Luc Tartar

An Introduction to Sobolev Spaces and Interpolation Spaces

索伯列夫空间和插值空间导论

Springer

世界图书出版公司
www.wpcbj.com.cn

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图书在版编目（CIP）数据

索伯列夫空间和插值空间导论 = An Introduction to Sobolev Spaces and Interpolation Spaces：英文/（美）塔塔（Tartar，L.）著. —影印本. —北京：世界图书出版公司北京公司，2012.8

ISBN 978-7-5100-5043-5

Ⅰ.①索… Ⅱ.①塔… Ⅲ.①索伯列夫空间—英文②插值—英文
Ⅳ.①O177.3②O174.42

中国版本图书馆 CIP 数据核字（2012）第 186189 号

书　　名：	An Introduction to Sobolev Spaces and Interpolation Spaces
作　　者：	Luc Tartar
中 译 名：	索伯列夫空间和插值空间导论
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河市国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@wpcbj.com.cn
开　　本：	24 开
印　　张：	10.5
版　　次：	2013 年 1 月
版权登记：	图字：01-2012-5689
书　　号：	978-7-5100-5043-5　　定　价：35.00 元

In memory of Sergei SOBOLEV, 1908–1989

He pioneered the study of some functional spaces which are crucial in the study of the partial differential equations of continuum mechanics and physics, and the first part of these lecture notes is about these spaces, named after him.

In memory of Jacques-Louis LIONS, 1928–2001

He participated in the development of Sobolev spaces, in part with Enrico MAGENES, applying the general theory of interpolation spaces which he had developed with Jaak PEETRE, who further simplified the theory so that it became more easy to use, and the second part of these lecture notes is about these interpolation spaces.

To Lucia

To my children
Laure, Michaël, André, Marta

Preface

After publishing an introduction to the Navier[1]-Stokes[2,3] equation and oceanography [18], the revised version of my lecture notes for a graduate course that I had taught in the spring of 1999, I want to follow with another set of lecture notes for a graduate course that I had taught in the spring of 2000; that course was divided into two parts, the first part on Sobolev[4] spaces, and the second part on interpolation spaces. The first version had been available on the Internet, and after a few years, I find it useful to make the text available to a larger audience by publishing a revised version.

When I was a student at Ecole Polytechnique, which was still in Paris, France, on the "Montagne Sainte Geneviève",[5] I had the chance to have

[1] Claude Louis Marie Henri NAVIER, French mathematician, 1785–1836. He worked in Paris, France.

[2] Sir George Gabriel STOKES, Irish-born mathematician, 1819–1903. He worked in London, and in Cambridge, England, holding the Lucasian chair (1849–1903).

[3] Reverend Henry LUCAS, English clergyman and philanthropist, 1610–1663.

[4] Sergei L'vovich SOBOLEV, Russian mathematician, 1908–1989. He worked in Leningrad, in Moscow, and in Novosibirsk, Russia. I first met him when I was a student, in Paris in 1969, then at the International Congress of Mathematicians in Nice in 1970, and conversed with him in French, which he spoke perfectly (all educated Europeans did learn French in the beginning of the 20th century). I only met him once more, when I traveled with a French group from INRIA (Institut National de la Recherche en Informatique et Automatique) in 1976 to Akademgorodok near Novosibirsk, Russia, where he worked. There is now a Sobolev Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk, Russia.

[5] Geneviève, patroness of Paris, c 419 or 422–512.

Laurent SCHWARTZ[6–8] as my main teacher in mathematics in the first year (1965–1966), and the course contained an introduction[9] to his theory of distributions,[10] but I only heard about Sobolev spaces in my second year (1966–1967), in a seminar organized by Jacques-Louis LIONS[11–13] for interested students, in addition to his course on numerical analysis. I learnt a little more in his courses at the university in the following years, and I read a course [13] that he had taught in 1962 in Montréal, Québec (Canada), and I also read a book [1] by Shmuel AGMON,[14] corresponding to a course that he had taught at Rice[15] University, Houston, TX.

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[6] Laurent SCHWARTZ, French mathematician, 1915–2002. He received the Fields Medal in 1950. He worked in Nancy, in Paris, France, at École Polytechnique, which was first in Paris (when I had him as a teacher in 1965–1966), and then in Palaiseau, France, and at Université Paris VII (Denis Diderot), Paris, France.

[7] John Charles FIELDS, Canadian mathematician, 1863–1932. He worked in Meadville, PA, and in Toronto, Ontario (Canada).

[8] Denis DIDEROT, French philosopher and writer, 1713–1784. He worked in Paris, France, and he was the editor-in-chief of the Encyclopédie. Université Paris 7, Paris, France, is named after him.

[9] Which means that he only considered questions of convergence for sequences, and he did not teach anything about the topologies of $\mathcal{D}$ or $\mathcal{D}'$, which I first learnt in his book [15].

[10] Laurent SCHWARTZ has described something about his discovery of the concept of distributions in his biography [16].

[11] Jacques-Louis LIONS, French mathematician, 1928–2001. He received the Japan Prize in 1991. He worked in Nancy and in Paris, France, holding a chair (analyse mathématique des systèmes et de leur contrôle, 1973–1998) at Collège de France, Paris, France. I first had him as a teacher at Ecole Polytechnique in 1966–1967, and I did research under his direction, until my thesis in 1971. The laboratory dedicated to functional analysis and numerical analysis which he initiated, funded by CNRS (Centre National de la Recherche Scientifique) and Université Paris VI (Pierre et Marie Curie), is now named after him, the Laboratoire Jacques-Louis Lions.

[12] Pierre CURIE, French physicist, 1859–1906, and his wife Marie SKŁODOWSKA-CURIE, Polish-born physicist, 1867–1934, jointly received the Nobel Prize in Physics in 1903, and she also received the Nobel Prize in Chemistry in 1911. They worked in Paris, France. Université Paris 6, Paris, France, is named after them.

[13] Alfred NOBEL, Swedish industrialist and philanthropist, 1833–1896. He created a fund to be used as awards for people whose work most benefited humanity.

[14] Shmuel AGMON, Israeli mathematician, born in 1922. He worked at The Hebrew University, Jerusalem, Israel.

[15] William Marsh RICE, American financier and philanthropist, 1816–1900.

I first read about interpolation spaces (in a Hilbert[16,17] setting) in a book that Jacques-Louis LIONS had written with Enrico MAGENES[18] [14], and then he gave me his article with Jaak PEETRE[19] to read for the theory in a Banach[20,21] setting, and later he asked me to solve some problems about interpolation for my thesis in 1971, and around that time I did read a few articles on interpolation, although I can hardly remember in which of the many articles of Jaak PEETRE I may have read about some of his results. For the purpose of this course, I also consulted a book by BERGH[22,23] & LÖFSTRÖM[24] [2].

I also learnt in other courses, by Jacques-Louis LIONS or others, in seminars, and the usual process went on, learning, forgetting, inventing a new proof or reinventing one, when asked a question by a fellow researcher or a student, so that for many results in this course I can hardly say if I have read them or filled the gaps in statements that I had heard, and my memory may be inaccurate on some of these details. Some of the results may have been obtained in my own research work, which is concerned with partial differential equations from continuum mechanics or physics, and my personal reason for being interested in the subject of this course is that some of the questions studied have appeared in a natural way in a few practical problems. Of course, although a few problems of continuum mechanics or physics have led to some of the mathematical questions described in this course, many have been added for the usual reason that mathematicians are supposed to discover general structures hidden behind particular results, and describe something

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16 David HILBERT, German mathematician, 1862–1943. He worked in Königsberg (then in Germany, now Kaliningrad, Russia) and in Göttingen, Germany. The term Hilbert space was coined by his student VON NEUMANN, when he worked on his mathematical foundation of quantum mechanics.

17 János (John) VON NEUMANN, Hungarian-born mathematician, 1903–1957. He worked in Berlin, in Hamburg, Germany, and at IAS (Institute for Advanced Study), Princeton, NJ.

18 Enrico MAGENES, Italian mathematician, born in 1923. He worked in Pavia, Italy.

19 Jaak PEETRE, Estonian-born mathematician, born in 1935. He worked in Lund, Sweden.

20 Stefan BANACH, Polish mathematician, 1892–1945. He worked in Lwów (then in Poland, now Lvov, Ukraine). There is a Stefan Banach International Mathematical Center in Warsaw, Poland. The term Banach space was introduced by FRÉCHET.

21 Maurice René FRÉCHET, French mathematician, 1878–1973. He worked in Poitiers, in Strasbourg and in Paris, France. I do not know who introduced the term Fréchet space.

22 Jöran BERGH, Swedish mathematician, born in 1941. He has worked in Lund, and at Chalmers University of Technology, Göteborg, Sweden.

23 William CHALMERS Jr., Swedish merchant, 1748–1811.

24 Jörgen LÖFSTRÖM, Swedish mathematician, born in 1937. He worked at Chalmers University of Technology, Göteborg, Sweden.

more general after having done a systematic study, akin to a cleaning process. For those who do not yet know much about continuum mechanics or physics, I recommend looking first at more classical descriptions of the problems, for example by consulting the books which have been prepared under the direction of Robert DAUTRAY[25] and Jacques-Louis LIONS [4–12]. For those who already know something about continuum mechanics or physics, I recommend looking at my other lecture notes for reading about the defects which I know about classical models, because other authors rarely mention these defects even when they have heard about them: I suppose that it is the result of having been raised as the son of a (Calvinist) Protestant minister that I learnt and practiced the point of view that one should not follow the path of the majority when reason clearly points to a different direction. However, although I advocate using reason for criticizing without concessions the points of view that are taught in order to find better "truths", one should observe that this approach is more suited to mathematicians than to engineers or physicists; actually, not all "mathematicians" have been trained well enough for following that path, and that might explain why some people initially trained as mathematicians write inexact statements, which they often do not change even after being told about their mistakes, which others repeat then without knowing that they propagate errors; if their goal had not been to mislead others, a better strategy would have been to point out that some statements were only conjectures.

I have decided to write my lecture notes with some information given in footnotes about the people who have participated in the creation of the knowledge related to the subject of the course, and I have mentioned in [18] a few reasons for doing that: I had great teachers[26] like Laurent SCHWARTZ and Jacques-Louis LIONS, and I have met many mathematicians, for whom I use their first names in the text, but I have tried to give some simple biographical data for all people quoted in the text in order to situate them both in time and in space, the famous ones as well as the almost unknown ones; I have seen so many ideas badly attributed and I have tried to learn more about the mathematicians who have introduced some of the ideas which I was taught when I was a student, and to be as accurate as possible concerning the work of all.[27] Another reason is that I enjoy searching for clues, even about questions that might be thought irrelevant for my goals; I might be stopped by a word.

[25] Ignace Robert DAUTRAY (KOUCHELEVITZ), French physicist, born in 1928.

[26] Although I immediately admired their qualities, like pedagogical skill, I later became aware of some of their defects, the discussion of which I shall postpone until I decide to publish all the letters that I wrote to them.

[27] Although I have never read much, it would be quite inefficient for me to change my method of work for the moment, because too many people have recently shown a tendency to badly quote their sources. In some cases, information that I had proven something in the 1970s has been ignored, for the apparent reason that I had told that to people who wanted to avoid mentioning my name, the strange thing being that instead of trying to find someone who would have done similar

wondering about its etymology, or by a new name, wondering about who this person was, or even by a name which has been attached to a well-known institution and I want to discover who was that forgotten person in honor of whom the institution is named; the Internet has given me the possibility to find such answers, sometimes as the result of many searches which had only given small hints, and I hope that I shall be told about all the inaccuracies that are found in my text.

I was glad to learn a few years ago the motto of Hugo of Saint Victor[28] "Learn everything, and you will see afterward that nothing is useless", and to compare it with what I had already understood in my quest about how creation of knowledge occurs. I have often heard people say about a famous physicist from the past, that luck played an important role in his discovery, but the truth must be that if he had not known beforehand all the aspects of his problem he would have missed the importance of the new hint that had occurred, and so this instance of "luck" reminds me of the saying "aide toi, le ciel t'aidera" (God helps those who help themselves). Those who present chance as an important factor in discovery probably wish that every esoteric subject that they like be considered important and funded, but that is not at all what the quoted motto is about. My reasons for publishing lecture notes is to tell the readers some of what I have understood; the technical mathematical aspects of the course are one thing, the scientific questions behind the theories are another, but there is more than that, a little difficult to express in words: I will have succeeded if many become aware, and go forward on the path of discovery, not mistaking research and development, knowing when and why they do one or the other, and keeping a higher goal in mind when for practical reasons they decide to obey the motto of the age for a while, "publish or perish".

When I was a graduate student in Paris, my advisor invited me a few times to join a dinner held for a visitor, who had usually talked in the Lions–Schwartz seminar, which met every Friday at IHP (Institut Henri Poincaré[29]). It was before Université de Paris split into many smaller universities, which happened in 1970 or 1971, and I had heard my advisor mention a special fund from ZAMANSKI,[30] the dean of "Faculté des Sciences". The buildings for sciences were then known as "Halle aux Vins", because they were being built on a place previously used for the wine market, and it was only after all the wine merchants had moved to Bercy, on the other bank of the river

work before me they sometimes preferred to quote one of their friends who had used the result in the 1990s, without any mention of an author for it.

[28] Hugo VON BLANKENBURG, German-born theologian, 1096–1141. He worked at the monastery of Saint Victor in Paris, France.

[29] Jules Henri POINCARÉ, French mathematician, 1854–1912. He worked in Paris, France. There is an Institut Henri Poincaré (IHP), dedicated to mathematics and theoretical physics, part of Université Paris VI (Pierre et Marie Curie), Paris, France.

[30] Marc ZAMANSKI, French mathematician, 1915–1996. He worked in Paris, France.

Seine, that the complex of buildings became known as "Jussieu",[31] and I wonder if part of the plan was not to support the local restaurants, who were losing a lot of their customers because of the transfer of the wine market to Bercy. It was at one of these dinners, which all took place in the restaurant "Chez Moissonnier", rue des fossés Saint Bernard,[32–34] that I first met Sergei SOBOLEV, probably in 1969, but I had not been aware that he had given a talk; my understanding of English was too poor at the time for conversing with visitors, but fortunately Sergei SOBOLEV spoke French, perfectly.[35]

I met Sergei SOBOLEV a second time, at the International Congress of Mathematicians in Nice in 1970, and as he was waiting in front of me in a line at a cafeteria, I took the occasion to ask him a question, about what he thought were interesting mathematical areas to study, and I must have mentioned applications, because his answer was that it was difficult to know what could become useful, as even questions in number theory had found applications.

I met Sergei SOBOLEV a third time in 1976, when I traveled to Novosibirsk with a group from INRIA (Institut National de Recherche en Informatique[36,37] et Automatique[38]), and he was working then on cubature formulas [17], i.e., quadrature formulas in three dimensions, a subject which I did not find interesting enough to enquire about it. Apart from the fact that he seemed eager not to be too involved with the political establishment, which may explain why he worked alone, the subject may have actually been more important in the Soviet Union than in the West, as I learnt during the same

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[31] Antoine Laurent DE JUSSIEU, French botanist, 1748–1836. He worked in Paris, France.

[32] Bernard DE FONTAINES, French monk, 1090–1153. He founded the monastery of Clairvaux, France, and is known as Saint Bernard de Clairvaux. He was canonized in 1174 by ALEXANDER III, and PIUS VIII bestowed on him the title of Doctor of the Church in 1830.

[33] Orlando BANDINELLI, Italian Pope, 1105–1181. Elected Pope in 1159, he took the name ALEXANDER III.

[34] Francesco Xaverio CASTIGLIONE, Italian Pope, 1761–1830. Elected Pope in 1829, he took the name PIUS VIII.

[35] Until the beginning of the 20th century, every educated person in Europe learnt French. I was told that Sergei SOBOLEV was born into an aristocratic family, and that without the 1917 revolution in Russia he would have become a duke.

[36] Informatique is the French word for computer science, and ordinateur is the French word for computer, but these words were in use much before DE GAULLE created a special committee for coining French words that had to be used in replacement of the American words invented in technology.

[37] Charles DE GAULLE, French general and statesman, 1890–1970. Elected President of the Republic in 1959 (by the two legislative chambers), he had then a new constitution for France accepted (5th republic), and he was reelected by direct election in 1965; he resigned in 1969.

[38] Automatique is the French word for control theory.

trip, when my good friend Roland GLOWINSKI[39] told me about some discussions with Nicolai YANENKO,[40] who had said that a numerical scheme mentioned by the French team did not work; together with Jean CEA,[41] he was trying to understand why the scheme did not work on Russian computers, while it worked well in France (although it was not a very efficient scheme), and after a few days, the explanation was found, which was that the Russian computers did not have double precision, a feature which had existed for some time on the American computers used in France, but such computers could not be imported into the Soviet Union, because of the embargo decided by United States, a consequence of the Cold War; it could have been precisely the limitations of the computers which made the study of good cubature formulas useful.

In his description of lives of great men, PLUTARCH[42] told a story about ARCHIMEDES[43,44] and Cicero.[45] At the time when Cicero became governor of Sicily, he wanted to visit the tomb of ARCHIMEDES, and the people of Syracuse had no idea where it was, but Cicero knew something about the tomb, which permitted him to discover it: ARCHIMEDES had wanted to have on his tomb a reminder of what he thought was his best result, that the surface of a sphere of radius R is equal to the lateral surface of a tangent cylinder of same height (i.e., with a circular base of radius R and height $2R$), which is then $4\pi R^2$, so Cicero's aides just had to go around the cemeteries of Syracuse and look for a tomb with a sphere and a cylinder on it.

Apparently, there was no mathematical result that Jacques-Louis LIONS was really proud of having proven, because after his death people who had been in contact with him insisted that what he had been most proud of was one of his successes in manipulating people. In 1984, in a discussion with Laurent SCHWARTZ, I had said that I was not good at following complicated proofs and he had said that one cannot follow another person's mind and that only Jacques-Louis LIONS was capable of following every proof. Roger

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39 Roland GLOWINSKI, French-born mathematician, born in 1937. He worked at Université Paris VI (Pierre et Marie Curie), Paris, France, and he works now in Houston, TX.

40 Nikolai Nikolaevich YANENKO, Russian mathematician, 1921–1984. He worked in Novosibirsk, Russia.

41 Jean CÉA, French mathematician, born in 1932. He worked in Rennes, and in Nice, France.

42 Mestrius PLUTARCHUS, Greek biographer, 46–120.

43 ARCHIMEDES, Greek mathematician, 287 BCE–212 BCE. He worked in Siracusa (Syracuse), then a Greek colony, now in Italy.

44 BCE = before common era; those who insist in linking questions of date with questions of religion may consider that it means "before Christian era".

45 Marcus TULLIUS Cicero, Roman orator and politician, 106 BCE–43 BCE.

PENROSE[46–48] expresses the same fact that in listening to someone he actually tries to guess what the other person says, because he cannot in general follow the details of the other person's reasoning, which is the way I feel; however, he wrote that in a book whose whole argument looked quite silly to me (and Jacques-Louis LIONS also made the same comment about the book), but I read it entirely because my good friend Constantine DAFERMOS[49–51] had offered it to me, which I interpreted as meaning that I should have a critical opinion about "artificial intelligence". What had prompted that information from Laurent SCHWARTZ about Jacques-Louis LIONS was that I always insist that my proofs are not difficult (and that the importance is more in the analysis about what kind of result to look for), because I always try to simplify what I have done, so that it can be easily understood; Jacques-Louis LIONS had this quality of looking for simplifying proofs, and he sometimes asked me to find a general proof after he had obtained a particular result whose proof looked much too complicated to him to be the "right one".

I recall a remark of Jacques-Louis LIONS that a framework which is too general cannot be very deep, and he had made this comment about semi-group theory; he did not deny that the theory is useful, and the proof of the Hille[52,53]–Yosida[54] theorem is certainly more easy to perform in the abstract setting of a Banach space than in each particular situation, but the result applies to equations with very different properties that the theory cannot

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46 Sir Roger PENROSE, English mathematician, born in 1931. He received the Wolf Prize (in Physics!) in 1988. He has worked in London and in Oxford, England, where he held the Rouse Ball professorship.

47 Ricardo WOLF, German-born diplomat and philanthropist, 1887–1981. He emigrated to Cuba before World War I; from 1961 to 1973 he was Cuban Ambassador to Israel, where he stayed afterwards. The Wolf foundation was established in 1976 with his wife, Francisca SUBIRANA-WOLF, 1900–1981, "to promote science and art for the benefit of mankind".

48 Walter William Rouse BALL, English mathematician, 1850–1925. He worked in Cambridge, England.

49 Constantine M. DAFERMOS, Greek-born mathematician, born in 1941. He worked at Cornell University, Ithaca, NY, and he works now at Brown University, Providence, RI.

50 Ezra CORNELL, American philanthropist, 1807–1974.

51 Nicholas BROWN Jr., American merchant, 1769–1841.

52 Einar Carl HILLE (HEUMAN), Swedish-born mathematician, 1894–1980. He worked in Princeton, NJ, and at Yale University, New Haven, CT.

53 Elihu YALE, American-born English philanthropist, Governor of Fort St George, Madras, India, 1649–1721.

54 Kôsaku YOSIDA, Japanese mathematician, 1909–1990. He worked in Tokyo, Japan, where I met him during my first trip to Japan, in the fall of 1976.

distinguish. After having written an article with Jean LERAY,[55,56] who had wanted to use the generalization of the Brouwer[57] topological degree that he had obtained with SCHAUDER,[58] Jacques-Louis LIONS had observed that the regularity hypotheses for applying the Leray–Schauder theory are too strong and that a better approach is to use the Brouwer topological degree for finite-dimensional approximations, and then pass to the limit, taking advantage of the particular properties of the problem for performing that limiting process. Maybe Jacques-Louis LIONS also thought that the theory of interpolation spaces lacks depth because it is a general framework based on two arbitrary Banach spaces, but certainly the theory is worth studying, and I decided to teach it after the first part on Sobolev spaces for that reason.

In an issue of the Notices of the American Mathematical Society, Enrico MAGENES had recalled the extreme efficiency of Jacques-Louis LIONS when they worked together, in particular for the study of their famous interpolation space $H^{1/2}_{00}(\Omega)$, but Peter LAX[59,60] had recalled an interesting encounter, showing that early in his career Jacques-Louis LIONS was already playing the functional analysis card against continuum mechanics. I had opposed my former advisor on this question, because I wanted to understand more about continuum mechanics and physics, and to develop new tools for going further in the study of the partial differential equations of continuum mechanics or physics, and in particular to show the limitations of the classical tools from functional analysis, but despite our opposition on these important questions, I have chosen to dedicate these lecture notes to Jacques-Louis LIONS, because he played an important role in the development of the theory of interpolation spaces, although the theory would have been quite difficult to use without the simplifying work of Jaak PEETRE.

Jaak PEETRE wrote to me a few years ago that he had obtained the same results on interpolation as Jacques-Louis LIONS, who had kindly proposed that they publish an article together. It seems to me that Jaak PEETRE was

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[55] Jean LERAY, French mathematician, 1906–1998. He shared the Wolf Prize in 1979 with WEIL. He worked in Nancy, France, in a prisoner of war camp in Austria (1940–1945), and in Paris, France; he held a chair (théorie des équations différentielles et fonctionnelles, 1947–1978) at Collège de France, Paris, France.

[56] André WEIL, French-born mathematician, 1906–1998. He received the Wolf Prize in 1979 (shared with Jean LERAY). He worked in Aligarh, India, in Haverford, PA, in Swarthmore, PA, in São Paulo, Brazil, in Chicago, IL, and at IAS (Institute for Advanced Study), Princeton, NJ.

[57] Luitzen Egbertus Jan BROUWER, Dutch mathematician, 1881–1966. He worked in Amsterdam, The Netherlands.

[58] Juliusz Pawel SCHAUDER, Polish mathematician, 1899–1943. He worked in Lwów (then in Poland, now Lvov, Ukraine).

[59] Peter David LAX, Hungarian-born mathematician, born in 1926. He received the Wolf Prize in 1987, and the Abel Prize in 2005. He works at NYU (New York University), New York, NY.

[60] Niels Henrik ABEL, Norwegian mathematician, 1802–1829.

following the influence of M. RIESZ[61–65] in Lund, Sweden, while Jacques-Louis LIONS was following the path opened by the characterization of traces of functions in $W^{1,p}$ by Emilio GAGLIARDO,[66] who had also worked with Nachman ARONSZAJN.[67] The interpolation spaces studied in the joint article of Jacques-Louis LIONS and Jaak PEETRE seem to depend upon three parameters, and it is an important simplification of Jaak PEETRE to have shown that they actually only depend upon two parameters, one parameter $\theta \in (0,1)$ which they had already introduced, and a single parameter[68] $p \in [1,\infty]$. The theory of interpolation spaces as it can be taught now thus owes much more to Jaak PEETRE than to Jacques-Louis LIONS, a point which I had not emphasized enough before. In some instances, the name of Jacques-Louis LIONS appears for questions of interpolation related to his joint works with Enrico MAGENES, and that corresponds to applying the already developed theory of interpolation spaces to questions of partial differential equations.

A few years ago, I had needed the support of a few friends to find the strength to decide to revise the lecture notes which I had already written, and to complete the writing of some unfinished ones, in view of publishing them to attain a wider audience. I carried out the first revision of this course in August 2002, and I want to thank Thérèse BRIFFOD for her hospitality at that time.

I would not have been able to complete the publication of my first lecture notes and to feel able to start revising again this second set of lecture notes without the support of Lucia OSTONI, and I want to thank her for that and for much more, giving me the stability that I had lacked so much in the last

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[61] Marcel RIESZ, Hungarian-born mathematician, 1886–1969 (the younger brother of Frederic RIESZ). He worked in Stockholm and in Lund, Sweden.

[62] Frigyes (Frederic) RIESZ, Hungarian mathematician, 1880–1956. He worked in Kolozsvár (then in Hungary, now Cluj-Napoca, Romania), in Szeged and in Budapest, Hungary. He introduced the spaces L^p in honor of LEBESGUE and the spaces $\mathcal{H}^p$ in honor of HARDY, but no spaces are named after him, and the Riesz operators have been introduced by his younger brother Marcel RIESZ.

[63] Henri Léon LEBESGUE, French mathematician, 1875–1941. He worked in Rennes, in Poitiers, and in Paris, France; he held a chair (mathématiques, 1921–1941) at Collège de France, Paris, France.

[64] Godfrey Harold HARDY, English mathematician, 1877–1947. He worked in Cambridge, in Oxford, England, holding the Savilian chair of geometry in 1920–1931, and in Cambridge again, holding the Sadleirian chair of pure mathematics (established in 1701 by Lady SADLEIR) in 1931–1942.

[65] Sir Henry SAVILE, English mathematician, 1549–1622. He founded professorships of geometry and astronomy at Oxford.

[66] Emilio GAGLIARDO, Italian mathematician, born in 1930. He worked in Pavia, Italy.

[67] Nachman ARONSZAJN, Polish-born mathematician, 1907–1980. He worked in Lawrence, KS, where I visited him during my first visit to United States, in 1971.

[68] Jaak PEETRE has even observed that one can use $p \in (0,\infty]$, and he developed the theory of interpolation for quasi-normed spaces.

twenty-five years, so that I could feel safer in resuming my research of giving a sounder mathematical foundation to 20th century continuum mechanics and physics.

I want to thank my good friends Carlo SBORDONE and Franco BREZZI for having proposed to publish my lecture notes in a series of Unione Matematica Italiana.

Milano,[69] July 2006

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[69] Two months after writing this preface, I was elected a foreign member of Istituto Lombardo Accademia di Scienze e Lettere, Milano.

Detailed Description of Lectures

a.b refers to definition, lemma or theorem # b in lecture # a, while (a.b) refers to equation # b in lecture # a.

Lecture 1, Historical background: Dirichlet principle: (1.1) and (1.2); Radon measures: (1.3).

Lecture 2, The Lebesgue measure, convolution: Convolution: 2.1 and (2.1)–(2.3); condition on supports: (2.4); operators τ_a: 2.2 and (2.5); commutation of τ_a with convolution: 2.3 and (2.6) and (2.7); an example of a C^∞ function: 2.4 and (2.8).

Lecture 3, Smoothing by convolution: Smoothing sequences: 3.1 and (3.1); convergence of smoothing sequences: 3.2; density of $C_c^\infty(R^N)$: 3.3.

Lecture 4, Truncation; Radon measures; distributions: Truncating sequences: 4.1; smoothing characteristic functions: 4.2 and (4.1); Radon measures and distributions: 4.3 and (4.2)–(4.5); derivation of distributions: 4.4 and (4.6)–(4.8).

Lecture 5, Sobolev spaces; multiplication by smooth functions: $W^{m,p}(\Omega)$: 5.1 and (5.1); $W^{m,p}(\Omega)$ is a Banach space and $H^m(\Omega)$ is a Hilbert space: 5.2 and (5.2) and (5.3); convergences in C_c^∞ and in $\mathcal{D}'$: (5.4) and (5.5); principal value of $\frac{1}{x}$: (5.6)–(5.9); multiplication by smooth functions: 5.3 and (5.10); Leibniz's formula for functions: 5.4 and (5.11) and (5.12), and for distributions: 5.5 and (5.13).

Lecture 6, Density of tensor products; consequences: Tensor product of functions: (6.1); density of tensor products: 6.2 and (6.1)–(6.4); $x_j T = 0$ for all j means $T = C\,\delta_0$: 6.3 and (6.5) and (6.6); Ω connected and $\frac{\partial T}{\partial x_j} = 0$ for all j means T constant: 6.4 and (6.7)–(6.12); $C_c^\infty(R^N)$ is dense in $W^{m,p}(R^N)$ for $1 \le p < \infty$: 6.5 and (6.13) and (6.14); $W_0^{m,p}(\Omega)$: 6.6; Sobolev's embedding theorem: 6.7.

Lecture 7, Extending the notion of support: Being 0 on an open subset: 7.1 and (7.1) and (7.2); partitions of unity: 7.2; being 0 on a union of open subsets: 7.3; support of Radon measures or of distributions: 7.4; $W_{loc}^{m,p}(\Omega)$: 7.5; product of functions in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$: 7.6; Lipschitz functions and $W^{1,\infty}(\Omega)$: 7.7 and 7.8.

Lecture 8, Sobolev's embedding theorem, $1 \le p < N$: A domain with a cusp, where Sobolev's embedding does not hold: 8.1; interpolation inequality for L^p: 8.2 and (8.1); $W^{1,p}(R^N) \subset L^q(R^N)$ implies $p \le q \le p_*$: 8.3 and (8.2)–(8.6); elementary solutions: 8.4; the Laplacian and its elementary solution: (8.7) and (8.8); the formula used by SOBOLEV: (8.9); $W^{1,1}(R) \subset C_0(R)$: 8.5 and (8.10)–(8.12); a lemma used by GAGLIARDO and by NIRENBERG: 8.6 and (8.13)–(8.16); estimates in various $L^q(R^N)$ spaces by the Gagliardo–Nirenberg method: 8.7 and (8.17)–(8.20).

Lecture 9, Sobolev's embedding theorem, $N \le p \le \infty$: Sobolev's embedding theorem estimates for the case $p = N$: 9.1 and (9.1)–(9.5), and for the case $p > N$: (9.6); estimates using a parametrix of the Laplacian: (9.7)–(9.9); $W^{1,p}(R^N) \subset C^{0,\gamma}(R^N)$ for $p > N$: 9.2 and (9.10)–(9.13).

Lecture 10, Poincaré's inequality: Questions of units: (10.1)–(10.4); Poincaré's inequality: 10.1; when Poincaré's inequality holds, or does not hold: 10.2.
Lecture 11, The equivalence lemma; compact embeddings: Equivalence lemma: 11.1; when compactness holds, or does not hold: 11.2.
Lecture 12, Regularity of the boundary; consequences: $\mathcal{D}(\overline{\Omega})$: 12.1; continuous or Lipschitz boundary: 12.2; $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$: 12.3 and (12.1)–(12.5); linear continuous extension from $W^{1,p}(\Omega)$ into $W^{1,p}(R^N)$: 12.4 and (12.6) and (12.7); linear continuous extension from $W^{m,p}(R^N_+)$ into $W^{m,p}(R^N)$: 12.5 and (12.8) and (12.9).
Lecture 13, Traces on the boundary: Traces for smooth functions in one chart: 13.1 and (13.1) and (13.2); the trace operator γ_0: 13.2; trace of a product: 13.3; Hardy's inequality: 13.4 and (13.3)–(13.6); trace 0 and $W_0^{1,p}(\Omega_F)$ in one chart: 13.5, 13.6 and (13.7)–(13.9); trace 0 and $W_0^{1,p}(\Omega)$: 13.7.
Lecture 14, Green's formula: The exterior normal: 14.1 and (14.1); Green's formula in one chart: 14.2, 14.3 and (14.2) and (14.3); Green's formula: 14.4; the injection of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact: 14.5.
Lecture 15, The Fourier transform: Fourier series: (15.1); Fourier integral: 15.1 and (15.2)–(15.4); basic properties of the Fourier transform: (15.5)–(15.7); Fourier transform of a Radon measure: (15.8) and (15.9); $\mathcal{S}(R^N)$: 15.2 and (15.10); Plancherel's formula: (15.11); Fourier transform on $\mathcal{S}'(R^N)$: 15.3 and (15.12); basic properties of the extended Fourier transform: (15.13); $\mathcal{F}1 = \delta_0$: 15.4 and (15.14); $\mathcal{F}$ is an isomorphism on $\mathcal{S}(R^N)$, on $\mathcal{S}'(R^N)$, and on $L^2(R^N)$: 15.5 and (15.15)–(15.17); $\mathcal{F}$ on $H^1(R^N)$: 15.6 and (15.18); $H^s(R^N)$: 15.7; $H_0^m(\Omega)$ and $H^{-m}(\Omega)$: 15.8; characterization of $H^{-1}(\Omega)$: 15.9; $C_c^\infty(R^N)$ is dense in $H^s(R^N)$: 15.10; $\mathcal{F}(\gamma_0 u)$: 15.11 and (15.19) and (15.20).
Lecture 16, Traces of $H^s(R^N)$: $\gamma_0\big(H^s(R^N)\big) = H^{s-(1/2)}(R^{N-1})$ for $s > \frac{1}{2}$: 16.1 and (16.1)–(16.5); traces of derivatives: 16.2 and (16.6)–(16.9); $H^s(R^N)$ for $0 < s < 1$: 16.3 and (16.10)–(16.14); extension of Lipschitz functions: (16.15).
Lecture 17, Proving that a point is too small: $\frac{u}{r} \in L^2(R^N)$ for $u \in H^1(R^N)$ and $N \geq 3$: 17.1 and (17.1) and (17.2); functions in $C_c^\infty(R^N)$ vanishing near 0 are dense in $H^1(R^N)$ for $N \geq 3$: 17.2 and (17.3), and for $N = 2$: 17.3; estimating the norm of $\frac{u}{r \log(r/R_0)}$ in $L^2(\Omega)$ for $u \in H_0^1(\Omega)$ and $\Omega \subset R^2$: 17.4 and (17.4)–(17.6).
Lecture 18, Compact embeddings: $\langle T, \varphi\rangle = 0$ if derivatives of φ vanish on the compact support of T up to the order of T: 18.1 and (18.1) and (18.2); distributions with support a point: 18.2 and (18.3); a criterion for compactness in $L^p(\Omega)$: 18.3; compact injection from $W^{1,p}(\Omega)$ into $L^p(\Omega)$: 18.4 and (18.4); Poincaré's inequality for $H_0^1(\Omega)$ if $meas(\Omega) < \infty$: 18.5.
Lecture 19, Lax–Milgram lemma: Elliptic, parabolic, hyperbolic models using Δ: (19.1)–(19.3); second-order elliptic equations with variable coefficients: (19.4) and (19.5); conservation of charge: (19.6); Lax–Milgram lemma: 19.1 and (19.7)–(19.9); variational formulations: (19.10)–(19.14); Neumann condition: (19.15); Lagrange multiplier for $\int_\Omega u\,dx = 0$: (19.16) and (19.17), and for $\int_{\partial\Omega} u\,dH^{N-1} = 0$: (19.18) and (19.19).