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L.V. Ovsyannikov

Lectures on the Theory of Group Properties of Differential Equations

微分方程群性质理论讲义

Edited by N. H. Ibragimov

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Weifen Fangcheng Qunxingzhi Lilun Jiangyi

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Editor's preface

When I studied at Novosibirsk State University (Russian) I was lucky to have such brilliant teachers in mathematics as M.A. Lavrentyev, S.L. Sobolev, A.I. Mal'tsev, Yu.G. Reshetnyak and others. But it were L.V. Ovsyannikov's lectures in Ordinary differential equations, Partial differential equations, Gas dynamics and Group properties of differential equations that were of the most benefit for me. I attended his course "Group properties of differential equations" when I was a third-year student. His lectures provided a clear introduction to Lie group methods for determining symmetries of differential equations and a variety of their applications in gas dynamics and other nonlinear models as well as Ovsyannikov's remarkable contribution to this classical subject. His lectures were spectacular not only due to the brilliant presentation of the material but also due to absolutely new discoveries for us. I remember one of our most emotional student's repeated exclamations like "Wonderful, ... Incredible!" every time when Ovsyannikov revealed most unusual properties of symmetries or unexpected methods.

His lecture notes of this course were published in 1966 with the print of 300 copies only. Since then the Notes were neither reprinted nor translated into English, though they contain the material that is useful for students and teachers but cannot be found in modern texts. For example, theory of partially invariant solutions developed by Ovsyannikov and presented in §3.5, §3.6 is useful for investigating mathematical models described by systems of nonlinear differential equations. It is important to make this classical text available to everyone interested in modern group analysis.

In order to adapt the text for modern students I made several minor changes in the English translation. In particular, sections have been divided into subsections and few misprints have been corrected. Part of the problems formulated in §3.7 have been completely or partially solved since 1966. But we did not make any comments on this matter in the present translation.

January 2013 Nail H. Ibragimov

Preface

The theory of differential equations has two aspects of investigation, namely local and global, no matter whether the equations arise from applied problems of physics and mechanics or from abstract speculations (which is rather frequent in modern mathematics). The local aspect is characterized by dealing with the inner structure of a family of solutions and its investigation in a neighborhood of a certain point. The global approach deals with solutions defined in some domain and having a given behavior on its boundary.

It would certainly be erroneous to oppose these directions to each other. However, it is no good to ignore the differences in approaches either. While the global approach necessitates the functional analytic apparatus, the local viewpoint allows one to get along with algebraic means only. A brilliant example of a profound local consideration is the famous Cauchy-Kovalevskaya theorem which is, in fact, an algebraic statement. Moreover, it is an easy matter to notice that the theory of boundary value problems also makes an essential application of various algebraic properties of the whole family of solutions. Therefore, the local aspect of the algebraic theory of differential equations is quite vital.

The theory of group properties of differential equations descried in the present lecture notes is a typical example of a local theory. It is especially valuable in investigating nonlinear differential equations, for its algorithms act here as reliably as for linear cases.

In spite of the fact that the fundamentals of the theory of group properties were elaborated in works of the Norwegian mathematician Sophus Lie more than a hundred years ago, its development is desirable nowadays as well.

Methodological peculiarity of the present text is that its first chapter uses only the simplest algebraic apparatus of one-parameter groups, which is especially advisable for researchers engaged in applied fields. This allows one to solve the problem of finding a group admitted by a given system of differential equations completely. The second chapter is tailored to provide a deeper insight into the subject resulting from solving determining equations. The group structure of the family of solutions itself is discussed in the third chapter, which also suggests some new elements for the theory. The latter are related to the notions of a partially invariant manifold of the group, its defect of invariance, and the problem of reduction of partially invariant solutions. The final section §3.7 suggests a qualitative formulation of several problems demonstrating possibilities for further development of the theory with no claim to be complete.

The present lecture notes are written hot on the traces of a special course given by the author in Novosibirsk State University during the 1965/1966 academic year. Such a prompt decision was made to have the lecture notes published by the spring examinations. Therefore, the lectures may appear to be "raw" to many extent, and the author is ready to be completely responsible for that.

The quick release of the lecture notes would be impossible without the support of administration of the university. A major technical work in preparing the manuscript was done by the students V.G. Firsov, E.Z. Borovskaya, T.E. Kuzmina, N.I. Naumenko, M.L. Kochubievskaya and others. The author is sincerely grateful to all these people.

Novosibirsk, Russia, May 1966 L.V. Ovsyannikov

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Chapter 1
One-parameter continuous transformation groups admitted by differential equations

1.1 One-parameter continuous transformation group

1.1.1 Definition

Let us consider transformations T of an N-dimensional Euclidian space E^N into itself, so that one has

$$x' = Tx \in E^N$$

for $x \in E^N$. If the point x has coordinates $x^1, \ldots, x^N$ then this transformation can be given by the system of equalities

$$x'^i = f^i(x) = f^i(x^1, \ldots, x^N) \quad (i = 1, \ldots, N). \tag{1.1.1}$$

The functions f are assumed to be thrice continuously differentiable and locally invertible, which means that there exists an inverse transformation T^{-1} for the transformation T in some neighborhood of the point $x' = Tx$, so that $x = T^{-1}x'$.

The product of transformations T_1T_2 is the transformation T consisting in successive transformations T_2 and then T_1. The identity transformation plays the role of the unit transformation with respect to the product.

The above product is written in terms of functions f by the following equations:

$$f^i(x) = f_1^i(f_2^1(x), \ldots, f_3^N(x)) \quad (i = 1, \ldots, N).$$

This operation of multiplication is associative, i.e.

$$T_1(T_2T_3) = (T_1T_2)T_3.$$

Note that the inversion of the product of transformations is given by

$$(T_1T_2)^{-1} = T_2^{-1}T_1^{-1}.$$

We will consider a family of transformations $\{T_a\}$ with the above properties depending on a real parameter a that varies within an interval Δ.

The family $\{T_a\}$ is said to be *locally closed with respect to the product* if there exists a subinterval $\Delta' \subset \Delta$ such that

$$T_b T_a \in \{T_a\}$$

for any $a, b \in \Delta'$. This leads to a function $c = \varphi(a,b)$ which determines the *multiplication law* for transformations of $\{T_a\}$ according to the formula

$$T_b T_a = T_c.$$

The transformation T_a is written in coordinates in the following form similar to Eqs. (1.1.1):

$$T_a : \; x'^i = f^i(x,a) \quad (i = 1, \ldots, N). \tag{1.1.2}$$

The product of transformations is written in terms of the functions (1.1.2) and the multiplication law $\varphi(a,b)$ as follows:

$$T_b T_a = T_{\varphi(a,b)} : f^i(f(x,a),b) = f^i(x, \varphi(a,b)) \quad (i = 1, \ldots, N). \tag{1.1.3}$$

Definition 1.1. The family $\{T_a\}$ is called a *local one-parameter continuous transformation group* if it is locally closed with respect to the product and if the interval Δ' can be chosen so that the following conditions hold.

1° There exists the unique value $a_0 \in \Delta'$ such that T_{a_0} is an identity transformation.

2° The function $\varphi(a,b)$ is thrice continuously differentiable and the equation $\varphi(a,b) = a_0$ has the unique solution $b = a^{-1}$ for any $a \in \Delta'$.

Condition 2° means that the operation of inversion of transformations $(T_a)^{-1} = T_{a^{-1}}$ is possible in $\{T_a\}$.

Hereafter the symbol a^{-1} indicates only a definite value of the parameter and not the inverse value of the number a, so that $a^{-1} \neq \dfrac{1}{a}$.

The choice of the interval Δ' is not unique, generally speaking. If such an interval is selected one can take any smaller interval instead of Δ'. It means that we are interested only in some sufficiently small neighborhood of a_0. The operations of multiplication and inversion of transformations T_a are feasible only for values of the parameter a from the above neighborhood. Therefore, the object introduced by Definition 1.1 is termed as a *local* group. In what follows the sufficient closeness of all considered values of the parameters $a, b \ldots$ to the value a_0 is provided.

Further on, the term "group G_1" will be used to indicate a local one-parameter continuous transformation group.

1.1.2 Canonical parameter

Generally introduction of the new parameter $\bar{a} = \bar{a}(a)$, where $\bar{a}(a)$ is a thrice continuously differentiable monotonous function, changes φ, Δ and Δ'.

In what follows, we assume that $a_0 = 0$ without loss of generality. Note that in this case the definition leads to the following properties of the function $\varphi(a,b)$:

$$\varphi(0,0) = 0, \quad \varphi(a,0) = a, \quad \varphi(0,b) = b,$$
$$\varphi(a,a^{-1}) = \varphi(a^{-1},a) = 0. \tag{1.1.4}$$

The parameter a is said to be canonical if the multiplication law is given by

$$\varphi(a,b) = a + b.$$

Then $a^{-1} = -a$ and equations (1.1.3) take the form

$$f^i(f(x,a),b) = f^i(x,a+b) \quad (i = 1,\dots,N). \tag{1.1.5}$$

Theorem 1.1. A canonical parameter can be introduced in any one-parameter group.

Proof. Let $T_c = T_b T_a$, so that $c = \varphi(a,b)$. Let us give a small increment Δb to the parameter b, then c receives a small increment Δc, so that

$$\varphi(a,b+\Delta b) = c + \Delta c.$$

For transformations it is written by the formula

$$T_{b+\Delta b} T_a = T_{c+\Delta c}.$$

Multiplying the right-hand side by

$$T_c^{-1} = T_a^{-1} T_b^{-1},$$

one obtains

$$T_{c+\Delta c} T_c^{-1} = T_{b+\Delta b} T_b^{-1}$$

due to the associative multiplication law. The equality has the form

$$\varphi(c^{-1}, c+\Delta c) = \varphi(b^{-1}, b+\Delta b) \tag{1.1.6}$$

in terms of the function φ.

Let

$$V(b) = \left.\frac{\partial \varphi(a,b)}{\partial b}\right|_{a=b^{-1}}.$$

Taylor's formula and the equation $\varphi(b^{-1},b) = 0$ yield

$$\varphi(b^{-1}, b+\Delta b) = V(b)\Delta b + O(|\Delta b|^2).$$

Applying the above equation to Eq. (1.1.6) and invoking that $|\Delta c| = O|\Delta b|$ one obtains

$$V(c)\Delta c = V(b)\Delta b + O(|\Delta b|^2). \tag{1.1.7}$$

Dividing both sides of Eq. (1.1.7) by Δb and taking the limit $\Delta b \to 0$, one arrives at the differential equation

$$V(c)\frac{dc}{db} = V(b) \tag{1.1.8}$$

with the initial condition

$$c|_{b=0} = a.$$

Furthermore, equations (1.1.4) show that $V(0) = 1$.

Let us introduce the function

$$\bar{a}(a) = \int_0^a V(s)ds.$$

Then the function $c = \varphi(a,b)$, determined by the relations

$$\bar{a}(c) = \bar{a}(a) + \bar{a}(b), \tag{1.1.9}$$

is a solution to Eq. (1.1.8).

The function $\bar{a}(a)$ is obviously monotonous and thrice continuously differentiable with respect to a. Taking it as a new parameter, one obtains that the new parameter is canonical due to Eq. (1.1.9).

Corollary 1.1. Any one-parameter transformation group is Abelian. Indeed, if a is a canonical parameter then, according to the definition, one has

$$T_bT_a = T_{a+b} = T_{b+a} = T_aT_b.$$

1.1.3 Examples

Example 1.1. *Translations* on a straight line:

$$x' = x + a.$$

Here

$$\varphi(a,b) = a + b.$$

Translations in an N-dimensional space in the direction of the vector $\lambda = (\lambda^1, \ldots, \lambda^N)$ are given by

$$x'^i = x^i + \lambda^i a \quad (i = 1, \ldots, N).$$

Example 1.2. *Dilations*:

$$x' = ax \quad (\Delta = (0,\infty)).$$

Here

$$\varphi(a,b) = ab.$$

Assuming that $a = e^{\bar{a}}$, one arrives to the canonical parameter $\bar{a}$.

The group G_1 of dilations in an N-dimensional space has the form

$$x'^i = a^{v^i} x^i,$$

where v^i =const. $(i = 1,\ldots,N)$.

Example 1.3. *Group of rotations* in the plane (x,y) :

$$x' = \sqrt{1-a^2}x + ay, \quad y' = -ax + \sqrt{1-a^2}y.$$

It is clear from the geometric meaning of these transformations that the transition to the canonical parameter is given by the formula $a = \sin\bar{a}$. In this parameter the rotation transformations take the standard form

$$x' = x\cos\bar{a} + y\sin\bar{a}, \quad y' = y\cos\bar{a} - x\sin\bar{a},$$

where $\bar{a} \in (-\pi,\pi)$.

Example 1.4. The transformations

$$x' = \frac{x}{1-ax}, \quad y' = \frac{y}{1-ax}$$

form the group of projective transformations on the (x,y) plane. Here a is a canonical parameter.

1.1.4 Auxiliary functions of groups

In what follows, we assume that the parameter a in groups G_1 under consideration is chosen to be canonical unless otherwise indicated.

Let us relate the auxiliary functions

$$\xi^i(x) = \left.\frac{\partial f^i(x,a)}{\partial a}\right|_{a=0} \quad (i = 1,\ldots,N) \tag{1.1.10}$$

to the group G_1 given by Eqs. (1.1.2).

Theorem 1.2. The functions $f^i(x,a)$ defining a group of transformations satisfy the system of differential equations

$$\frac{\partial f^i}{\partial a} = \xi^i(f) \quad (i = 1, \ldots, N) \tag{1.1.11}$$

with the initial conditions

$$f^i|_{a=0} = x^i \quad (i = 1, \ldots, N). \tag{1.1.12}$$

Conversely, given any system of sufficiently smooth (continuously differentiable) functions $\xi^i(x)$, the functions $f^i(x,a)$ obtained by solving the problem (1.1.11)—(1.1.12) determines the group G_1.

Proof. Let us give a small increment Δa to the parameter a and write the equation $T_{a+\Delta a} = T_{\Delta a} T_a$ in terms of the functions f^i :

$$f^i(x, a + \Delta a) = f^i(f(x,a)\Delta a).$$

The Taylor expansion of both sides of the above equation with respect to Δa has the form

$$f^i(x, a + \Delta a) = f^i(x,a) + \frac{\partial f^i}{\partial a}\Delta a + O(|\Delta a|^2),$$

$$f^i(f(x,a), \Delta a) = f^i(x,a) + \frac{\partial f^i}{\partial \Delta a}(f)\Big|_{\Delta a = 0} \Delta a + O(|\Delta a|^2).$$

Equating the right-hand sides of the system, dividing by Δa and letting $\Delta a \to 0$, one obtains Eqs. (1.1.11). Equations (1.1.12) hold due to the fact that T_0 is the identity transformation. Thus, the direct statement is proved.

Conversely, let $\xi^i(x)$ be a given set of continuously differentiable functions. Equations (1.1.11) provide a system of ordinary differential equations with the independent variable a. The assumptions of the theorem guarantee that this system has a unique solution in a neighborhood of $a = 0$. The solution provides a one-parameter family of transformations. Let us demonstrate that it is a group G_1. It is manifest from Eqs. (1.1.12) that we have the identity transformation when $a = 0$. Let us prove that the equation $T_b T_a = T_{a+b}$ holds in a certain neighborhood of the values $a = 0$. In terms of the function f we have to show that

$$f^i(f(x,a), b) = f^i(x, a + b).$$

Let

$$y^i(b) = f^i(f(x,a), b), \quad z^i(b) = f^i(x, a + b).$$

Calculating the derivatives of these functions one obtains

$$\frac{\partial y^i}{\partial b} = \frac{\partial f^i(f,b)}{\partial b} = \xi^i(y)$$

since the functions f^i satisfy Eqs. (1.1.11). Moreover, by virtue of Eqs. (1.1.12) one has $f^i(f(x,a),0) = f^i(x,a)$, i.e.

$$y^i(0) = f^i(x,a).$$

On the other hand, the same reasoning shows that

$$\frac{dz^i}{db} = \frac{\partial f^i(x,a+b)}{\partial b} = \xi^i(z), \quad z^i(0) = f^i(x,a).$$

Thus, functions $y^i(b)$ and $z^i(b)$ satisfy one and the same system of differential equations and the same initial data. The theorem on uniqueness of a solution of a system of differential equations guarantees that

$$y^i(b) = z^i(b).$$

The existence of the inverse transformation follows from the fact that letting $a^{-1} = -a$ one obtains

$$T_a(T_a)^{-1} = T_a T_{-a} = T_0.$$

The later is the identity transformation by virtue of Eqs. (1.1.12).

Theorem 1.2 can be used for constructing local one-parameter transformation groups.

1.2 Infinitesimal operator of the group

1.2.1 Definition and examples

We will use the usual convention on summation with respect to repeated indices. Namely, if the upper and the lower indices are the same in a sum, the sign $\sum$ is omitted for the sake of brevity. For instance, instead of the expression

$$\sum_{i=1}^{N}\sum_{j=1}^{N} A^{ij} a_i b_j,$$

we write $A^{ij} a_i b_j$.

Definition 1.2. An infinitesimal operator of the group G_1 is the linear differential operator

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{1.2.1}$$

where $\xi^i(x)$ are determined in (1.1.10). Functions $\xi^i(x)$ are coordinates of the operator X.

Let us write out the operators of the groups G_1 for Examples 1.1-1.4 from §1.1.

Example 1.5. The operator of the translation group $x' = x + a$ along the x-axis is