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Linguistic Decision Making: Theory and Methods

Zeshui Xu

(基于语言信息的决策理论与方法)

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Preface

The complexity and uncertainty of objective thing and the fuzziness of human thought result in decision making with linguistic information in the fields of society, economy, medicine, management and military affairs, etc., such as personnel evaluation, military system performance evaluation, online auctions, supply chain management, venture capital and medical diagnostics.

In real-life, there are many situations, such as evaluating university faculty for tenure and promotion and evaluating the performance of different kinds of stocks and bonds, in which the information cannot be assessed precisely in numerical values but may be in linguistic variables. That is, variables whose values are not numbers but words or sentences in a natural or artificial language. For example, when evaluating the "comfort" or "design" of a car, linguistic labels like "good", "fair" and "poor" are usually used, and evaluating a car's speed, linguistic labels like "very fast", "fast" and "slow" can be used. Therefore, how to make a scientific decision with linguistic information is an interesting and important research topic that has been attracting more and more attention in recent years.

To date, a lot of methods have been proposed for dealing with linguistic information. These methods are mainly as follows:

(1) The methods based on extension principle, which make operations on the fuzzy numbers that support the semantics of the linguistic labels.

(2) The methods based on symbols, which make computations on the indexes of the linguistic terms.

Both the above methods develop some approximation processes to express the results in the initial expression domains, which produce the consequent loss of information and hence the lack of precision.

(3) The methods based on fuzzy linguistic representation model, which represent the linguistic information with a pair of values called 2-tuple, composed by a linguistic term and a value of the symbolic translation. Together with the model, the methods also give some computational techniques to deal with the 2-tuples without loss of information. But the model needs some transformation between a counting of information and the linguistic 2-tuple by a function in the aggregation process, and thus, the model is somewhat cumbersome in representation.

(4) The methods which compute with words directly.

Compared with the methods (1)~(3), the methods (4) can not only avoid losing any linguistic information, but also are straightforward and very convenient in calculation, and thus, are more practical in actual applications.

In recent years, the author has made an in-depth and systematical research on the methods (4) and their applications. The research results mainly include linguistic evaluation scales, linguistic aggregation operators, uncertain linguistic aggregation operators, dynamic linguistic aggregation operators, the priority theory and methods of linguistic preference relations, uncertain linguistic preference relations and incomplete linguistic preference relations, interactive approach to linguistic multi-attribute decision making, linguistic multi-attribute group decision making methods, dynamic linguistic multi-attribute decision making methods, uncertain linguistic multi-attribute decision making methods, and their applications in solving a variety of practical problems, such as the partner selection of supply chain management, personnel appraisal, investment decision making, military system efficiency dynamic evaluation, venture capital project evaluation, and enterprise technology innovation capacity evaluation, etc. This book will give a thorough and comprehensive introduction to these results, which mainly consists of the following parts:

The preface to this book gives a brief background introduction to the current study on the theory and methods of linguistic decision making, and summarizes the main contents and structure.

Chapter 1 mainly introduces the basis of linguistic decision making–Linguistic evaluation scales. Linguistic evaluation scales are classified into two types: additive linguistic evaluation scales and multiplicative linguistic evaluation scales. The unbalanced additive linguistic label sets and unbalanced multiplicative linguistic label sets are highlighted.

Chapter 2 introduces the aggregation techniques for linguistic information. A comprehensive survey of the existing main linguistic aggregation operators is provided.

Chapter 3 mainly introduces the concepts of linguistic preference relations, uncertain linguistic preference relations, incomplete linguistic preference relations, consistent linguistic preference relations, acceptable linguistic preference relations, and their desirable properties. Then the decision making approaches based on these linguistic preference relations are also overviewed.

Chapter 4 mainly introduces the approaches to linguistic multi-attribute decision making. Based on a variety of linguistic aggregation operators, such as the linguistic weighted averaging operator, dynamic linguistic weighted averaging operator, linguistic weighted geometric operator and dynamic linguistic weighted geometric operator, etc., a series of methods and models for multi-attribute decision making under linguistic environments are established, including the maximizing deviation procedure, ideal point-based model, goal programming model, interactive decision making approach, and multi-period multi-attribute decision making method, etc. Furthermore, most of these results are extended to accommodate multi-attribute decision making under uncertain linguistic environments.

This book is suitable for the engineers, technicians and researchers in the fields of fuzzy mathematics, operations research, information science, management science and systems engineering, etc. It can also be used as a textbook for the senior undergraduate and graduate students in the relevant professional institutions of higher learning.

Zeshui Xu
Nanjing
December, 2011

Some Commonly Used Symbols

S_i	Linguistic evaluation scale
$\bar{S}_i$	Extended linguistic evaluation scale
s_α, s_β	Linguistic labels
$\tilde{s}_i$	Uncertain linguistic variable
$\boldsymbol{w}, \boldsymbol{\omega}, \boldsymbol{\zeta}, \boldsymbol{\upsilon}, \boldsymbol{\xi}$	Weight vectors
x_i, e_k, G_k	Alternative, decision maker, attribute
$X, E, G, \Omega, \Theta, H, \tilde{S}_i$	Sets
d, ρ	Distance, similarity degree
$\langle u_i,\ s_{\alpha_i}\rangle, \langle u_i,\ \tilde{s}_i\rangle$	2-tuples
$\langle v_i, u_i, s_{\alpha_i}\rangle, \langle v_i, u_i, \tilde{s}_i\rangle$	3-tuples
$\boldsymbol{A}, \boldsymbol{A}_k, \boldsymbol{B}, \boldsymbol{B}_k, \tilde{\boldsymbol{A}}, \tilde{\boldsymbol{A}}_k, \tilde{\boldsymbol{B}}$	Matrices
$\tilde{\boldsymbol{B}}_k, \boldsymbol{R}, \tilde{\boldsymbol{R}}, \boldsymbol{R}_k, \tilde{\boldsymbol{R}}_k, \boldsymbol{P}$	Matrices
$a_{ij}, a_{ij}^{(k)}, b_{ij}, b_{ij}^{(k)}, r_{ij}, \tilde{r}_{ij}, r_{ij}^{(k)}, \tilde{r}_{ij}^{(k)}, p_{ij}$	Elements
c_j	Close degree
$\alpha_0, \tau_0, \eta_0, \beta_0$	Thresholds
ε_j^+	Upper deviation variable
ε_j^-	Lower deviation variable

Contents

Chapter 1
Linguistic Evaluation Scales

The complexity and uncertainty of objective thing and the fuzziness of human thought result in decision making with linguistic information in a wide variety of practical problems, such as personnel evaluation, military system performance evaluation, online auctions, supply chain management, venture capital, and medical diagnostics. In such problems, a realistic approach may be to use linguistic assessments instead of numerical values by means of linguistic variables, that is, variables whose values are not numbers but words or sentences in a natural or artificial language (Fan and Wang, 2003; 2004; Fan et al., 2002; Herrera and Herrera-Viedma, 2003; 2000a; 2000b; 1997; Herrera and Martínez, 2001a; 2001b; 2000a; 2000b; Herrera and Verdegay, 1993; Herrera et al., 2005; 2003; 2001a; 2001b; 2000; 1997; 1996a; 1996b; 1995; Herrera-Viedma, 2001; Herrera-Viedma and Peis, 2003; Herrera-Viedma et al., 2005; 2004; 2003; Wang and Chuu, 2004; Xu, 2010; 2009a; 2009b; 2009c; 2008; 2007a; 2007b; 2007c; 2007d; 2007e; 2006a; 2006b; 2006c; 2006d; 2006e; 2006f; 2006g; 2006h; 2006i; 2005a; 2005b; 2005c; 2005d; 2004a; 2004b; 2004c; 2004d; 2004e; 2004f; Xu and Da, 2003; 2002; Zadeh and Kacprzyk, 1999a; 1999b). This may arise for different reasons (Chen and Hwang, 1992): ① the information may be unquantifiable due to its nature; and ② the precise quantitative information may not be stated because either it is unavailable or the cost of its computation is too high and an "approximate value" may be tolerated. For example, when evaluating the "comfort" or "design" of a car, linguistic labels like "good", "fair" and "poor" are usually used, and evaluating a car's speed, linguistic labels like "very fast", "fast" and "slow" can be used (Bordogna et al., 1997; Levrat et al., 1997).

Considering that a proper linguistic evaluation scale should be predefined when a decision maker needs to provide his/her preferences over an object with linguistic labels (Carlsson and Fullér, 2000; Herrera et al., 1996a; Torra, 1996; Xu, 2004a; 2004b; 2004c; 2004d; 2004e; 2004f; Yager, 1995; 1992), in the following sections, we shall introduce some common linguistic evaluation scales:

1.1 Additive Linguistic Evaluation Scales

Yager (1995; 1992) defined an ordinal scale as

$$L = \{L_i | i = 1, 2, \cdots, m\} \tag{1.1}$$

such that $L_i > L_j$ if $i > j$.

Later, Cordón et al. (2002), Herrera et al. (2001a; 2000; 1996a), Herrera-Viedma et al. (2004; 2003), Martínez et al. (2005) and Torra (2001) introduced a finite and totally ordered discrete additive linguistic evaluation scale (linguistic label set):

$$S_1 = \{s_\alpha | \alpha = 0,\ 1,\ \cdots,\ \tau\} \tag{1.2}$$

where s_α represents a possible value for a linguistic label. In particular, s_0 and s_τ denote the lower and upper limits of linguistic labels used by a decision maker in practical applications, τ is a positive integer. The cardinality value of S_1 is odd such as 7 and 9, the limit of cardinality is 11 or not more than 13, it must be small enough so as not to impose useless precision on the decision makers, and it must also be rich enough in order to allow a discrimination of the performances of each object in a limited number of grades (Bordogna et al., 1997). The linguistic label s_α has the following characteristics:

(1) The set is ordered: $s_\alpha > s_\beta$, if $\alpha > \beta$;

(2) The negation operator is defined: $\text{neg}(s_\alpha) = s_\beta$ such that $\alpha + \beta = \tau$.

For example, a set of seven linguistic labels S_1 (Bordogna et al., 1997) could be (Figure 1.1)

$$S_1 = \{s_0 = \text{none},\ \ s_1 = \text{very low},\ \ s_2 = \text{low},\ \ s_3 = \text{medium},\ \ s_4 = \text{high},\ \ s_5 = \text{very high},\ \ s_6 = \text{perfect}\}$$

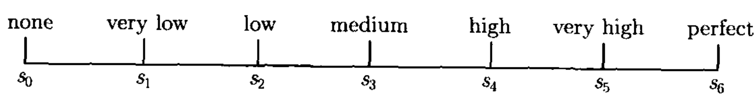

Figure 1.1 Additive linguistic evaluation scale S_1 ($\tau = 6$) (Xu, 2009c)

In the process of aggregating information, the aggregated result may not match any of the original linguistic labels in the additive linguistic evaluation scale S_1. In order to preserve all the given information, Dai et al. (2008) extended the discrete linguistic evaluation scale S_1 to a continuous linguistic evaluation scale:

$$\bar{S}_1 = \{s_\alpha | \alpha \in [0, q]\} \tag{1.3}$$

where q ($q > \tau$) is a sufficiently large positive integer. If $s_\alpha \in S_1$, then s_α is called an original linguistic label; otherwise, s_α is termed an extended (or virtual) linguistic label. The extended linguistic labels also meet the characteristics (1) and (2) described above.

In general, a decision maker uses the original linguistic labels to evaluate objects (or alternatives), and the virtual linguistic labels can only appear in calculations.

Based on the extended additive linguistic evaluation scale $\bar{S}_1$, Dai et al. (2008) introduced some operational laws:

Definition 1.1 (Dai et al., 2008) Let $s_\alpha, s_\beta \in \bar{S}$ and $\lambda \in [0,1]$. Then

(1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;

(2) $\lambda s_\alpha = s_{\lambda\alpha}$.

However, in the process of operations, if we take $s_2 =$ low and $s_4 =$ high, then

$$s_2 \oplus s_4 = s_6 \tag{1.4}$$

i.e., the aggregated result of the linguistic labels "low" and "high" is "perfect". This is not in accordance with actual situations.

To overcome the issue above, Dai et al. (2008) improved the additive linguistic evaluation scale above, and gave a subscript-symmetric linguistic evaluation scale:

$$S_2 = \{s_\alpha | \alpha = -\tau, \cdots, -1, 0, 1, \cdots, \tau\} \tag{1.5}$$

where the mid linguistic label s_0 represents an assessment of "indifference", and with the rest of the linguistic labels being placed symmetrically around it. In particular, $s_{-\tau}$ and s_τ are the lower and upper limits of linguistic labels used by a decision maker in practical applications, τ is a positive integer, and S_2 satisfies the following conditions (Xu, 2005c):

(1) If $\alpha > \beta$, then $s_\alpha > s_\beta$;

(2) The negation operator is defined: $\text{neg}(s_\alpha) = s_{-\alpha}$, especially, $\text{neg}(s_0) = s_0$.

For example, when $\tau = 3$, S_2 can be taken as (Figure 1.2)

$$S_2 = \{s_{-3} = \text{none},\ s_{-2} = \text{very low},\ s_{-1} = \text{low},\ s_0 = \text{medium},\ s_1 = \text{high},\ s_2 = \text{very high},\ s_3 = \text{perfect}\}$$

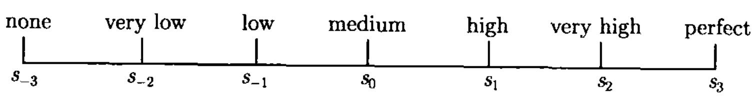

Figure 1.2 Additive linguistic evaluation scale S_2 ($\tau = 3$)

For the convenience of calculations and in order to preserve all the given information, Xu (2005c) extended the discrete linguistic evaluation scale S_2 to a continuous linguistic evaluation scale $\bar{S}_2 = \{s_\alpha | \alpha \in [-q, q]\}$, where q ($q > \tau$) is a sufficiently large positive integer. The operational laws of linguistic labels in $\bar{S}_2$ are similar to (1) and (2) in Definition 1.1. In this case, if we take $s_{-2} =$ low and $s_2 =$ high, then we have

$$s_{-2} \oplus s_2 = s_0 \tag{1.6}$$

i.e., the aggregated result of the linguistic labels "low" and "high" is "medium", which is clearly more in accord with actual situations than the case described in (1.4).

The linguistic labels in the above linguistic evaluation scales are uniformly and symmetrically distributed. However, in real-life, the unbalanced linguistic information may appear due to the nature of the linguistic variables used in the problems (Herrera and Herrera-Viedma, 2003). Therefore, to develop unbalanced linguistic label sets is an interesting and important research topic. In the following, we shall pay attention to this issue:

Xu (2000) gave a simulation-based evaluation of four common numerical evaluation scales (1-9 scale (Saaty, 1980), 9/9-9/1 scale (Wang and Ma, 1993), 10/10-18/2 scale (Wang and Ma, 1993) and exponential scale (Shu and Liang, 1990)) from different angles (Table 1.1). The results show that 10/10-18/2 scale is of the best performance.

Table 1.1 Four common numerical scale (Xu, 2000)

	1-9 scale	Exponential scale	9/9–9/1 scale	10/10–18/2 scale
Equal importance	1	$9^{(0)}$ (1)	9/9 (1)	10/10 (1)
Moderate importance of one over another	3	$9^{(1/9)}$ (1.277)	9/7 (1.286)	12/8 (5)
Essential or strong importance	5	$9^{(3/9)}$ (2.08)	9/5 (6)	14/6 (2.333)
Very strong importance	7	$9^{(6/9)}$ (4.327)	9/3 (3)	16/4 (4)
Extreme importance	9	$9^{(9/9)}$ (9)	9/1 (9)	18/2 (9)
	k	$9^{(k/9)}$ (4.327)	$9/(10-k)$	$(9+k)/(11-k)$

In the process of practical applications, such as selecting projects for different kinds of funding policies, and evaluating university faculty for tenure and promotion, we find that as the unbalanced linguistic information appears, the absolute value of the deviation between the indices of two adjoining linguistic labels should increase as the absolute value of the indices of the linguistic labels steadily increase. Motivated by this idea and based on the 10/10-18/2 scale, we introduce a linguistic evaluation scale from the viewpoints of simplicity, feasibility and practicability (Dai et al., 2008; Xu, 2009c), shown as below:

$$S_3 = \left\{ s_\alpha \middle| \alpha = -\frac{2(\tau-1)}{\tau+2-\tau}, -\frac{2(\tau-1-1)}{\tau+2-(\tau-1)}, \cdots, 0, \cdots, \frac{2(\tau-1-1)}{\tau+2-(\tau-1)}, \frac{2(\tau-1)}{\tau+2-\tau} \right\} \tag{1.7}$$

The right part (after s_0) of S_3 can be written as

$$S_3^+ = \left\{ s_\alpha \middle| \alpha = \frac{\tau+i}{\tau+2-i} - 1 = \frac{2(i-1)}{\tau+2-i}, \quad i = 2, \cdots, \tau-1, \tau \right\} \tag{1.8}$$

and the left part (before s_0) of S_3 can be written as

$$S_3^- = \left\{ s_\alpha \middle| \alpha = -\frac{2(i-1)}{\tau+2-i}, \quad i = \tau, \tau-1, \cdots, 2 \right\} \tag{1.9}$$

Obviously, (1.7) can be simplified as

$$S_3 = \left\{ s_\alpha \middle| \alpha = -(\tau-1), \ -\frac{2}{3}(\tau-2), \cdots, \ 0, \cdots, \ \frac{2}{3}(\tau-2), \ \tau-1 \right\} \tag{1.10}$$

where the mid linguistic label s_0 represents an assessment of "indifference", and with the rest of the linguistic labels being placed symmetrically around it. Especially, $s_{-(\tau-1)}$ and $s_{(\tau-1)}$ are the lower and upper limits of linguistic labels used by a decision maker in practical applications. τ is a positive integer, and the cardinality value of S_3 is $2\tau - 1$. The linguistic labels in S_3 have the following characteristics:

(1) If $\alpha > \beta$, then $s_\alpha > s_\beta$;

(2) The negation operator is defined: $\text{neg}(s_\alpha) = s_{-\alpha}$, especially, $\text{neg}(s_0) = s_0$.

For example, if $\tau = 1$, then we can take S_3 as (Figure 1.3)

$$S_3 = \{s_{-1} = \text{low}, \quad s_0 = \text{medium}, \quad s_1 = \text{high}\}$$

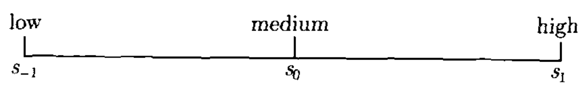

Figure 1.3 Additive linguistic evaluation scale S_3 ($\tau = 1$) (Xu, 2009c)

When $\tau = 4$, S_3 could be (Figure 1.4)

$$S_3 = \{s_{-3} = \text{none}, \ s_{-4/3} = \text{very low}, \ s_{-1/2} = \text{low}, \ s_0 = \text{medium}, \ s_{1/2} = \text{high}, \ s_{4/3} = \text{very high}, \ s_3 = \text{perfect}\}$$

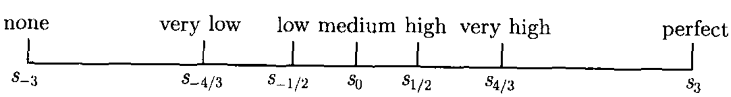

Figure 1.4 Additive linguistic evaluation scale S_3 ($\tau = 4$) (Xu, 2009c)

When $\tau = 5$, S_3 could be (Figure 1.5)

$S_3 = \{s_{-4} = \text{none}, \ s_{-2} = \text{very low}, \ s_{-1} = \text{low}, \ s_{-0.4} = \text{slightly low}, \ s_0 = \text{medium},$

$s_{0.4}$ = slightly high, s_1 = high, s_2 = very high, s_4 = perfect}

Similarly, we extend the discrete linguistic evaluation scale S_3 into a continuous linguistic evaluation scale $\bar{S}_3 = \{s_\alpha | \alpha \in [-q, q]\}$, where q $(q > \tau - 1)$ is a sufficiently large positive integer. The operational laws of linguistic labels in $\bar{S}_3$ are similar to (1) and (2) in Definition 1.1.

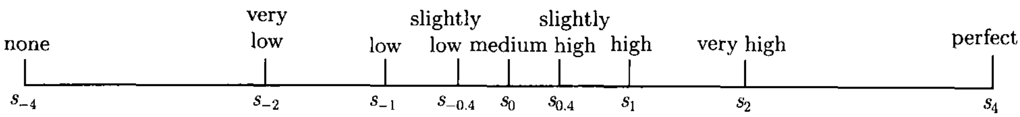

Figure 1.5 Additive linguistic evaluation scale S_3 $(\tau = 5)$ (Xu, 2009c)

Theorem 1.1 (Dai et al., 2008) For the linguistic evaluation scale S_3, the absolute value of the deviation between the indices of two adjoining linguistic labels should increase as the absolute values of the indices of the linguistic labels steadily increase.

Proof We first consider the linguistic label set S_3^+. Let s_{α_1} and s_{α_2} be two adjoining linguistic labels in S_3^+. Without loss of generality, let the indices of the linguistic labels s_{α_1} and s_{α_2} be

$$\alpha_1 = \frac{2(i-1)}{\tau+2-i}, \quad \alpha_2 = \frac{2(i+1-1)}{\tau+2-(i+1)}$$

respectively, then

$$\begin{aligned} |\alpha_1 - \alpha_2| &= \frac{2(i+1-1)}{\tau+2-(i+1)} - \frac{2(i-1)}{\tau+2-i} = \frac{2i}{\tau-i+1} - \frac{2(i-1)}{\tau+2-i} \\ &= \frac{2(\tau+1)}{(\tau+2-i)(\tau+1-i)} \end{aligned} \tag{1.11}$$

From (1.11), we can see that with the increase of i, the numerator of (1.11) remains unchanged, but its denominator decreases, and thus, the absolute value of the deviation, $|\alpha_1 - \alpha_2|$, also increases correspondingly. We can prove the other cases in a similar way.

In the above, we have introduced three types of additive linguistic evaluation scales S_i $(i = 1, 2, 3)$, where S_1 and S_2 are the uniform additive linguistic evaluation scales, while S_3 is an unbalanced additive linguistic evaluation.

In the next section, we shall introduce some multiplicative linguistic evaluation scales.

1.2 Multiplicative Linguistic Evaluation Scales

Xu (2004e) defined a multiplicative linguistic evaluation scale as follows:

$$S_4 = \left\{ s_\alpha | \alpha = \frac{1}{\tau}, \cdots, \frac{1}{2}, 1, 2, \cdots, \tau \right\} \tag{1.12}$$

where s_α is a linguistic label. In particular, $s_{1/\tau}$ and s_τ are the lower and upper limits of the linguistic labels used by the decision makers in actual applications, τ is a positive integer, and S_4 has the following characteristics:

(1) If $\alpha > \beta$, then $s_\alpha > s_\beta$;

(2) The reciprocal operator is defined: $\text{rec}(s_\alpha) = s_\beta$, such that $\alpha\beta = 1$, especially, $\text{rec}(s_1) = s_1$.

For example, when $\tau = 4$, S_4 could be (Figure 1.6)

$$\begin{aligned} S_4 = \{ & s_{1/4} = \text{none}, \quad s_{1/3} = \text{very low}, \quad s_{1/2} = \text{low}, \quad s_1 = \text{medium}, \\ & s_2 = \text{high}, \quad s_3 = \text{very high}, \quad s_4 = \text{perfect} \} \end{aligned}$$

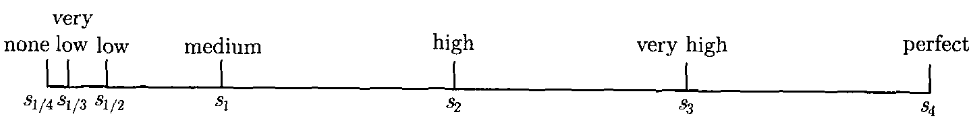

Figure 1.6 Multiplicative linguistic evaluation scale S_4 ($\tau = 4$)

The multiplicative linguistic evaluation scale S_4 has the following property:

Theorem 1.2 Let S_4 be defined as (1.12), and let the right and left parts of S_4 be defined as

$$S_4^+ = \{ s_\alpha | \alpha = 1, 2, \cdots, \tau - 1, \tau \} \tag{1.13}$$

and

$$S_4^- = \left\{ s_\alpha \middle| \alpha = \frac{1}{\tau}, \frac{1}{\tau - 1}, \cdots, \frac{1}{2}, 1 \right\} \tag{1.14}$$

respectively. Then

(1) The absolute value of the deviation between the indices of each two adjoining linguistic labels in S_4^+ is a constant;

(2) The absolute value of the deviation between the indices of two adjoining linguistic labels in S_4^+ should increase as the absolute values of the indices of the linguistic labels steadily increase.

Proof (1) Now we rewrite (1.13) as

$$S_4^+ = \{ s_{\alpha_i} | \alpha_i = i, \quad i = 1, 2, \cdots, \tau \} \tag{1.15}$$

Then

$$|\alpha_{i+1} - \alpha_i| = (i + 1) - i = 1, \quad i = 1, 2, \cdots, \tau - 1 \tag{1.16}$$

Hence, for any $i = 1, 2, \cdots, \tau - 1$, $|\alpha_{i+1} - \alpha_i|$ is a constant.

(2) We rewrite (1.14) as

$$S_4^- = \left\{ s_{\alpha_i} \middle| \alpha_i = \frac{1}{\tau - (i-1)},\ i = 1, 2, \cdots, \tau \right\} \tag{1.17}$$

Then

$$|\alpha_{i+1} - \alpha_i| = \frac{1}{\tau - (i+1-1)} - \frac{1}{\tau - (i-1)} = \frac{1}{(\tau - i + 1)(\tau - i)}, \quad i = 1, 2, \cdots, \tau - 1 \tag{1.18}$$

We can see from (1.18) that $|\alpha_{i+1} - \alpha_i|$ increases with the increase of i. This completes the proof.

Xu (2009a) defined another multiplicative linguistic evaluation scale:

$$S_5 = \left\{ s_\alpha \middle| \alpha = \frac{1}{\tau}, \frac{2}{\tau}, \cdots, \frac{\tau - 1}{\tau}, 1, \frac{\tau}{\tau - 1}, \cdots, \frac{\tau}{2}, \tau \right\} \tag{1.19}$$

where $s_{1/\tau}$ and s_τ are the lower and upper limits of the linguistic labels used by the decision makers in actual applications, τ is a positive integer, and S_5 has the following characteristics:

(1) If $\alpha > \beta$, then $s_\alpha > s_\beta$;

(2) The reciprocal operator is defined: $\text{rec}(s_\alpha) = s_\beta$, such that $\alpha\beta = 1$, especially, $\text{rec}(s_1) = s_1$.

The multiplicative linguistic evaluation scale S_5 has the following property:

Theorem 1.3 (Xu, 2009a) Let S_5 be defined as (1.19), and let the right and left parts of S_5 be defined as

$$S_5^+ = \left\{ s_\alpha \middle| \alpha = 1, \frac{\tau}{\tau - 1}, \cdots, \frac{\tau}{2}, \tau \right\} \tag{1.20}$$

and

$$S_5^- = \left\{ s_\alpha \middle| \alpha = \frac{1}{\tau}, \frac{2}{\tau}, \cdots, \frac{\tau - 1}{\tau}, 1 \right\} \tag{1.21}$$

respectively. Then

(1) The greater the value of the index of a linguistic label in S_5^+, the greater the value of the deviation between the indices of this linguistic label and its adjoining linguistic label in S_5^+;

(2) The greater the value of the index of a linguistic label in S_5^-, the smaller the value of the deviation between the indices of this linguistic label and its adjoining linguistic label in S_5^-.

Proof (1) For convenience of description, we rewrite S_5^+ as

$$S_5^+ = \left\{ s_{\alpha_i} \middle| \alpha_i = \frac{\tau}{\tau - (i-1)},\ i = 1, 2, \cdots, \tau \right\} \tag{1.22}$$