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$\mathcal{D}_{\langle M_k \rangle}$ OPERATORS AND SPECTRAL OPERATORS

WANG SHENGWANG

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1. $\mathcal{D}_{\langle M_k \rangle}$ operators with their spectrum on the complex plane. Throughout this paper, all notations are the same as [1, 2] and the sequence $\{M_k\}$ satisfies (M. 1), (M. 2) and (M. 3)^[1], i.e., logarithmic convexity, non-quasianalyticity and differentiability. By means of $\{M_k\}$, we can define the associated function $M(t_1, t_2)$ (cf. [7])

$$M(t_1, t_2) = \sup_{\substack{k_i \geq 0 \\ (i=1, 2)}} \left(\sum_{i=1}^2 k_i \ln |t_i| - \ln M_{k_1+k_2} \right) \quad (t_i \neq 0, i=1, 2),$$

and the space $\mathcal{D}_{\langle M_k \rangle}$ of two variables

$$\mathcal{D}_{\langle M_k \rangle} = \left\{ \varphi \mid \varphi \in \mathcal{D}; \|\varphi\|_\nu = \sup_{\substack{s \in \mathbb{R}^2 \\ k_i \geq 0 \\ (i=1, 2)}} \left| \frac{\partial^{k_1+k_2} \varphi(s)}{\partial s_1^{k_1} \partial s_2^{k_2}} \right| / \nu^k M_k < +\infty \right. \\ \left. \text{for some integer } \nu > 0 \right\},$$

where $s = (s_1, s_2)$, $k = k_1 + k_2$. It is evident that for any $\varphi \in \mathcal{D}_{\langle M_k \rangle}$

$$\sup_{\substack{s \in \mathbb{R}^2 \\ k \geq 0}} \left| \frac{\partial^k \varphi(s)}{\nu^k M_k} \right| < +\infty$$

for some integer $\nu > 0$, where $\partial = \frac{1}{2} \left(\frac{\partial}{\partial s_1} - i \frac{\partial}{\partial s_2} \right)$, and $|\partial^k \varphi(s)| \leq \|\varphi\|_\nu \nu^k M_k$. $\|\cdot\|_\nu$ will be called ν -norm. For the definition and properties of bounded $\mathcal{D}_{\langle M_k \rangle}$ operators with their spectrum on the complex plane, we refer the reader to see [3, 4]. Let $\mathbf{X}$ be a Banach space, $B(\mathbf{X})$ be the ring of all linear bounded operators defined on $\mathbf{X}$. If $T \in B(\mathbf{X})$ is a $\mathcal{D}_{\langle M_k \rangle}$ operator, we have $T = T_1 + iT_2$, $T_1 = U_{\text{Re}t}$, $T_2 = U_{\text{Im}t}$, where U is a spectral ultradistribution of T . Since $\text{supp}(U)$ is compact, U may be easily extended to the whole space $\mathcal{E}_{\langle M_k \rangle}$.

By few computations, as a function of (s_1, s_2) , $e^{i(t_1 s_1 + t_2 s_2)}$ satisfies: for given $\mu_l > 0$ ($l=1, 2$), there exist $A > 0$ and an integer $\nu > 0$ such that

$$\|e^{i(t_1 s_1 + t_2 s_2)}\|_\nu \leq A e^{M(\mu_1 t_1, \mu_2 t_2)},$$

where $\|e^{i(t_1 s_1 + t_2 s_2)}\|_\nu$ denotes the ν -norm of $e^{i(t_1 s_1 + t_2 s_2)}$. For every $\varphi \in \mathcal{D}_{\langle M_k \rangle}$, there exist $h_l > 0$ ($l=1, 2$) and $A' > 0$ such that

$$|\hat{\varphi}(t_1, t_2)| \leq A' e^{-M(h_1 t_1, h_2 t_2)},$$

where

$$\hat{\varphi}(t_1, t_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(t_1 s_1 + t_2 s_2)} \varphi(s_1, s_2) ds_1 ds_2$$

is the Fourier transform of $\varphi(s_1, s_2)$. Using the same argument as in [1] theorem 3, we can easily prove that one of the spectral ultradistributions of $\mathcal{D}_{\langle M_k \rangle}$ operator T can be expressed as

$$U_\varphi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(t_1 T_1 + t_2 T_2)} \hat{\varphi}(t_1, t_2) dt_1 dt_2.$$

Let T be a spectral operator, $S, N, E(\cdot)$ be the scalar part, radical part and the spectral measure of T respectively. The following theorem gives a sufficient condition of a spectral operator to be a $\mathcal{D}_{\langle M_k \rangle}$ operator.

Theorem 1. Let $T \in B(X)$ be a spectral operator satisfying

$$\sup_{k>0} \sup_{\substack{|\mu_j| \leq 1 \\ \delta_j \in \mathcal{B} \\ j=1, 2, \dots, k}} \left(\left\| \frac{N^n}{n!} \sum_{j=1}^k \mu_j E(\delta_j) \right\| M_n \right)^{\frac{1}{n}} \rightarrow 0 \quad (n \rightarrow \infty), \quad (1)$$

where $\mathcal{B}$ denotes the class of Borel subsets in the complex plane, then T is a $\mathcal{D}_{\langle M_k \rangle}$ operator and one of its spectral ultradistributions can be expressed as

$$U_\varphi = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int \partial^n \varphi(s) dE(s).$$

Proof Let $\varphi \in \mathcal{D}_{\langle M_k \rangle}$ satisfy

$$|\partial^n \varphi| \leq \|\varphi\|_\nu \nu^n M_n \quad (n=0, 1, 2, \dots),$$

by (1), we have

$$\left\| \frac{N^n}{n!} \sum_{j=1}^k \mu_j E(\delta_j) \right\| \leq \frac{A}{(2\nu)^n M_n},$$

where $A > 0$ only depends on $\nu, |\mu_j| \leq 1$. By a simple computation, we get

$$\sum_{n=0}^{\infty} \left\| \frac{N^n}{n!} \int \partial^n \varphi dE \right\| \leq 2A \|\varphi\|_\nu. \quad (2)$$

Put

$$U_\varphi = \sum_{n=0}^{\infty} \frac{N^n}{n!} \int \partial^n \varphi dE,$$

then $U_1 = I, U_s = T$ and

$$\begin{aligned} U_{\varphi\psi} &= \sum_{n=0}^{\infty} \frac{N^n}{n!} \int \partial^n (\varphi\psi) dE = \sum_{n=0}^{\infty} N^n \sum_{k=0}^n \frac{1}{k!(n-k)!} \int \partial^k \varphi \partial^{n-k} \psi dE \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} N^k \int \partial^k \varphi dE \cdot \sum_{j=0}^{\infty} \frac{1}{j!} N^j \int \partial^j \psi dE = U_\varphi U_\psi, \end{aligned}$$

i.e., $U: \mathcal{D}_{\langle M_k \rangle} \rightarrow B(X)$ is a continuous homomorphism. Therefore T is a $\mathcal{D}_{\langle M_k \rangle}$ operator. Theorem is proved.

Corollary 1. If $N, E(\cdot)$ satisfy

$$\left(\frac{M_n}{n!} \vee (N^n E) \right)^{\frac{1}{n}} \rightarrow 0 \quad (n \rightarrow \infty),$$

then T is a $\mathcal{D}_{\langle M_k \rangle}$ operator.

Corollary 2. Let N be a quasivipotent, then if and only if

$$\left(\frac{\|N^n\|}{n!} M_n \right)^{\frac{1}{n}} \rightarrow 0 \quad (n \rightarrow \infty), \quad (3)$$

$S+N$ is a $\mathcal{D}_{\langle M_k \rangle}$ operator for every scalar operator S commuting with N .

We shall call N a $\{M_k\}$ -quasinilpotent if it satisfies (3) (cf. [4]). The following proposition gives some properties of a $\{M_k\}$ -quasinilpotent.

Proposition. *Let N be a quasinilpotent, the following assertions are equivalent:*

- (i) N is a $\{M_k\}$ -quasinilpotent.
- (ii) For every $\lambda > 0$, there exists $B_\lambda > 0$ such that¹⁾

$$\|R(\zeta, N)\| \leq B_\lambda e^{M^* \left(\frac{\lambda}{|\zeta|}\right)} \quad (|\zeta| \text{ is sufficiently small}).$$

- (iii) For every $\mu > 0$, there exists $A_\mu > 0$ such that

$$\|e^{izN}\| \leq A_\mu e^{M(\mu|z|)}.$$

Proof By [9] proposition 4.5, the equivalence of (i), (iii) is evident (cf. [4]). It remains to prove the equivalence of (ii), (iii).

(ii) $\Rightarrow$ (iii). By putting $\lambda = \frac{\mu}{2}$ in (ii), $r = \delta_\mu(|z|)$ (cf. [1]) and using lemma 4 in [1], we have

$$\begin{aligned} \|e^{izN}\| &\leq \frac{1}{2\pi} \int_{|\zeta|=r} |e^{iz\zeta}| \|R(\zeta, N)\| |d\zeta| \leq B_\lambda r e^{|z|r} e^{M^* \left(\frac{\lambda}{r}\right)} \\ &\leq B_\lambda e^{|z|\delta_\mu(|z|)} e^{M^* \left(\frac{\mu}{2\delta_\mu(|z|)}\right)} \leq 2B_\lambda e^{M(\mu|z|)}, \end{aligned}$$

where $0 < r < 1$.

(iii) $\Rightarrow$ (ii). Since $\|R(\zeta, N)\| = \|R(|\zeta| e^{\frac{\pi}{2}i}, e^{(\frac{\pi}{2}-\arg \zeta)i} N)\|$, we can easily obtain (ii) by applying the sufficient part of theorem 5 in [1] to the operator $e^{(\frac{\pi}{2}-\arg \zeta)i} N$.

2. $\mathcal{D}_{\langle M_k \rangle}$ operators with their spectrum on the real line. In this section, all functions in $\mathcal{D}_{\langle M_k \rangle}$ are of one variable, hence if $T \in B(X)$ is a $\mathcal{D}_{\langle M_k \rangle}$ operator, then $\sigma(T) \subset \mathbb{R}$, the real line. Now we consider the conditions to guarantee a bounded $\mathcal{D}_{\langle M_k \rangle}$ operator T to be a spectral. In the sequel, the conjugate space of X will be denoted by X^* .

If $f \in \mathcal{D}'_{\langle M_k \rangle}$, from [8, 9], there exist countable many regular measures μ_n ($n \geq 0$) satisfying

$$\langle f, \varphi \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \varphi^{(n)}(t) d\mu_n(t) \quad (4)$$

and for every $h > 0$, there exists $A > 0$ such that

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} M_n \int |d\mu_n| \leq A. \quad (5)$$

By a similar method used in [10], it can be easily proved that:

- 1° for $f \in \mathcal{D}'_{\langle M_k \rangle}$, if $f' = 0$, then $f = \text{const}$;
- 2° for $f \in \mathcal{D}'_{\langle M_k \rangle}$, if f' is a measure, then f is a function of bounded variation in every finite interval.

1) In (ii), we have to suppose that $\left\{\frac{M_k}{k!}\right\}$ is logarithmically convex. As for $M^* \left(\frac{\lambda}{|\zeta|}\right)$, $M(\mu|z|)$, we refer the reader to see [1].

In general, the sequence $\{\mu_n\}$ ($n \geq 0$) in (4), (5) is not unique. Now we introduce the following:

Definition. Let n_0 be a positive integer, $f \in \mathcal{D}'_{\langle M_k \rangle}$ with compact support (i. e., $f \in \mathcal{S}'_{\langle M_k \rangle}$) is called n_0 -singular, if for all $n \geq n_0$, there exist μ_n in (4), (5) with $\text{Supp}(\mu_n)$ contained in a closed subset F satisfying $\text{mes } F = 0$. If $n_0 = 1$, f is called singular. Suppose that $T \in B(X)$ is a $\mathcal{D}_{\langle M_k \rangle}$ operator, U is its spectral ultradistribution, we say that T is n_0 -singular (singular), if for every $x \in X$, $x^* \in X^*$, $x^* U x$ is n_0 -singular (singular).

In the sequel, we shall often suppose that $f \in \mathcal{D}'_{\langle M_k \rangle}$ has compact support and when f is n_0 -singular, μ_n ($n \geq n_0$) will denote what satisfy the conditions described in the above definition. Therefore all of these μ_n ($n \geq n_0$) are singular with respect to Lebesgue measure, but the inverse is false.

Lemma 1. If $f \in \mathcal{D}_{\langle M_k \rangle}$ is n_0 -singular, then for every $n \geq n_0$, μ_n is unique, especially, μ_0 is also unique when $n_0 = 1$.

Proof It suffices to prove that when $f = 0$, then $\mu_n = 0$ ($n \geq n_0$). In fact, $f = 0$ is equivalent to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n^{(n)} = 0 \quad \text{or} \quad \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mu_n^{(n-1)} \right)' = -\mu_0. \quad (6a)$$

Since $\mathcal{D}_{\langle M_k \rangle}$ is differentiable, it follows that $g_1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mu_n^{(n-1)} \in \mathcal{D}'_{\langle M_k \rangle}$. Since all of $\text{supp}(\mu_n)$ ($n \geq 0$) are contained in a neighbourhood of $\text{supp}(f)$, we may suppose that $\text{supp}(g_1)$ is compact. By (6a), g_1 is a function of bounded variation. Similarly, $\sum_{n=k}^{\infty} \frac{(-1)^n}{n!} \mu_n^{(n-k)} = g_k$ are also functions of bounded variation for all $k > 1$. By the hypothesis, we can easily see that the subset where $g_k \neq 0$ is of Lebesgue measure zero, hence as ultradistributions in $\mathcal{D}'_{\langle M_k \rangle}$, $g_k = 0$ ($k \geq n_0$), i. e.,

$$\left. \begin{aligned} (-1)^{n_0} \frac{\mu_{n_0}}{n_0!} + (-1)^{n_0+1} \frac{\mu_{n_0+1}'}{(n_0+1)!} + \dots &= 0, \\ (-1)^{n_0+1} \frac{\mu_{n_0+1}}{(n_0+1)!} + \dots &= 0, \\ \dots\dots\dots &= 0. \end{aligned} \right\} \quad (6b)$$

(6b) shows that $\mu_{n_0} = \mu_{n_0+1} = \dots = 0$. If $n_0 = 1$, by $\mu_n = 0$ ($n \geq 1$) and (6a), we have $\mu_0 = 0$. Thus the lemma is proved.

For $\varphi \in \mathcal{D}_{\langle M_k \rangle}$, $\hat{\varphi} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-its} \varphi(t) dt$ and $\check{\varphi}(s) = \int_{-\infty}^{\infty} e^{its} \varphi(t) dt$ will express the Fourier and inverse Fourier transform of φ respectively. When f is an ultradistribution, $\hat{f}$, $\check{f}$ also have the same meaning.

Remark If ultradistributions $f = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n^{(n)}}{n!}$ and $g = \sum_{n=0}^{\infty} \frac{(-1)^n \nu_n^{(n)}}{n!}$ are singular, where μ_n, ν_n ($n \geq 1$) satisfy those conditions described in the preceding definition,

then from $\check{f} = \check{g}$ (i. e., $\sum_{n=0}^{\infty} \frac{(it)^n}{n!} \check{\mu}_n(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \check{\nu}_n(t)$), we can deduce $f = g$, by the above lemma, we have $\mu_n = \nu_n$, hence $\check{\mu}_n = \check{\nu}_n$ ($n \geq 0$), i. e., from $\check{f} = \check{g}$, we get $\check{\mu}_n = \check{\nu}_n$.

Lemma 2. For $f \in \mathcal{D}'_{(M_k)}$ has compact support, we have $\check{f}(s) = \langle f_t, e^{its} \rangle$.

Proof For every $\varphi \in \mathcal{D}_{(M_k)}$, $\varphi(t) = \int_{-\infty}^{\infty} e^{its} \hat{\varphi}(s) ds$ is the limit of the integral sum $\sum_{j=0}^n e^{its_j} \hat{\varphi}(s_j) \Delta s_j$ with respect to the topology of $\mathcal{S}_{(M_k)}$, hence

$$\begin{aligned} \int_{-\infty}^{\infty} \langle f_t, e^{its} \rangle \hat{\varphi}(s) ds &= \lim_{j \rightarrow \infty} \sum_{j=1}^n \langle f_t, e^{its_j} \rangle \hat{\varphi}(s_j) \Delta s_j \\ &= \left\langle f_t, \lim_{j \rightarrow \infty} \sum_{j=1}^n e^{its_j} \hat{\varphi}(s_j) \Delta s_j \right\rangle = \langle f, \varphi \rangle. \end{aligned}$$

Lemma 3. Suppose that X is reflexive, T is a bounded singular $\mathcal{D}_{(M_k)}$ operator, U is its spectral ultradistribution, then there exist operator-valued measures $u_n(\cdot)$ ($n \geq 0$) such that

(i) each of $u_n(\cdot)$ is bounded and strongly countably additive and for every $h > 0$ there exists $A > 0$ such that

$$\|u_n(\delta)\| \leq A n! h^n / M_n \quad (n \geq 0) \quad (7)$$

for every Borel subset δ on the real line;

(ii) for every $\varphi \in \mathcal{D}_{(M_k)}$

$$U_{\varphi} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \varphi^{(n)}(t) du_n(t), \quad (8)$$

in which every integral converges in the sense of strong operator topology and the series converges in the sense of uniform operator topology.

Proof For $x \in X$, $x^* \in X^*$, $\|x\| \leq 1$, $\|x^*\| \leq 1$, the class of ultradistributions $x^* U x$ is bounded in $\mathcal{S}'_{(M_k)}$ and their supports are contained in $\sigma(T)$. By [8, 9], there exist regular measures $\mu_n(\cdot; x, x^*)$ ($n \geq 0$) satisfying: for every $h > 0$, there exists $A > 0$ such that

$$\|\mu_n(\cdot; x, x^*)\| \leq A n! h^n / M_n \quad (n \geq 0) \quad (9)$$

uniformly for all $\|x\| \leq 1$, $\|x^*\| \leq 1$, where $\|\mu\|$ denotes the total variation of μ . In addition, we have

$$x^* U_{\varphi} x = \sum_{n=0}^{\infty} \frac{1}{n!} \int \varphi^{(n)}(t) d\mu_n(t; x, x^*),$$

in which the series converges absolutely and uniformly with respect to all $\|x\| \leq 1$, $\|x^*\| \leq 1$. By the singularity of T , we may suppose that for every $n \geq 1$, $\text{supp}(\mu_n(\cdot; x, x^*))$ is contained in a fixed closed subset F with Lebesgue measure zero. From lemma 1, $\mu_n(\cdot; x, x^*)$ ($n \geq 0$) is unique, hence for every Borel subset δ , $\mu_n(\delta; x, x^*)$ is a bounded bilinear functional of x, x^* . By the reflexivity of X , there exists for every $n \geq 0$ a bounded linear operator $u_n(\delta)$ defined on X such that

$$x^* u_n(\delta) x = \mu_n(\delta; x, x^*).$$

From (9), $u_n(\cdot)$ satisfies (i). Evidently, we may suppose that $\text{supp}(u_n)$ ($n \geq 0$) is contained in a fixed neighbourhood G of $\sigma(T)$. As for (ii), using the following inequality

$$\left| x^* \int \varphi^{(n)}(t) du_n(t) x \right| = \left| \int \varphi^{(n)}(t) d\mu_n(t; x, x^*) \right| \\ \leq A \sup_{t \in G} |\varphi^{(n)}(t)| n! h^n \|x\| \|x^*\| / M_W,$$

we get the result that the series in (8) converges with respect to the uniform operator topology. Finally, by [12] Theorem IV. 10.8 and Definition IV. 10.7, every integral in (8) converges in the strong operator topology.

Lemma 4. *Under the hypotheses of the preceding lemma, we have*

$$\check{u}_n(\tau) \check{u}_m(\sigma) = \check{u}_{n+m}(\tau + \sigma) \quad (m, n \geq 0), \quad (10)$$

in which τ, σ are real numbers.

Proof For every $x \in X, x^* \in X^*$, we have

$$x^* U_{e^{i(\tau+\sigma)t}} x = x^* U_{e^{i\tau t}} U_{e^{i\sigma t}} x = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} x^* \check{u}_n(\tau) U_{e^{i\sigma t}} x; \quad (11)$$

$$x^* U_{e^{i(\tau+\sigma)t}} x = \sum_{k=0}^{\infty} \frac{[i(\tau+\sigma)]^k}{k!} x^* \check{u}_k(\tau+\sigma) x = \sum_{k=0}^{\infty} \sum_{n=0}^k \frac{(i\tau)^n}{n!} \frac{(i\sigma)^{k-n}}{(k-n)!} x^* \check{u}_k(\tau+\sigma) x \\ = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} \sum_{m=0}^{\infty} \frac{(i\sigma)^m}{m!} x^* \check{u}_{n+m}(\tau+\sigma) x = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} \int e^{i\sigma t} dx^* v_n(t) x, \quad (12)$$

in which

$$v_n(\delta) = \sum_{m=0}^{\infty} \frac{(i\sigma)^m}{m!} \int_{\delta} e^{i\sigma t} du_{n+m}(t) \quad (13)$$

depends on σ and δ is a Borel subset. Putting σ fixed, from (9), for every $h > 0$ there exists $A > 0$ such that for all $\|x\| \leq 1, \|x^*\| \leq 1$,

$$\|x^* v_n(t) x\| \leq \sum_{m=0}^{\infty} \frac{|\sigma|^m}{m!} \|x^* u_{n+m}(t) x\| \leq A \sum_{m=0}^{\infty} \frac{|\sigma|^m}{m!} (n+m)! \left(\frac{h}{2}\right)^{n+m} / M_{n+m} \\ \leq (An! h^n / M_n) \left(\sum_{m=0}^{\infty} \frac{|\sigma|^m (n+m)!}{m! n! 2^{n+m}} h^m / M_m \right) \\ \leq (An! h^n / M_n) \sum_{m=1}^{\infty} (|\sigma| h)^m / M_m.$$

Since $B = \sum_{m=0}^{\infty} \frac{(|\sigma| h)^m}{M_m} < +\infty$, it follows that

$$\|x^* v_n(t) x\|_V \leq ABn! h^n / M_W$$

uniformly for all $\|x\| \leq 1, \|x^*\| \leq 1$. Therefore $v_n(\cdot)$ ($n=0, 1, 2, \dots$) are bounded strongly countably additive operator-valued measures. Evidently, the utradistribution

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x^* v_n(t) x)^{(n)}$ is singular. By (11), (12) and the remark after lemma 1, we have

$$x^* \check{u}_n(\tau) U_{e^{i\sigma t}} x = x^* \check{v}_n(\tau) x,$$

i. e.,

$$\sum_{m=0}^{\infty} \frac{(i\sigma)^m}{m!} x^* \check{u}_n(\tau) \check{u}_m(\sigma) x = \sum_{m=0}^{\infty} \frac{(i\sigma)^m}{m!} x^* \check{u}_{n+m}(\tau+\sigma) x. \quad (14)$$

For fixed τ , the two sides of (14) are the inverse Fourier transform of $\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}$ $\cdot (x^* \check{u}_n(\tau) u_m(\cdot) x)^{(m)}$, $\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (x^* w_{n+m}(\cdot) x)^{(m)}$ respectively, where $w_{n+m}(\delta) = \int_{\delta} e^{i\tau t} \cdot du_{n+m}(t)$, δ is a Borel subset. Still by the remark after lemma 1, we get (10). Lemma 4 is thus proved.

Lemma 5. Putting $E(\cdot) = u_0(\cdot)$, $N = u_1(R)$, we have

(i) $E(\cdot)$ is a spectral measure;

(ii) for every Borel subset δ and every $n \geq 0$,

$$u_n(\delta) = N^n E(\delta) = E(\delta) N^n;$$

(iii) N is a quasinilpotent satisfying

$$\lim_{n \rightarrow \infty} \left(\frac{\|N^n\|}{n!} M_n \right)^{\frac{1}{n}} = 0. \quad (15)$$

Proof (i) From lemma 3 (i), it remains to prove that

$$E(\delta) E(\varepsilon) = E(\delta \cap \varepsilon), \quad E(\delta) + E(\varepsilon) - E(\delta) E(\varepsilon) = E(\delta \cup \varepsilon), \quad (16)$$

for all Borel subsets δ, ε . Putting $n = m = 0$ in (10), we obtain $\check{E}(\tau) \check{E}(\sigma) = \check{E}(\tau + \sigma)$, i. e.,

$$\int e^{i\tau t} dE(t) \check{E}(\sigma) = \int e^{i(\tau+\sigma)t} dE(t) = \int e^{i\tau t} d_t \left(\int_{-\infty}^t e^{i\sigma t} dE(\tau) \right).$$

Since the inverse Fourier transform is 1-1, it follows that

$$E(\delta) \check{E}(\sigma) = \int_{\delta} e^{i\sigma t} dE(t). \quad (17)$$

(17) may be written as

$$\int e^{i\sigma t} dE(\delta) E(t) = \int_{\delta} e^{i\sigma t} dE(t),$$

still by the property (1-1) of the inverse Fourier transform, we have $E(\delta) E(\varepsilon) = E(\delta \cap \varepsilon)$. Finally, by the additivity of $E(\cdot)$,

$$E(\delta \cup \varepsilon) = E(\delta) + E(\varepsilon \setminus (\delta \cap \varepsilon)) = E(\delta) + E(\varepsilon) - E(\delta \cap \varepsilon).$$

(16) holds.

(ii) and (iii) Putting $\sigma = 0$, $m = 1$ and substituting n by $n - 1$ in (10), we have

$$\check{u}_n(\tau) = \check{u}_{n-1}(\tau) N. \quad (18)$$

Similarly, $\check{u}_n(\tau) = N \check{u}_{n-1}(\tau)$. Let $n = 0, 1, 2, \dots$, it follows that

$$\check{u}_n(\tau) = \check{E}(\tau) N^n = N^n \check{E}(\tau).$$

By the property (1-1) of the inverse Fourier transform again, we get

$$u_n(\cdot) = E(\cdot) N^n = N^n E(\cdot),$$

especially, $N^n = u_n(R)$. Therefore by (9), for every $h > 0$,

$$\|N^n\| = \|u_n(R)\| \leq A n! h^n / M_n,$$

i. e., (15) holds.

Summarizing the above discussions, we obtain

Theorem 2. Suppose X is reflexive. $T \in B(X)$ is a singular $\mathcal{D}_{(M_n)}$ operator if and

only if T is a spectral operator satisfying

(i) For every $x \in X$ and $x^* \in X^*$, $\text{supp}(x^* N^n E(\cdot) x)$ is contained in a fixed closed subset F of Lebesgue measure zero for all $n \geq 1$ (F may depend on x, x^*), where $E(\cdot), N$ are the spectral measure and radical part of T respectively;

(ii) N satisfies (15).

Corollary. Suppose X is reflexive. $T \in B(X)$ is a singular $\mathcal{D}_{(M_n)}$ operator and $\text{mes } \sigma(T) = 0$ if and only if T is a spectral operator satisfying

(i) $\text{mes } \text{supp}(E(\cdot)) = 0$;

(ii) N satisfies (15).

Theorem 2 is an extension of some results of [5]¹⁾.

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¹⁾ But the author has not seen the full proof of [5].