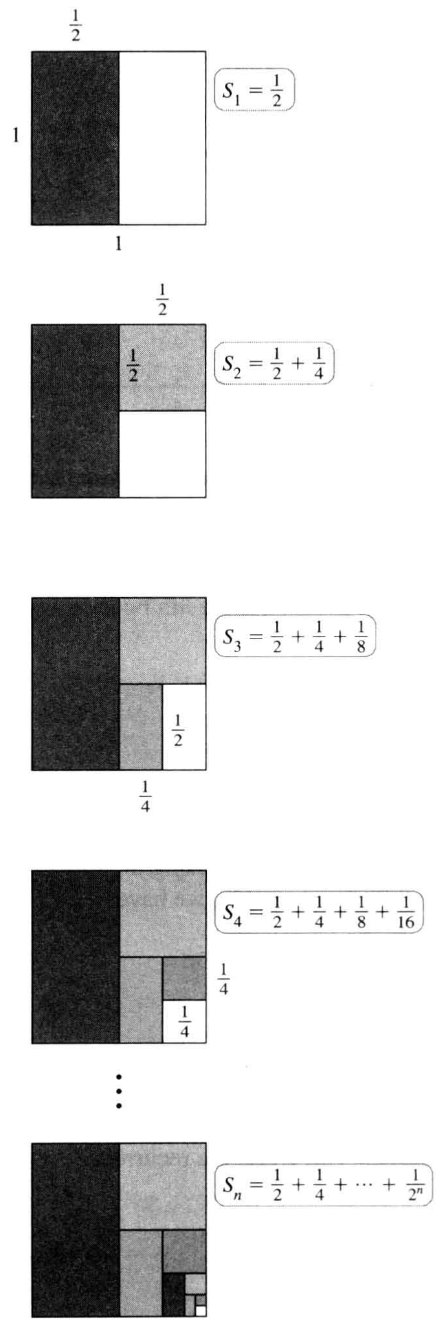

FIGURE 9.9

The area of the colored region in the second figure is S_1 plus the area of the smaller blue square, which is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Therefore,

$$S_2 = \frac{1}{2} + \frac{1}{4}.$$

The area of the colored region in the third figure is S_2 plus the area of the smaller green rectangle, which is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Therefore,

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

Continuing in this manner, we find that

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}.$$

If this process is continued indefinitely, the area of the colored region S_n approaches the area of the square, which is 1. So, it is plausible that

$$\lim_{n \to \infty} S_n = \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}_{\text{sum continues indefinitely}} = 1.$$

This example shows that it is possible to sum an infinite set of numbers and obtain a finite number—in this case, the sum is 1. The sequence $\{S_n\}$ generated in this example is extremely important. It is called a *sequence of partial sums*, and its limit is the value of the infinite series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

EXAMPLE 7 **Working with series** Consider the infinite series

$$0.9 + 0.09 + 0.009 + 0.0009 + \cdots,$$

where each term of the sum is $1/10$ of the previous term.

a. Find the sum of the first one, two, three, four, and five terms of the series.

b. What value would you assign to the infinite series $0.9 + 0.09 + 0.009 + \cdots$?

SOLUTION

a. Let S_n denote the sum of the first n terms of the given series. Then,

$$\begin{aligned} S_1 &= 0.9 \\ S_2 &= 0.9 + 0.09 = 0.99 \\ S_3 &= 0.9 + 0.09 + 0.009 = 0.999 \\ S_4 &= 0.9 + 0.09 + 0.009 + 0.0009 = 0.9999 \\ S_5 &= 0.9 + 0.09 + 0.009 + 0.0009 + 0.00009 = 0.99999. \end{aligned}$$

b. Notice that the sums $S_1, S_2, \ldots, S_n$ form a sequence $\{S_n\}$, which is a *sequence of partial sums*. As more and more terms are included, the values of S_n approach 1. Therefore, a reasonable conjecture for the value of the series is 1:

$$\underbrace{\underbrace{\underbrace{0.9}_{S_1 = 0.9} + 0.09}_{S_2 = 0.99} + 0.009}_{S_3 = 0.999} + 0.0009 + \cdots = 1$$

Related Exercises 49–52 ◀

QUICK CHECK 3 Reasoning as in Example 7, what is the value of $0.3 + 0.03 + 0.003 + \cdots$? ◀

The general nth term of the sequence in Example 7 can be written as

$$S_n = \underbrace{0.9 + 0.09 + 0.009 + \cdots + 0.0\ldots9}_{n \text{ terms}} = \sum_{k=1}^{n} 9 \cdot 0.1^k.$$

➤ Recall the summation notation introduced in Chapter 5: $\sum_{k=1}^{n} a_k$ means $a_1 + a_2 + \cdots + a_n$.

We observed that $\lim_{n\to\infty} S_n = 1$. For this reason, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} 9 \cdot 0.1^k}_{S_n} = \underbrace{\sum_{k=1}^{\infty} 9 \cdot 0.1^k}_{\text{new object}} = 1.$$

By letting $n \to \infty$ a new mathematical object $\sum_{k=1}^{\infty} 9 \cdot 0.1^k$ is created. It is an infinite series and it is the *limit of the sequence of partial sums*.

➤ The term *series* is used for historical reasons. When you see *series*, you should think *sum*.

DEFINITION Infinite Series

Given a set of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an **infinite series**. Its **sequence of partial sums** $\{S_n\}$ has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \text{for } n = 1, 2, 3, \ldots \end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series **converges** to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} a_k}_{S_n} = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also **diverges**.

QUICK CHECK 4 Do the series $\sum_{k=1}^{\infty} 1$ and $\sum_{k=1}^{\infty} k$ converge or diverge? ◀

EXAMPLE 8 **Sequence of partial sums** Consider the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

a. Find the first four terms of the sequence of partial sums.

b. Find an expression for S_n and make a conjecture about the value of the series.

SOLUTION

a. The sequence of partial sums can be evaluated explicitly:

$$S_1 = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$$

$$S_2 = \sum_{k=1}^{2} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \sum_{k=1}^{3} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$S_4 = \sum_{k=1}^{4} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

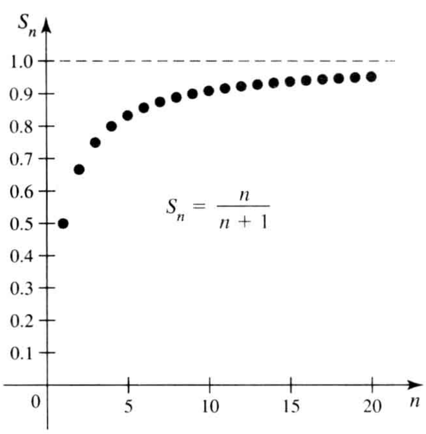

FIGURE 9.10

b. Based on the pattern in the sequence of partial sums, a reasonable conjecture is that $S_n = \frac{n}{n+1}$, for $n = 1, 2, 3, \ldots$, which produces the sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$ (Figure 9.10). Because $\lim_{n\to\infty} \frac{n}{n+1} = 1$, we conclude that

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

Related Exercises 53–56 ◄

QUICK CHECK 5 Find the first four terms of the sequence of partial sums for the series $\sum_{k=1}^{\infty} (-1)^k k$. Does the series converge or diverge? ◄

Summary

This section has shown that there are three key ideas to keep in mind.

- A *sequence* $\{a_1, a_2, \ldots, a_n, \ldots\}$ is an ordered *list* of numbers.
- An *infinite series* $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ is a *sum* of numbers.
- The *sequence of partial sums* $S_n = a_1 + a_2 + \cdots + a_n$ is used to evaluate the series $\sum_{k=1}^{\infty} a_k$.

For sequences, we ask about the behavior of the individual terms as we go out farther and farther in the list; that is, we ask about $\lim_{n\to\infty} a_n$. For infinite series, we examine the sequence of partial sums related to the series. If the sequence of partial sums $\{S_n\}$ has a limit, then the infinite series $\sum_{k=1}^{\infty} a_k$ converges to that limit. If the sequence of partial sums does not have a limit, the infinite series diverges.

Table 9.2 shows the correspondences between sequences/series and functions, and between summing and integration. For a sequence, the index n plays the role of the independent variable and takes on integer values; the terms of the sequence $\{a_n\}$ correspond to the dependent variable.

With sequences $\{a_n\}$, the idea of accumulation corresponds to summation, whereas with functions, accumulation corresponds to integration. A finite sum is analogous to integrating a function over a finite interval. An infinite series is analogous to integrating a function over an infinite interval.

Table 9.2

	Sequences/Series	**Functions**
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	Integers e.g., $n = 0, 1, 2, 3, \ldots$	Real numbers e.g., $\{x: x \geq 0\}$
Accumulation	Sums	Integrals
Accumulation over a finite interval	$\sum_{k=0}^{n} a_k$	$\int_0^n f(x)\,dx$
Accumulation over an infinite interval	$\sum_{k=0}^{\infty} a_k$	$\int_0^{\infty} f(x)\,dx$

SECTION 9.1 EXERCISES

Review Questions

1. Define *sequence* and give an example.
2. Suppose the sequence $\{a_n\}$ is defined by the explicit formula $a_n = 1/n$, for $n = 1, 2, 3, \ldots$. Write out the first five terms of the sequence.
3. Suppose the sequence $\{a_n\}$ is defined by the recurrence relation $a_{n+1} = na_n$, for $n = 1, 2, 3, \ldots$, where $a_1 = 1$. Write out the first five terms of the sequence.
4. Define *finite sum* and give an example.
5. Define *infinite series* and give an example.
6. Given the series $\sum_{k=1}^{\infty} k$, evaluate the first four terms of its sequence of partial sums $S_n = \sum_{k=1}^{n} k$.
7. The terms of a sequence of partial sums are defined by $S_n = \sum_{k=1}^{n} k^2$, for $n = 1, 2, 3, \ldots$. Evaluate the first four terms of the sequence.
8. Consider the infinite series $\sum_{k=1}^{\infty} \frac{1}{k}$. Evaluate the first four terms of the sequence of partial sums.

Basic Skills

9–12. Explicit formulas *Write the first four terms of the sequence* $\{a_n\}_{n=1}^{\infty}$.

9. $a_n = 1/10^n$
10. $a_n = n + 1/n$
11. $a_n = 1 + \sin(\pi n/2)$
12. $a_n = 2n^2 - 3n + 1$