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Nail H. Ibragimov

Transformation Groups and Lie Algebras

变换群和李代数

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Author
Nail H. Ibragimov
Department of Mathematics and Science
Blekinge Institute of Technology
S-371 79 Karlskrona, Sweden

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Preface

The term *transformation group* refers to the following properties of a collection G of invertible transformations $\bar{x} = T(x)$ of certain objects x :

1°. G contains the identity transformation I.
2°. G contains the inverse T^{-1} of any $T \in G$.
3°. G contains the product T_2T_1 of any $T_1, T_2 \in G$.

Note that the identity transformation I is defined by the equation $I(x) = x$. The product T_2T_1 is defined as a successive action of T_1 and T_2, i.e.

$$\big(T_2T_1\big)(x) = T_2\big(T_1(x)\big).$$

Finally, the inverse T^{-1} is defined by the equations $T^{-1}T = TT^{-1} = I$.

The group property of G is closely connected with the *invariance* of sets of the objects x under the transformations $T \in G$. We can formulate the statement in the following form.

Proposition. Let S be a set of objects x and G be the collection of all invertible transformations T defined on S and mapping any $x \in S$ into $T(x) = \bar{x} \in S$. Then G is a group.

Proof. Let us verify that the group properties $1° - 3°$ hold. The validity of the property 1° is obvious because $x \in S$ implies $I(x) = x \in S$. Hence, $I \in G$. Furthermore, $T(x) = \bar{x} \in S$ implies that $T^{-1}(\bar{x}) = x \in S$, and hence $T^{-1} \in G$, i.e. the property 2° is also satisfied. Finally, to verify the property 3°, we note that if $T_1, T_2 \in G$, then the action $T_2\big(T_1(x)\big)$ is defined because $T_1(x) \in S$, and $T_2\big(T_1(x)\big) \in S$ because T_2 maps any element of S into an element of S. Hence, $T_1, T_2 \in G$. This completes the proof.

In particular, if x denotes a solution of a given differential equation $F = 0$ and S is the totality of the solutions of $F = 0$, then the above statement shows that the collection of all transformations mapping any solution of $F = 0$ into a solution of the same differential equation compose a group. It is called the *group admitted by the differential equation,* or the *symmetry group* of the equation in question.

Part I of these notes introduces the reader to the basic concepts of the classical theory of local transformation groups and their Lie algebras. It has been designed for the graduate course on *Transformation groups and Lie algebras* that I have been teaching at Blekinge Institute of Technology, Karlskrona, Sweden, since 2002. The aim of this course was to augment a preliminary knowledge on symmetries of differential equations obtained by students during the course *Differential equations* based on my book [17], *A practical course in differential equations and mathematical modelling*.

Part II of these notes provides an easy to follow introduction to the new topic. It is based on my talks at various conferences, in particular on the plenary lecture at the International Workshop on "Differential equations and chaos" (University of Witwatersrand, Johannesburg, South Africa, January 1996). The final form of the presentation of this material, used in the present book, was prepared for my lectures "Approximate transformation groups" delivered for MSc students at Blekinge Institute of Technology since 2009.

Each part of the book contains an Assignment provided by detailed solutions of all problems. I hope that these assignments will be useful both for students and teachers.

Nail H. Ibragimov

Contents

Part I
Local Transformation Groups

Calculations show that groups admitted by differential equations involve one or more parameters and depend continuously on these parameters. This circumstance led Lie to the concept of *continuous transformation groups*. Multi-parameter continuous transformation groups are composed by *one-parameter groups* depending on a single continuous parameter. Each one-parameter group is determined by its *infinitesimal transformation* or the corresponding first-order linear differential operator termed the *generator* of the one-parameter group. One-parameter transformation groups and their generators are connected by means of the so-called *Lie equations*. Since the existence of solutions of the Lie equations is guaranteed, in general, only for values of the group parameter in a small neighborhood of its initial value, one arrives at what is called *local groups* of continuous transformations.

The generators of multi-parameter transformation groups form specific linear spaces known as *Lie algebras*. Description of continuous transformation groups in terms of their Lie algebras simplifies the calculation and use of groups admitted by differential equations significantly. Namely, the generators of continuous groups admitted by a given differential equation are defined by solving an over-determined system of linear differential equations known as *determining equations*. The characteristic property of determining equations is that *the totality of their solutions spans a Lie algebra*.

Due to the fundamental role of one-parameter groups in Lie's theory of continuous groups, it is natural to begin the study of the general theory of transformation groups and symmetries of differential equations by considering one-parameter groups and their generators.

Chapter 1
Preliminaries

This chapter introduces the reader to a general idea of transformations and exhibits a variety of transformation groups. The duality between changes of frames of reference and point transformations is useful in group analysis. We discuss the idea of the duality in this chapter and will employ it in the next chapter for the prolongation of point transformation groups to derivatives.

1.1 Changes of frames of reference and point transformations

1.1.1 Translations

Consider, in the (x,y) plane, a point P having the coordinates (x,y) in the rectangular Cartesian reference frame with the axes Ox, Oy. Let $e = (e_1, e_2)$ be a fixed unit vector. Consider a new pair of rectangular axes $\overline{O}\overline{x}, \overline{O}\overline{y}$ parallel to the former axes such that $\overline{O}$ has the coordinates $(-ae_1, -be_2)$ with respect to the original frame of reference, where a is an arbitrary real parameter. Then the coordinates $(\bar{x},\bar{y})$ of the point P in the new frame of reference are given by

$$\bar{x} = x + ae_1, \quad \bar{y} = y + be_2. \tag{1.1.1}$$

An alternative interpretation of Eqs. (1.1.1) is as follows. One ignores the new axes $\overline{O}\overline{x}, \overline{O}\overline{y}$ and regards (x,y) and $(\bar{x},\bar{y})$ as the coordinates of points P and $\overline{P}$, respectively, each referred to the original frame Ox, Oy. Then Eqs. (1.1.1) define a transformation of the point $P(x,y)$ into the new position $\overline{P}(\bar{x},\bar{y})$ in the (x,y) plane. Accordingly, equations (1.1.1) determine the displacement (translation) of all points P of the plane through the distance a in the direction of the vector e.

1.1.2 Rotations

Consider again the rectangular Cartesian reference frame with the axes Ox, Oy. Let $O\bar{x}, O\bar{y}$ be the new pair of axes obtained by rotating the original axes round the origin

O counter-clockwise through an angle a. Let (x,y) and $(\bar{x},\bar{y})$ be the coordinates of a point P referred to the axes Ox, Oy and $O\bar{x}, O\bar{y}$, respectively. Then we have

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a. \tag{1.1.2}$$

Indeed, in the polar coordinates (r,θ), connected with the Cartesian coordinates by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1.1.3}$$

the rotation by the angle a about the origin clockwise is written

$$\bar{r} = r, \quad \bar{\theta} = \theta - a. \tag{1.1.4}$$

Equations (1.1.3), (1.1.4) yield the following transformation:

$$\bar{x} = \bar{r}\cos\bar{\theta} = r\cos(\theta - a), \quad \bar{y} = \bar{r}\sin\bar{\theta} = r\sin(\theta - a).$$

Expanding $\cos(\theta - a)$ and $\sin(\theta - a)$ and substituting $r\cos\theta = x,\ r\sin\theta = y$, one arrives at Eqs. (1.1.2).

An alternative interpretation of Eqs. (1.1.2) is as follows. We regard (x,y) and $(\bar{x},\bar{y})$ as the coordinates of the points P and $\overline{P}$, respectively, each referred to the same axes Ox, Oy. Then Eqs. (1.1.2) accomplish the rotation of all points of the plane about O clockwise through the angle a.

1.1.3 Galilean transformation

Everyone travelling by train can observe the duality between uniform motions of his local frame of reference (a train) and outside points (people or other objects on a depot). This remarkable exhibition of the duality, when one cannot determine who is actually moving, is known in the classical mechanics as *Galileo's relativity principle.* It is equivalent to the invariance of equations of motion of mechanical systems under the transformation

$$\bar{t} = t, \quad \overline{\boldsymbol{x}} = \boldsymbol{x} + t\boldsymbol{V}, \tag{1.1.5}$$

where $\boldsymbol{V}$ is the constant velocity. Differentiation of $\overline{\boldsymbol{x}}$ with respect to $\bar{t} = t$ yields

$$\overline{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{V}. \tag{1.1.6}$$

The transformation (1.1.6) of the velocity is a mathematical expression of Galileo's relativity principle. The transformation (1.1.5) is known as the Galilean transformation and lies at the core of the *Galilean group* which is one of the most important groups in non-relativistic physics.

1.2 Introduction of transformation groups

1.2.1 Definitions and examples

We will consider invertible transformations in an n-dimensional Euclidean space $\mathbb{R}^n$ defined, in coordinates, by equations of the form

$$\bar{x}^i = f^i(x), \quad i = 1, \dots, n, \tag{1.2.1}$$

where the vector-function $f = (f^1, \dots, f^n)$ is continuous together with its derivatives involved in further discussions. Since the transformation (1.2.1) is invertible, there exists the inverse transformation

$$x^i = (f^{-1})^i(\bar{x}), \quad i = 1, \dots, n. \tag{1.2.2}$$

Let us denote the transformation (1.2.1) by T and its inverse (1.2.2) by T^{-1}. Thus, T carries any point

$$x = (x^1, \dots, x^n) \in \mathbb{R}^n$$

into a new position

$$\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \mathbb{R}^n,$$

and T^{-1} returns $\bar{x}$ into the original position x. It is assumed that the coordinates x^i and $\bar{x}^i$ of points x and $\bar{x}$, respectively, are referred to one and the same coordinate system. The identical transformation

$$\bar{x}^i = x^i, \quad i = 1, \dots, n, \tag{1.2.3}$$

will be denoted by I.

Let T_1 and T_2 be two transformations of the form (1.2.1) with functions f_1^i and f_2^i, respectively. Their *product* T_2T_1 (termed also *composition* and denoted by $T_2 \circ T_1$) is defined as the consecutive application of these transformations and is given by

$$\bar{\bar{x}}^i = f_2^i(\bar{x}) = f_2^i(f_1(x)), \quad i = 1, \dots, n. \tag{1.2.4}$$

The geometric interpretation of the product is as follows. Since T_1 carries the point x to the point $\bar{x} = T_1(x)$, which T_2 carries to the new position $\bar{\bar{x}} = T_2(\bar{x})$, the effect of the product T_2T_1 is to carry x directly to its final location $\bar{\bar{x}}$, without a stopover at $\bar{x}$. Thus, equation (1.2.4) means that

$$\bar{\bar{x}} \stackrel{\text{def}}{=} T_2(\bar{x}) = T_2T_1(x). \tag{1.2.5}$$

In this notation, the definition of the inverse transformation (1.2.2) means

$$TT^{-1} = T^{-1}T = I. \tag{1.2.6}$$

Definition 1.1. A set G of transformations (1.2.1) in $\mathbb{R}^n$ containing the identity I is called a transformation group if it contains the inverse T^{-1} of every transformation $T \in G$ and the product $T_1 T_2$ of any transformations $T_1, T_2 \in G$. Thus, the attributes of the group G are:

$$I \in G, \quad \text{and} \quad T^{-1} \in G, \quad T_1 T_2 \in G \quad \text{whenever } T, T_1, T_2 \in G. \tag{1.2.7}$$

Example 1.1. The set $G = \{I, T_1, \ldots, T_5\}$ of the transformations

$$\begin{gathered} I : \bar{x} = x, \qquad T_1 : \bar{x} = 1 - x, \qquad T_2 : \bar{x} = \frac{1}{x}, \\ T_3 : \bar{x} = \frac{1}{1-x}, \quad T_4 : \bar{x} = \frac{x}{x-1}, \quad T_5 : \bar{x} = \frac{x-1}{x} \end{gathered} \tag{1.2.8}$$

on the straight line is a group containing six elements (see, e.g., [6], §9). The group properties (1.2.7) can be verified by computing the inverses and products of the transformations (1.2.8), e.g.

$$\begin{gathered} T_1^{-1} = T_1,\ T_2^{-1} = T_2,\ T_3^{-1} = T_5,\ T_4^{-1} = T_4,\ T_5^{-1} = T_3, \\ T_1^2 = I, \quad T_2^2 = I, \quad T_3^2 = T_5, \quad T_4^2 = I, \quad T_5^2 = T_3, \\ T_2 T_1 = T_3, \quad T_1 T_2 = T_5, \quad T_3 T_1 = T_2, \quad T_1 T_3 = T_4. \end{gathered} \tag{1.2.9}$$

Example 1.2. Consider the set G of all translations (displacements) T_a :

$$\bar{x} = x + a \tag{1.2.10}$$

on the straight line. Since $\bar{x} = x$ when $a = 0$, the set G contains the identity $I = T_0$. Furthermore, the combined effect of two translations, T_a and T_b, acting in succession, is to displace x through the distance $a + b$. Hence,

$$T_b T_a = T_{a+b}. \tag{1.2.11}$$

Equation (1.2.11) shows that

$$T_a^{-1} = T_{-a}.$$

Thus, the transformations (1.2.10) obey the group properties (1.2.7), and hence define a *one-parameter group* G, i.e. a group containing one arbitrary parameter a. This group is known as the *translation group* and provides one of the simplest illustrations to the following definition.

Definition 1.2. A set G of transformations T_a in $\mathbb{R}^n$ depending continuously on a parameter a, where a ranges over all real numbers from a given interval $U \subset R$, is called a *one-parameter group* if there is a unique value $a = a_0$ in U providing the identical transformation, $T_{a_0} = I$, and the following conditions hold for all $a, b \in U$:

$$T_a^{-1} = T_{a^{-1}} \in G, \quad T_b T_a = T_c \in G, \tag{1.2.12}$$

where $a^{-1}, c \in U$ and $c = \phi(a,b)$ is a continuous function.

Definition 1.3. A group containing a finite number of parameters and depending continuously on these parameters was termed by Lie a *finite continuous group*. A continuous group which depends on $r < \infty$ essential parameters is called an r-parameter group and is denoted by G_r.

Remark 1.1. A continuous group of a different type is provided by invertible transformations $\overline{x} = f(x)$ of a straight line. The set of all these transformations, where $f(x)$ ranges over all continuously differentiable functions satisfying the invertibility condition $f'(x) \neq 0$, is a group. Continuous groups involving arbitrary functions are called *infinite continuous groups*. Thus, $\overline{x} = f(x)$ is an example of an infinite continuous group involving one arbitrary function $f(x)$.

Example 1.3. The set G of the transformations

$$\overline{x} = a_1 + a_2 x, \quad a_2 \neq 0 \tag{1.2.13}$$

provides an example of a *two-parameter group*, i.e. a group involving two parameters, a_1, a_2. This group is known as the *general linear group*.

Let us verify that the group properties (1.2.7) are satisfied. We will introduce the vector valued parameter

$$a = (a_1, a_2)$$

and denote the transformation (1.2.13) by T_a. Thus,

$$T_a: \quad \overline{x} = a_1 + a_2 x. \tag{1.2.14}$$

The identity transformation is obtained by letting

$$a_1 = 0, \quad a_2 = 1. \tag{1.2.15}$$

Furthermore, the transformation T_b with the parameter $b = (b_1, b_2)$ maps the point $\overline{x}$ into $\overline{\overline{x}}$ defined as follows:

$$T_b: \quad \overline{\overline{x}} = b_1 + b_2 \overline{x} = b_1 + b_2(a_1 + a_2 x). \tag{1.2.16}$$

Hence, the combined effect of the transformations T_a and T_b acting in succession is

$$T_b T_a: \quad \overline{\overline{x}} = b_1 + b_2 a_1 + b_2 a_2 x = c_1 + c_2 x.$$

It means that

$$T_b T_a = T_c, \tag{1.2.17}$$

where $c = (c_1, c_2)$ is the vector valued parameter with the components

$$c_1 = b_1 + b_2 a_1, \quad c_2 = b_2 a_2. \tag{1.2.18}$$

Let us find the inverse transformation to (1.2.14). Comparing Eqs. (1.2.6), (1.2.17) and using Eqs. (1.2.15), (1.2.18), we see that the inverse transformation $T_b = T_a^{-1}$

to T_a is found by solving the equations

$$b_1 + b_2 a_1 = 0, \quad b_2 a_2 = 1,$$

whence

$$b_2 = \frac{1}{a_2}, \quad b_1 = -\frac{a_1}{a_2}.$$

One can verify by substituting these values of b_1, b_2 in Eqs. (1.2.15) that one has indeed $\bar{\bar{x}} = x$. Thus, $T_a^{-1} = T_{a^{-1}}$, where a^{-1} is the vector valued parameter

$$a^{-1} = \left(-\frac{a_1}{a_2}, \quad \frac{1}{a_2}\right).$$

Definition 1.4. Let G be a transformation group. Its subset $H \subset G$ is called a subgroup of G, if H possesses all group properties (1.2.7), i.e. $I \in H$ and $T^{-1} \in H$, $T_1 T_2 \in H$ whenever $T, T_1, T_2 \in H$.

Example 1.4. The set

$$H = \{I, T_1\}$$

is a subgroup of the group $G = \{I, T_1, \ldots, T_5\}$ of the transformations (1.2.8). Indeed, H contains the identity transformation I. Furthermore,

$$T_1^{-1}, T_1^2 \in H$$

because $T_1^2 = I$, and hence $T_1^{-1} = T_1$.

Example 1.5. The two-parameter group (1.2.13) has two one-parameter subgroups, namely, the translation group (1.2.10) with $a = a_1$,

$$\bar{x} = x + a,$$

and the dilation group

$$T_a: \quad \bar{x} = ax, \quad a \neq 0, \tag{1.2.19}$$

with the parameter $a = a_2$. The group properties for the dilation (1.2.19) can be examined as above. Namely, one can readily verify that the multiplication of the dilations T_a and T_b yields (compare with Eq. (1.2.11))

$$T_b T_a = T_{ab}. \tag{1.2.20}$$

Equation (1.2.20) shows that the inverse transformation to T_a is the dilation with the parameter $a^{-1} = 1/a$:

$$T_a^{-1} = T_{a^{-1}}.$$

The identity transformation is obtained by letting $a = 1$.

Definition 1.5. Two transformation groups are said to be *similar* if one can be obtained from another by an appropriate change of the variables x^i.