Hans U. Gerber

LIFE INSURANCE MATHEMATICS

Third Edition

人寿保险数学

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with exercises contributed by Samuel H. Cox

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Preface

Two major developments have influenced the environment of actuarial mathematics. One is the arrival of powerful and affordable computers; the once important problem of numerical calculation has become almost trivial in many instances. The other is the fact that today's generation is quite familiar with probability theory in an intuitive sense; the basic concepts of probability theory are taught at many high schools. These two factors should be taken into account in the teaching and learning of actuarial mathematics. A first consequence is, for example, that a recursive algorithm (for a solution) is as useful as a solution expressed in terms of commutation functions. In many cases the calculations are easy; thus the question "why" a calculation is done is much more important than the question "how" it is done. The second consequence is that the somewhat embarrassing deterministic model can be abandoned; nowadays nothing speaks against the use of the stochastic model, which better reflects the mechanisms of insurance. Thus the discussion does not have to be limited to expected values; it can be extended to the deviations from the expected values, thereby quantifying the risk in the proper sense.

The book has been written in this spirit. It is addressed to the young reader (where "young" should be understood in the sense of operational time) who likes applied mathematics and is looking for an introduction into the basic concepts of life insurance mathematics.

In the first chapter an overview of the theory of compound interest is given. In Chapters 2–6 various forms of insurance and their mechanisms are discussed in the basic model. Here the key element is the future lifetime of a life aged x, which is denoted by T and which is (of course!) a random variable. In Chapter 7 the model is extended to multiple decrements, where different causes for departure (for example death and disability) are introduced. In Chapter 8 insurance policies are considered where the benefits are contingent on more than one life (for example widows' and orphans' pensions). In all these chapters the discussion focuses on a single policy, which is possible in the stochastic model, as opposed to the deterministic model, where each policy is considered as a member of a large group of identical policies. In Chapter 9 the risk arising from a group of policies (a *portfolio*) is examined. It is shown how the distribution of the aggregate claims can be calculated recursively. Information about this distribution is indispensable when reinsurance is purchased. The topic of Chapter 10 is of great practical importance; for simplicity of presentation the expense loading is considered only in this chapter. Chapter 11 examines some statistical problems, for instance, how to estimate the distribution of T from observations. The book has been written without much compromise; however, the appendix should be a sign of the conciliatory nature of the author. For the very same reason the basic probability space $(\Omega, \mathcal{F}, P)$ shall be mentioned at least once: now!

The publication of this book was made possible by the support of the Fund for the Encouragement of Actuarial Mathematics of the Swiss Association of Actuaries; my sincere thanks go to the members of its committee, not in the least for the freedom granted to me. I would like to thank in particular Professor Bühlmann and Professor Leepin for their valuable comments and suggestions. Of course I am responsible for any remaining flaws.

For some years now a team of authors has been working on a comprehensive text, which was commissioned by the Society of Actuaries and will be published in 1987 in its definitive form. The cooperation with the coauthors Professors Bowers, Hickman, Jones and Nesbitt has been an enormously valuable experience for me.

Finally I would like to thank my assistant, Markus Lienhard, for the careful perusal of the galley proofs and Springer-Verlag for their excellent cooperation.

Lausanne, March 1986 *Hans U. Gerber*

Acknowledgement

I am indebted to my colleague, Dr. Walther Neuhaus (University of Oslo), who translated the text into English and carried out the project in a very competent and efficient way. We are also very grateful to Professor Hendrik Boom (University of Manitoba) for his expert advice.

Lausanne and Winnipeg, April 1990 *Hans U. Gerber*

Acknowledgement

The second edition contains a rich collection of exercises, which have been prepared by Professor Samuel H. Cox of Georgia State University of Atlanta, who is an experienced teacher of the subject. I would like to express my sincere thanks to my American colleague: due to his contribution, the book will not only find *readers* but it will find *users*!

Lausanne, August 1995 *Hans U. Gerber*

To Cecil Nesbitt

Foreword

Halley's Comet has been prominently displayed in many newspapers during the last few months. For the first time in 76 years it appeared this winter, clearly visible against the nocturnal sky. This is an appropriate occasion to point out the fact that Sir Edmund Halley also constructed the world's first life table in 1693, thus creating the scientific foundation of life insurance. Halley's life table and its successors were viewed as deterministic laws, i.e. the number of deaths in any given group and year was considered to be a well defined number that could be calculated by means of a life table. However, in reality this number is random. Thus any mathematical treatment of life insurance will have to rely more and more on probability theory.

By sponsoring this monograph the Swiss Association of Actuaries wishes to support the "modern" probabilistic view of life contingencies. We are fortunate that Professor Gerber, an internationally renowned expert, has assumed the task of writing the monograph. We thank the Springer-Verlag and hope that this monograph will be the first in a successful series of actuarial texts.

Zürich, March 1986

Hans Bühlmann
President
Swiss Association of Actuaries

Acknowledgement

The second edition has been sold out rapidly. This led to the present third edition, in which several misprints have been corrected. I am thankful to Sam Cox, Cheng Shixue, Wolfgang Quapp, André Dubey and Jean Cochet for their valuable advice.

At this occasion I would like to thank Springer and the Swiss Association of Actuaries for authorising the Chinese, Slovenian and Russian editions of Life Insurance Mathematics. I am indebted to Cheng Shixue, Yan Ying, Darko Medved and Valery Mishkin. From my own experience I know that translating a scientific text is a challenging task.

Lausanne, January 1997 *Hans U. Gerber*

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Chapter 1. The Mathematics of Compound Interest

1.1 Mathematical Bases of Life Contingencies

To life insurance mathematics primarily two areas of mathematics are fundamental: the theory of compound interest and probability theory. This chapter gives an introduction to the first topic. The probabilistic model will be introduced in the next chapter; however, it is assumed that the reader is familiar with the basic principles of probability theory.

1.2 Effective Interest Rates

An interest rate is always stated in conjunction with a *basic time unit*; for example, one might speak of an annual rate of 6%. In addition, the *conversion period* has to be stated: this is the time interval at the end of which interest is credited or "compounded". An interest rate is called *effective* if the conversion period and the basic time unit are identical; in that case interest is credited at the end of the basic time unit.

Let i be an effective annual interest rate; for simplicity we assume that i is the same for all years. We consider an account (or fund) where the initial capital F_0 is invested, and where at the end of year k an additional amount of r_k is invested, for $k = 1, \cdots, n$. What is the balance at the end of n years? Let F_k be the balance at the end of year k, including the payment of r_k. Interest credited on the previous year's balance is iF_{k-1}. Thus

$$F_k = F_{k-1} + iF_{k-1} + r_k, \; k = 1, \cdots, n\,. \tag{1.2.1}$$

We may write this recursive formula as

$$F_k - (1+i)F_{k-1} = r_k\,; \tag{1.2.2}$$

if we multiply this equation by $(1+i)^{n-k}$ and sum over all values of k, all but two terms on the left hand side vanish, and we obtain

$$F_n = (1+i)^n F_0 + \sum_{k=1}^{n} (1+i)^{n-k} r_k\,. \tag{1.2.3}$$

The powers of $(1+i)$ are called *accumulation factors*. The accumulated value of an initial capital C after h years is $(1+i)^h C$. Equation (1.2.3) illustrates an obvious result: the capital at the end of the interval is the accumulated value of the initial capital plus the sum of the accumulated values of the intermediate deposits.

The *discount factor* is defined as

$$v = \frac{1}{1+i}. \tag{1.2.4}$$

Equation (1.2.3) can now be written as

$$v^n F_n = F_0 + \sum_{k=1}^{n} v^k r_k. \tag{1.2.5}$$

Hence the present value of a capital C, due at time h, is $v^h C$.

If we write equation (1.2.1) as

$$F_k - F_{k-1} = iF_{k-1} + r_k \tag{1.2.6}$$

and sum over k we obtain

$$F_n - F_0 = \sum_{k=1}^{n} iF_{k-1} + \sum_{k=1}^{n} r_k. \tag{1.2.7}$$

Thus the increment of the fund is the sum of the total interest credited and the total deposits made.

1.3 Nominal Interest Rates

When the conversion period does not coincide with the basic time unit, the interest rate is called *nominal*. An annual interest rate of 6% with a conversion period of 3 months means that interest of 6%/4 = 1.5% is credited at the end of each quarter. Thus an initial capital of 1 increases to $(1.015)^4 = 1.06136$ at the end of one year. Therefore, an annual nominal interest rate of 6%, convertible quarterly, is equivalent to an annual effective interest rate of 6.136%.

Now, let i be a given annual effective interest rate. We define $i^{(m)}$ as the nominal interest rate, convertible m times per year, which is equivalent to i. Equality of the accumulation factors for one year leads to the equation

$$(1 + \frac{i^{(m)}}{m})^m = 1 + i, \tag{1.3.1}$$

which implies that

$$i^{(m)} = m[(1+i)^{1/m} - 1]. \tag{1.3.2}$$

The limiting case $m \to \infty$ corresponds to continuous compounding. Let

$$\delta = \lim_{m\to\infty} i^{(m)} ; \tag{1.3.3}$$

this is called the *force of interest* equivalent to i. Writing (1.3.2) as

$$i^{(m)} = \frac{(1+i)^{1/m} - (1+i)^0}{1/m}, \tag{1.3.4}$$

we see that δ is the derivative of the function $(1+i)^x$ at the point $x = 0$. Thus we find that

$$\delta = \ln(1+i) \tag{1.3.5}$$

or

$$e^{\delta} = 1 + i . \tag{1.3.6}$$

We can verify this result by letting $m \to \infty$ in (1.3.1) and using the definition (1.3.3).

Thus the accumulation factor for a period of h years is $(1+i)^h = e^{\delta h}$; the discount factor for the same period of time is $v^h = e^{-\delta h}$. Here the length of the period h may be any real number.

Intuitively it is obvious that $i^{(m)}$ is a decreasing function of m. We can give a formal proof of this by interpreting $i^{(m)}$ as the slope of a secant. see (1.3.4). and using the convexity of the function $(1+i)^x$. The following numerical illustration is for $i = 6\%$.

m	$i^{(m)}$
1	0.06000
2	0.05913
3	0.05884
4	0.05870
6	0.05855
12	0.05841
∞	0.05827

1.4 Continuous Payments

We consider a fund as in Section 1.2, but now we assume that payments are made continuously with an annual instantaneous rate of payment of $r(t)$. Thus the amount deposited to the fund during the infinitesimal time interval from t to $t + dt$ is $r(t)\,dt$. Let $F(t)$ denote the balance of the fund at time t. We assume that interest is credited continuously, according to a, possibly time-dependent, force of interest $\delta(t)$. Interest credited in the infinitesimal time interval from t to $t+dt$ is $F(t)\delta(t)\,dt$. The total increase in the capital during this interval is thus

$$dF(t) = F(t)\delta(t)\,dt + r(t)\,dt\,. \tag{1.4.1}$$

To solve the corresponding differential equation

$$F'(t) = F(t)\delta(t) + r(t)\,, \tag{1.4.2}$$

we write

$$\frac{d}{dt}[e^{-\int_0^t \delta(s)\,ds}F(t)] = e^{-\int_0^t \delta(s)\,ds}r(t)\,. \tag{1.4.3}$$

Integration with respect to t from 0 to h gives

$$e^{-\int_0^h \delta(s)\,ds}F(h) - F(0) = \int_0^h e^{-\int_0^t \delta(s)\,ds}r(t)\,dt\,. \tag{1.4.4}$$

Thus the value at time 0 of a payment to be made at time t (i.e. its *present value*) is obtained by multiplication with the factor

$$e^{-\int_0^t \delta(s)\,ds}\,. \tag{1.4.5}$$

From (1.4.4) we further obtain

$$F(h) = e^{\int_0^h \delta(s)\,ds}F(0) + \int_0^h e^{\int_t^h \delta(s)\,ds}r(t)\,dt\,. \tag{1.4.6}$$

Thus the value at time h of a payment made at time $t < h$ (its accumulated value) is obtained by multiplication with the factor

$$e^{\int_t^h \delta(s)\,ds}\,. \tag{1.4.7}$$

In the case of a constant force of interest, i.e. $\delta(t) = \delta$, the factors (1.4.5) and (1.4.7) are reduced to the discount factors and accumulation factors introduced in Section 1.2.

1.5 Interest in Advance

Until now it was assumed that interest was to be credited at the end of each conversion period (or *in arrears*). But sometimes it is useful to assume that interest is credited at the beginning of each conversion period. Interest credited in this way is also referred to as *discount*, and the corresponding rate is called *discount rate* or *rate of interest-in-advance*.

Let d be an annual effective discount rate. A person investing an amount of C will be credited interest equal to dC immediately, and the invested capital